The pleating lamination of convex co-compact hyperbolic manifolds

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The hyperbolic space

Definition

$\mathbb{H}^3 = \{\in \mathbb{R}^{3,1} \mid \langle x, x\rangle_{3,1} = -1 \ \& \ x_0 > 0\}$, with the induced metric.

Complete, simply connected manifold of constant curvature -1.
Klein: as the interior of the unit ball in $\mathbb{R}^3$.
Isometry group $PSL(2,\mathbb{C})$, acting conformally on $\mathbb{C}P^1 = \partial_\infty \mathbb{H}^3$.

Definition

A *hyperbolic 3-manifold* is a Riemannian manifold M of constant curvature -1 (or equivalently, locally modelled on $\mathbb{H}^3$).

If M is complete and oriented, $M = \mathbb{H}^3/\rho(\pi_1 M)$, where $\rho : \pi_1 M \to PSL(2,\mathbb{C})$.

Convex co-compact and quasifuchsian manifolds

Definition

A (non-compact) hyperbolic manifold is *convex co-compact* if it is complete and contains a non-empty, compact, geodesically convex subset $K \subset M$.

NB: $K \subset M$ is *geodesically convex* if any geodesic segment with endpoints in K is contained in K.

Definition

M is quasifuchsian if convex co-compact and homeomorphic to $S \times \mathbb{R}$, where S is a closed surface of genus ≥ 2.

We will focus on quasifuchsian manifolds, but things extend to convex co-compact manifolds.

Simultaneous uniformization

Let M be quasifuchsian, $M \simeq S \times \mathbb{R}$.
$M = \mathbb{H}^3/\rho(\pi_1 S)$, $\rho : \pi_1 S \to PSL(2, \mathbb{C})$.
Limit set: $\Lambda_\rho = \overline{\rho(\pi_1 S).x} \cap \partial_\infty \mathbb{H}^3$ is a Jordan curve (quasicircle).
$\mathbb{C}P^1 \setminus \Lambda_\rho = \Omega_- \cup \Omega_+$.
ρ acts properly discontinuously on $\Omega_\pm$, conformally.
$\partial_\infty M = (\Omega_+/\rho(\pi_1 S)) \cup (\Omega_-/\rho(\pi_1 S))$.
Therefore we get two conformal structures on S,
$(c_-, c_+) \in \mathcal{T}_S \times \mathcal{T}_S$.

Theorem (Bers double uniformization (1950))

The map $M \mapsto (c_, c_+)$, from $\mathcal{QF}$ to $\mathcal{T}_S \times \mathcal{T}_S$, is 1-1 (biholomorphism).

The convex core

Let M be quasifuchsian.

The intersection of two non-empty geodesically convex subsets is geodesically convex and non-empty. Therefore, M contains a smallest non-empty convex subset, its *convex core* $C(M)$.

M is *Fuchsian* if $C(M)$ is a totally geodesic surface. Otherwise, $C(M)$ is 3d (considered below).

$C(M)$ is a convex subset without extremal points (because minimal). Therefore, $\partial C(M)$ is the union of two *pleated surfaces*: totally geodesic except along geodesics where it is "pleated".

Therefore $\partial C(M)$ is equipped with:

- its induced metric, hyperbolic: $m_+, m_- \in \mathcal{T}_S$,
- its *measured pleating lamination*, a closed union of disjoint geodesics, equipped with a *transverse measure* which records the pleating. $({}_{-}, l_+) \in \mathcal{ML}_S \times \mathcal{ML}_S$.

Measured laminations on surfaces

Let S be a closed surface, of genus ≥ 2, with m hyperbolic metric.
Measured laminations are composed of:

- a geodesic lamination: closed union of disjoint, complete geodesics,
- a transverse measure (weight on transverse segments).

In fact does not require m, topological data (e.g. closed curves vs geodesics). Properties:

- $\mathcal{ML}_S$ homeomorphic to ball, piecewise linear structure,
- $dim(\mathcal{ML}_S) = \dim(\mathcal{T}_S) = 6g - 6$,
- $P\mathcal{ML}_S = \partial\mathcal{T}_S$ (Thurston compactification),
- $\mathcal{T}_S \times \mathcal{ML}_S \simeq T^*\mathcal{T}_S$.

Measured laminations correspond to HQD in the "hyperbolic" (vs "complex") Teichmüller theory.

Two questions

Thurston asked two questions (1980s).

Question (induced metrics)

Can any pair $(m_+, m_-) \in \mathcal{T}_S \times \mathcal{T}_S$ be uniquely realized as induced metric on $\partial C(M)$?

Question

Is M uniquely determined by (l_-, l_+)?

Analogy with e.g. hyperbolic polyhedra (compact, ideal):

- (Alexandrov) $P \subset \mathbb{H}^3$ compact is uniquely determined by the induced metric on ∂P, hyperbolic metric with cone singularities of angle $< 2\pi$ on S^2,

Motivation (cont'd)

- (Rivin) $P \subset \mathbb{H}^3$ ideal is uniquely determined by the induced metric on ∂P, hyperbolic metric with cusps on $S^2 \setminus \{x_1, \cdots, x_n\}$,
- (Andreev, Hodgson-Rivin) $P \subset \mathbb{H}^3$ compact is uniquely determined by dihedral angles (if acute) or *dual metric*,
- (Andreev, Rivin) $P \subset \mathbb{H}^3$ ideal is uniquely determined by dihedral angles.

Deeper motivation: quasifuchsian (resp. convex co-compact) manifolds connect the *complex* Teichmüller theory "at infinity" to the *hyperbolic* Teichmüller theory on the boundary of the convex core.

- Complex $\mathcal{T}_S$: complex structures, holomorphic quadratic differentials, ...
- Hyperbolic $\mathcal{T}_S$: hyperbolic structures, measured laminations, ...

Results

Theorem (Sullivan, Epstein-Marden, Labourie, 1990s)

Any $(m_+, m_-) \in \mathcal{T}_S \times \mathcal{T}_S$ *can be realized on* $\partial C(M)$.
Uniqueness ???

Theorem (Bonahon-Otal 2004)

(l_-, l_+) *can be realizes on* $\partial C(M)$ *iff it* fills *and has no curve with weight* $> \pi$.

Theorem (Dular-S. 2024)

M (non Fuchsian) quasifuchsian manifold is uniquely determined by (l_-, l_+).

A word on the proof

Definition

$\mathcal{L} : \mathcal{QF} \to \mathcal{FML}_\pi \subset \mathcal{ML}_S \times \mathcal{ML}_S, M \mapsto (l_-, l_+)$.

We want to prove that $\mathcal{L}$ is 1-1.

1. $\mathcal{L}$ is a limit of homeomorphisms.

Uses a foliation of $M \setminus C(M)$ by K-surfaces, $K \in (-1, 0)$ (Labourie 1992) and characterization of M by III on those K-surfaces (S. 2006). Defines $\mathcal{L}_K : \mathcal{QF} \to \mathcal{T}_S \times \mathcal{T}_S$, $M \to (III_{-,K}, III_{+,K})$. Then $\mathcal{L}_K \to \mathcal{L}$ as $K \to -1$.

2. $\mathcal{L}^{-1}(\{l\})$ is compact (Bonahon-Otal 2004, "closing lemma").
3. Since $\mathcal{L}$ is a limit of homeos and ANR, $\mathcal{L}^{-1}(\{l\})$ is contractible (Finney 1967, Daverman 1986).
4. $\mathcal{L}^{-1}(\{l\})$ is a real analytic variety (follows from Bonahon).
5. A compact, contractible real analytic variety is a point (Sullivan).