From 4D cross-ratio systems to constant curvature surfaces

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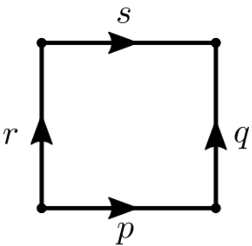
Joint work with T. Hoffmann and A. O. Sageman-Furnas

Conference on Discrete Geometric Structures

September 3, 2024

A quad equation



Edge-based $p, q, r, s \in \mathcal{A}$ for some unit associative algebra $\mathcal{A}$

such that

$(\lambda + q)(\lambda + p) = (\lambda + s)(\lambda + r)$ for all scalars λ

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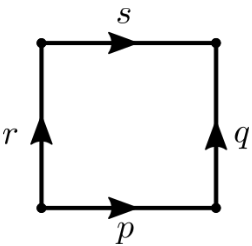

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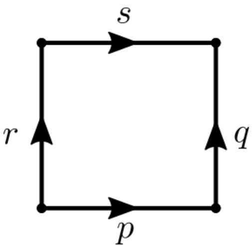

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Goal: study geometric applications of this equation!

Some (discrete) examples

$$(\lambda + q)(\lambda + p) = (\lambda + s)(\lambda + r)$$

- Cross-ratio systems [Nijhoff '97]
- K-Nets/ Pseudospherical nets [Bobenko,Pinkall '96]
- Lund-Regge systems [Schief '07]
- Euler top [Moser, Veselov '91]
- Hashimoto flow [Pinkall,Springborn,Weißmann '07][Hoffmann '08]
- Bicycle transformation [Tabachnikov, Tsukerman '13]
- Isothermic nets [Bobenko,Pinkall '96][Hertrich-Jeromin '00]
- Polygon recutting [Adler '93][Izosimov '23]
- Closed linkages [Hegedüs,Schicho,Schröcker '13],...
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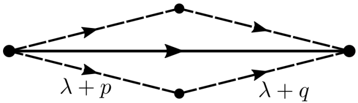
Please tell me if you know more!

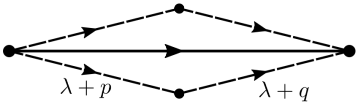
Factorization



$L = (\lambda + q)(\lambda + p)$

Quadratic polynomial L belongs to diagonal

Factorization



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Converse: Can we factorize polynomial equations?

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- more complicated combinatorics

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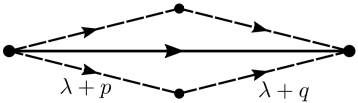

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Example: CMC Lax representation [Bobenko,Pinkall '99]

$$\begin{pmatrix} a & \lambda^2 b + \frac{1}{\lambda^2}\frac{1}{b} \\ -\frac{1}{\lambda^2}\bar{b} - \lambda^2 \frac{1}{\bar{b}} & \bar{a} \end{pmatrix} \stackrel{?}{=} (\lambda + \frac{1}{\lambda} s)(\lambda + \frac{1}{\lambda} r)$$

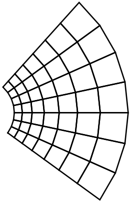
Result: CMC can be interpreted as slice in a 4D cross-ratio system.

$\rightarrow$ Topic of this talk!

Overview

Complex plane

Cross-ratio system
$f : \mathbb{Z}^2 \to \mathbb{C}$



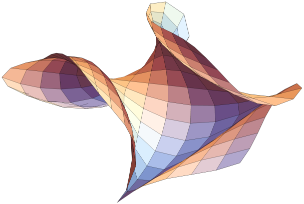
$\Longrightarrow$

associated
family

$\Longrightarrow$

Euclidean space

Constant curvature surface
$f^t : \mathbb{Z}^2 \to \mathbb{R}^3$



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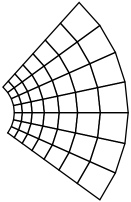

represented by matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

$\Longrightarrow$ associated family $\Longrightarrow$

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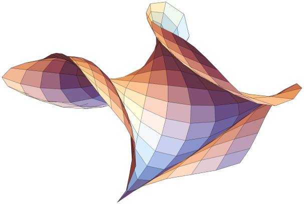


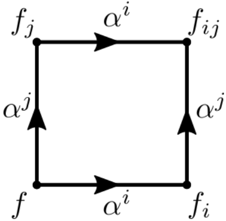
represented by quaternions

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathbb{H} \subseteq \mathbb{C}^{2\times 2}$$

Cross-ratio systems

Definition (Edge labelling)

A map α : $\{$edges of $\mathbb{Z}^n\} \to \mathbb{C}$ where values on opposite edges agree.



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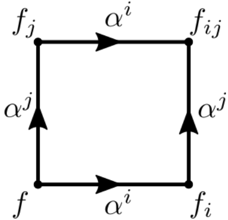
Cross-ratio of four points in $\mathbb{C}$:

$$cr(a,b,c,d) = \frac{a-b}{b-c}\frac{c-d}{d-a}$$

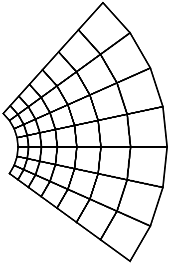


Definition (Cross-ratio system)

A map $f : \mathbb{Z}^n \to \mathbb{C}$ with

$$cr(f, f_i, f_{ij}, f_j) = \frac{(\alpha^i)^2}{(\alpha^j)^2}$$

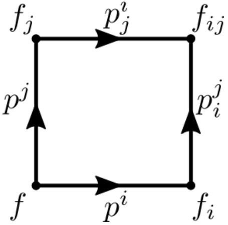
for some edge labelling α.



Holomorphic map: $cr = -1$

Matrix representation

Define $p^i := \begin{pmatrix} 0 & f_i - f \\ -\frac{(\alpha^i)^2}{f_i - f} & 0 \end{pmatrix}$ on each edge.



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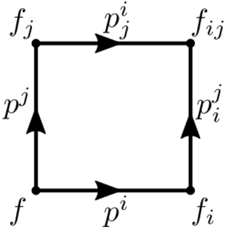

Theorem

f is a cross-ratio system if and only if

$$(\lambda + \frac{1}{\lambda} p^j_i)(\lambda + \frac{1}{\lambda} p^i) = (\lambda + \frac{1}{\lambda} p^i_j)(\lambda + \frac{1}{\lambda} p^j)$$

for all $\lambda \in \mathbb{C}$.

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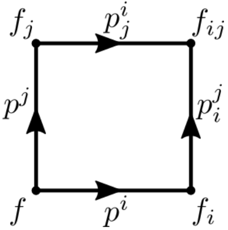

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This is a linear Lax representation of cross-ratio systems as in [Nijhoff '97].

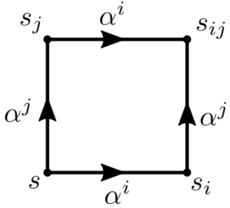
The Hirota system

Define vertex based map $s : \mathbb{Z}^n \to \mathbb{C}$:

$$s(0, ..., 0) = 1, \qquad f_i - f = \alpha^i s_i s$$

Cross-ratio equation becomes:

$$\frac{s_{ij}}{s} = \frac{\alpha^j s_j - \alpha^i s_i}{\alpha^j s_i - \alpha^i s_j}$$



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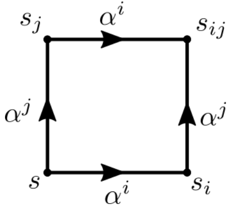
$$\lambda + \frac{1}{\lambda} p^i = \begin{pmatrix} \lambda & \frac{1}{\lambda}\alpha^i s_i s \\ -\frac{1}{\lambda}\frac{\alpha^i}{s_i s} & \lambda \end{pmatrix}$$



Associated family

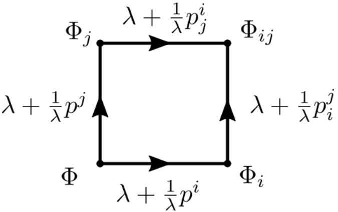
1. Choose a $\lambda : \mathbb{R} \to \mathbb{C}$
2. Define frame $\Phi : \mathbb{Z}^n \times \mathbb{R} \to \mathbb{C}^{2\times 2}$ by

$$\Phi(0, ..., 0, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Phi_i = (\lambda(t) + \frac{1}{\lambda(t)} p^i)\Phi$$

3. The associated family is given by

$$f^t = \Phi^{-1} \frac{d}{dt} \Phi$$



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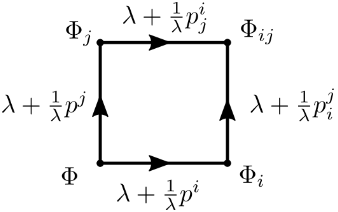
- f^t is no cross-ratio system
- $\lambda + \frac{1}{\lambda} p^i \in \mathbb{H} \Rightarrow f^t \in \mathbb{H}$



K-nets

Observation: For real $\lambda, \alpha^i \in \mathbb{R}$ and unitary $s \in \mathbb{S}^1$

$$\lambda + \frac{1}{\lambda} p^i = \begin{pmatrix} \lambda & \frac{1}{\lambda}\alpha^i s_i s \\ -\frac{1}{\lambda}\frac{\alpha^i}{s_i s} & \lambda \end{pmatrix}$$

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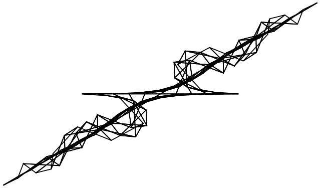
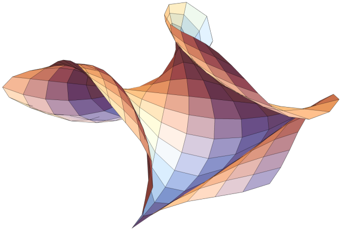


$\Longrightarrow$

associated family

$\Longrightarrow$



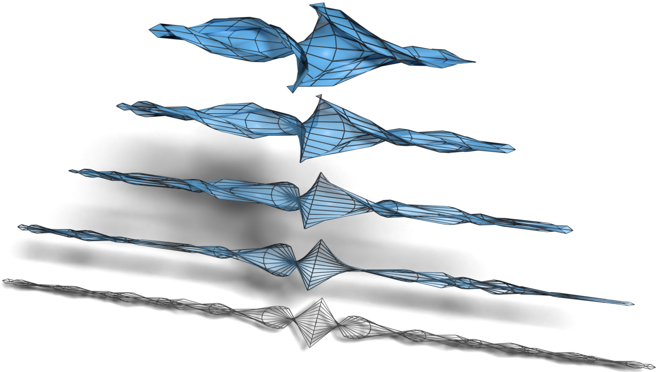
Cross-ratio system

with $cr = \frac{(\alpha^i)^2}{(\alpha^j)^2} > 0$

K-net

K-Nets

K-Nets: Discrete surfaces with Gaussian curvature $K = -1$.



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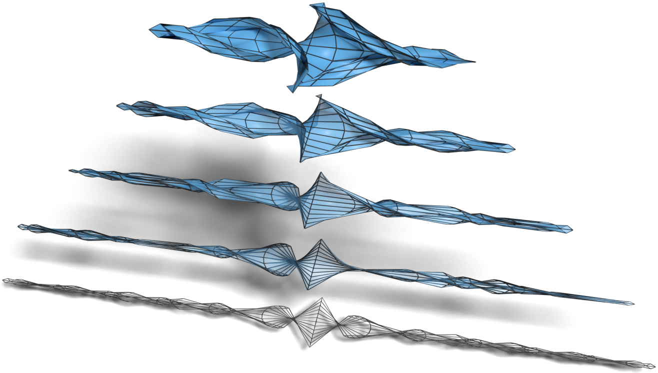
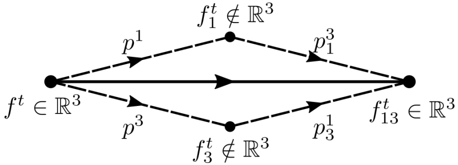


Can we find other types of surfaces from this construction?

More possibilities

Idea: We only require every second point of f^t to be real:



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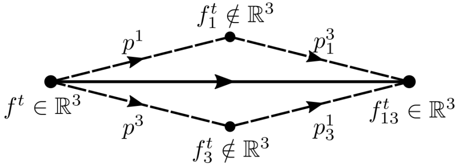

Then, it suffices if $L := (\lambda + \frac{1}{\lambda} p_1^3)(\lambda + \frac{1}{\lambda} p^1)$ is a quaternion!

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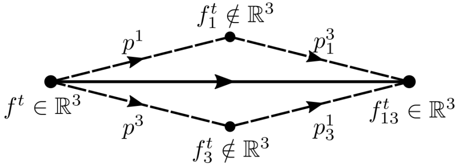

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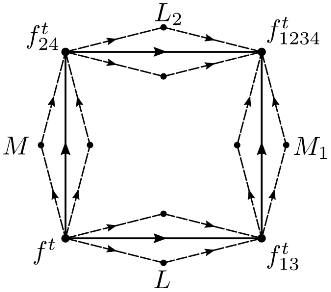
Real quad corresponds to 4D cube!



4D cross-ratio systems

4D Cross-ratio system:

$$f : \mathbb{Z}^4 \to \mathbb{C}$$

Diagonal net $\mathcal{D}$:

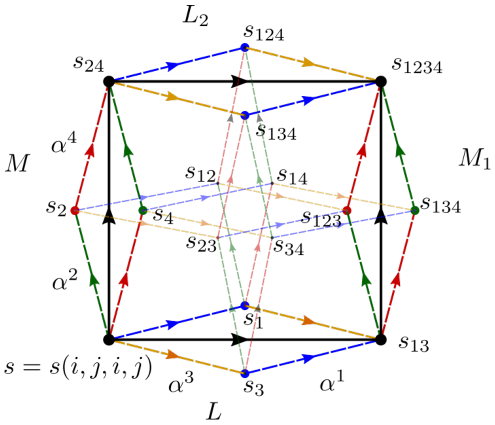
$$f(i,j,i,j) : \mathbb{Z}^2 \to \mathbb{C}$$

First diagonal direction:

$$\alpha^1, \alpha^3, s_1, s_3, L, ...$$

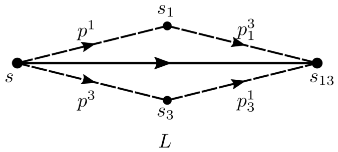
Second diagonal direction:

$$\alpha^2, \alpha^4, s_2, s_4, M, ...$$



Reductions

One direction at vertices $(i, j, i, j) \in \mathcal{D}$:



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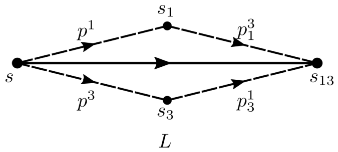

1. The C^- *lattice*: If

$$s_3 = \frac{1}{\bar{s}_1} \qquad \text{and} \qquad \alpha^3 = \bar{\alpha}^1$$

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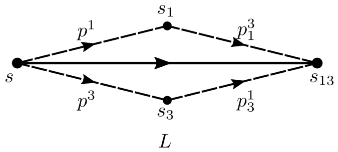

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2. The C^+ *lattice*: If

$$s_3 = \frac{1}{\bar{s}_1} \qquad \text{and} \qquad \alpha^3 = \frac{-1}{\bar{\alpha}^1}$$

then $\tilde{L} = \frac{1}{\beta} G_{13} L G^{-1}$ is a quaternion for all $\lambda \in \mathbb{S}^1$.

(Here: gauge $G = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$ and $\beta = \sqrt{-\alpha^3 \alpha^1}$ scaling factor)

Extension

Theorem

Every 2D cross-ratio system $f(\cdot,\cdot,0,0) : \mathbb{Z}^2 \to \mathbb{C}$ can be extended into a unique C^+ and a unique C^- lattice.

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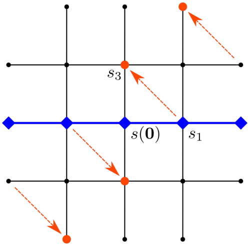
Construction:

1. 1,3-coordinate plane $f(\cdot,0,\cdot,0)$:



- $s_3=\frac{1}{\bar{s}_1}$
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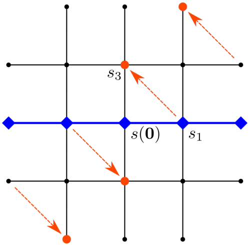

- $s_3 = \frac{1}{s_1}$
- Cross-ratio evolution

2. 2,4-coordinate plane $f(0,\cdot,0,\cdot)$ analogously.
3. Axes determine everything.

The C^+ lattice and CMC surfaces

Theorem

The associated family of a C^+ lattice yields a family of discrete nets with constant positive Gaussian curvature $K > 0$.

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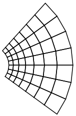
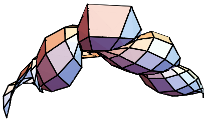
Holomorphic map



C^+-extension $\Longrightarrow$ 'DPW'

C^+-lattice $\Longrightarrow$

$\Longrightarrow$ associated family



constant positive Gaussian curvature surface

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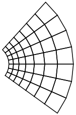
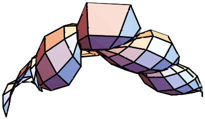


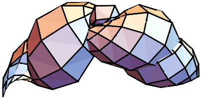
$\xRightarrow[\text{'DPW'}]{C^+\text{-extension}}$ C^+-lattice $\Longrightarrow$

$\xRightarrow[\text{associated family}]{}$



constant positive Gaussian curvature surface

$\xRightarrow[\pm\frac{1}{2}n]{}$

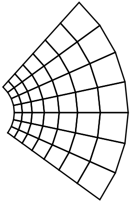


constant mean curvature (CMC) surface

The DPW method

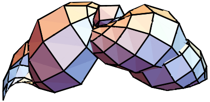
Construction of CMC surfaces from holomorphic data

- Smooth [Dorfmeister, Pedit, Wu '98]
- Discrete [Hoffmann '99][Ogata, Yasumoto '17]



Holomorphic data

$\Longrightarrow$

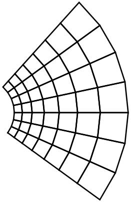


constant mean curvature (CMC) surface

The DPW method

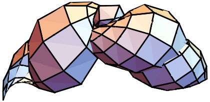
Construction of CMC surfaces from holomorphic data

- Smooth [Dorfmeister, Pedit, Wu '98]
- Discrete [Hoffmann '99][Ogata, Yasumoto '17]



Holomorphic data

$\Longrightarrow$



constant mean curvature
(CMC) surface

We can interpret DPW as extension of holomorphic data to a 4D cross-ratio system!

The C^- lattice and a surprising object

Theorem

The associated family of a C^- lattice yields a family of (algebraic) discrete nets with constant negative Gaussian curvature $K < 0$.

The C^- lattice and a surprising object

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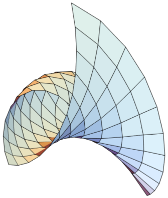
Holomorphic map $\overset{C^-\text{-extension}}{\Longrightarrow}$ C^--lattice $\Longrightarrow$

$\Longrightarrow$ associated family



constant negative Gaussian curvature surface?

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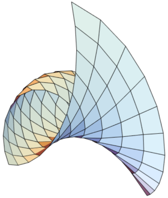
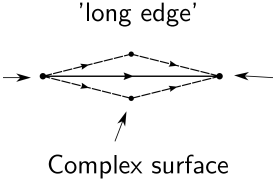
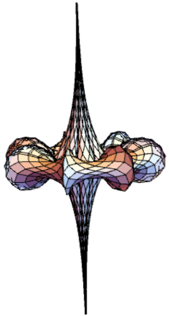


constant negative Gaussian curvature surface?

This does not look right!

Breather transformations

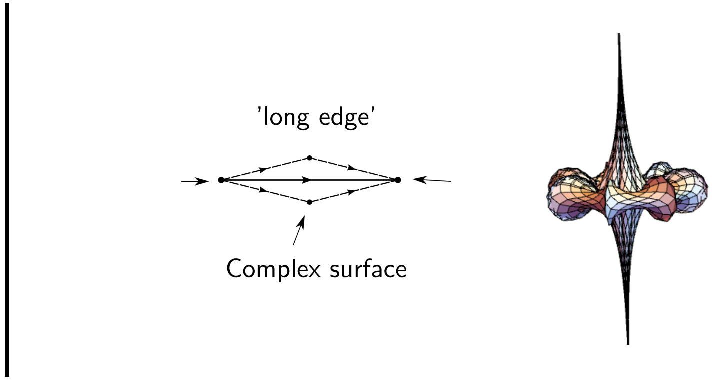




Straight line

Breather transformation

Breather transformations



Straight line

Breather transformation

Idea: Use C^- lattice to construct a lattice of breather transformations!

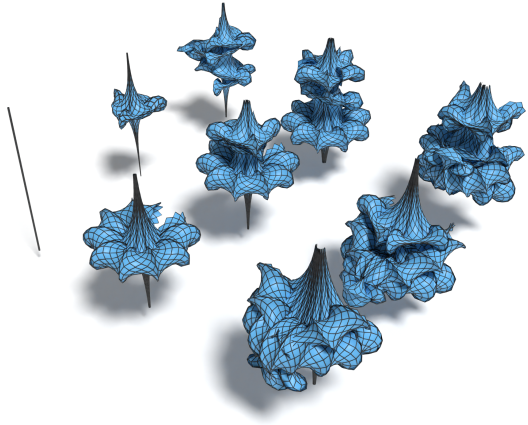
The breather lattice

Holomorphic
map

$+$

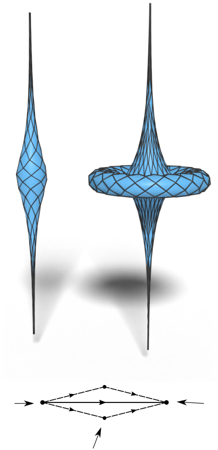
K-net

$\Longrightarrow$

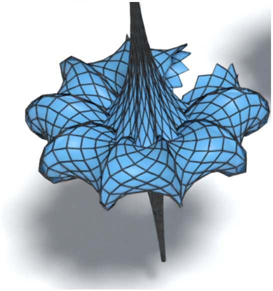


Factorization of a breather transformation

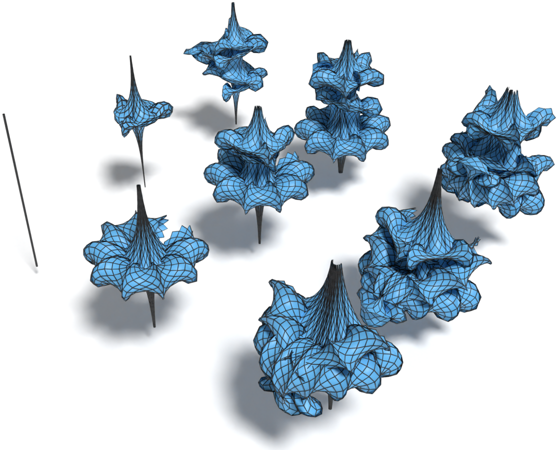
Straight line



Factors (no K-Nets)



Breather transformation



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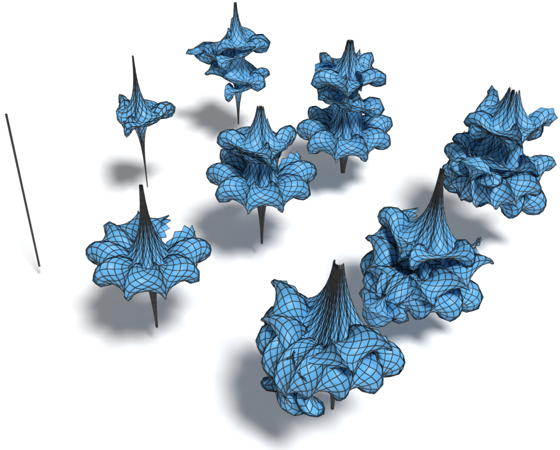

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Thank you!