Discrete Painlevé equations and pencils of quadrics in $\mathbb{P}^3$

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Painlevé equations

Painlevé equations (P_I–P_{VI}, 1902): 2nd order nonlinear ODEs $u'' = F(u', u, t)$ with the *Painlevé property* (solutions have no initial value dependent singularities other than poles). E.g.,

$$P_I : \quad u'' = 6u^2 + t,$$

$$P_{III} : \quad u'' = \frac{u'^2}{u} - \frac{u}{t} + \frac{1}{t}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}.$$

Integrability: *isomonodromic structure* (each Painlevé equation is a condition for a certain Fuchsian system with u-dependent coefficients to have t-independent monodromy).

Discrete Painlevé equations

- First appearance: Shohat (1936),

$$u_n(u_{n+1} + u_n + u_{n-1}) = \alpha n + \beta,$$

  equation for coefficients of the three-term recurrence for certain orthogonal polynomials (now called dP_I).
- Grammaticos, Ramani, Papageorgiou (1991): *singularity confinement* as a discrete analogue of Painlevé property. Systematic search for *discrete Painlevé equations*. Isomonodromic structure.
- Sakai (2001): geometric classification scheme, based on relation to *generalized Halphen surfaces* (blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points).

Example: qP_{III} vs a QRT recurrence

$$\mathrm{qP}_{III}: \qquad y_{n+1}y_{n-1} = \frac{y_n^2 - 1}{1 - c_n^{-2}y_n^2}, \quad c_n = c_0 q^{2n}.$$

A non-autonomous version of a *QRT recurrence*

$$y_{n+1}y_{n-1} = \frac{y_n^2 - 1}{1 - c^{-2}y_n^2},$$

which can be put as $f : \mathbb{C}^2 \to \mathbb{C}^2$ (a *QRT root*),

$$f: \quad (x, y) \mapsto (\widetilde{x}, \widetilde{y}) = \left(y, \frac{y^2 - 1}{x(1 - c^{-2}y^2)} \right).$$

Inverse map:

$$f^{-1} : (\widetilde{x}, \widetilde{y}) \mapsto (x, y) = \left(\frac{\widetilde{x}^2 - 1}{\widetilde{y}(1 - c^{-2}\widetilde{x}^2)}, \widetilde{x} \right).$$

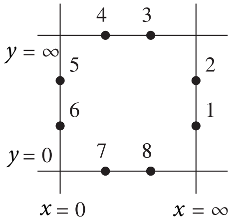
QRT as a birational map

Lift f to $\mathbb{P}^1 \times \mathbb{P}^1$. Then it has four indeterminacy points

$$p_1 = (\infty, c), \quad p_2 = (\infty, -c), \quad p_5 = (0, 1), \quad p_6 = (0, -1),$$

while f^{-1} has four indeterminacy points

$$p_3 = (c, \infty), \quad p_4 = (-c, \infty), \quad p_7 = (1, 0), \quad p_8 = (-1, 0).$$



QRT as a birational map

The eight singular points define a *pencil of biquadratic curves* in $\mathbb{P}^1 \times \mathbb{P}^1$, which are invariant under the map f:

$$C_\mu : \quad \Big\{ c^{-2}x^2y^2 - x^2 - y^2 + 1 - \mu xy = 0 \Big\}.$$

(Note that C_∞ is the union of four lines from the previous picture.)

Singularity confinement patterns for f read:

$$\begin{array}{ccccccc}
\{y=-c\} & \to & (-c,\infty) & \to & (\infty,c) & \to & \{x=c\}, \\
\{y=c\} & \to & (c,\infty) & \to & (\infty,-c) & \to & \{x=-c\}, \\
\{y=-1\} & \to & (-1,0) & \to & (0,1) & \to & \{x=1\}, \\
\{y=1\} & \to & (1,0) & \to & (0,-1) & \to & \{x=-1\}.
\end{array}$$

From a pencil of biquadratic curves to QRT map

One can construct f starting with the pencil C_μ.

- For a given (x, y), determine μ such that $(x, y) \in C_\mu$.
- Define the *vertical switch* i_1 and the *horizontal switch* i_2 as the second intersection point of C_μ with the line $x = \mathrm{const}$, resp. $y = \mathrm{const}$. One computes:

$$i_1(x,y) = \Big(x, \frac{x^2-1}{y(1-c^{-2}x^2)}\Big), \quad i_2(x,y) = \Big(\frac{y^2-1}{x(1-c^{-2}y^2)}, y\Big).$$

- Define the *QRT map* $F = i_1 \circ i_2$. If the pencil C_μ is symmetric under $s(x,y) = (y,x)$, define the *QRT root* $f = s \circ i_2 = i_1 \circ s$, so that $F = f^2$.

From a QRT recurrence to qP_{III}

One can consider qP_{III} as a sequence of maps of the type f, but for which (some of) the points $p_1, \ldots, p_8$ depend on n (*de-autonomization*).

Main requirement (which singles out the evolution $c_n = c_0 q^{2n}$): the same singularity confinement patterns.

No algebraic integrals of motion! However, universally accepted as an integrable system:

- vanishing algebraic entropy
- isomonodromic structure (hence, monodromy data serve as transzendental integrals of motion)

Discrete time Euler top

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{cases} \quad \rightsquigarrow \quad \begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2). \end{cases}$$

Features:

- Equations are linear w.r.t. $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^{\mathrm{T}}$:

$$A(x,\epsilon)\widetilde{x} = x, \qquad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix},$$

imply a rational map, which is *reversible* (therefore birational):

$$\widetilde{x} = \Phi(x,\epsilon) = A^{-1}(x,\epsilon)x, \quad \Phi^{-1}(x,\epsilon) = \Phi(x,-\epsilon).$$

Explicit formulas:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases}$$

where $\Delta(x,\epsilon) = \det A(x,\epsilon)$

$$= 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3.$$

Projective formulation

In homogeneous coordinates on $\mathbb{P}^3$:

$$\Phi : [x_1 : x_2 : x_3 : x_4] \mapsto [\widetilde{x}_1 : \widetilde{x}_2 : \widetilde{x}_3 : \widetilde{x}_4],$$

where

$$\begin{cases} \widetilde{x}_1 = x_1 x_4^2 + 2\epsilon\alpha_1 x_2 x_3 x_4 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2), \\ \widetilde{x}_2 = x_2 x_4^2 + 2\epsilon\alpha_2 x_3 x_1 x_4 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2), \\ \widetilde{x}_3 = x_3 x_4^2 + 2\epsilon\alpha_3 x_1 x_2 x_4 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2), \\ \widetilde{x}_4 = x_4^3 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)x_4 - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3. \end{cases}$$

A birational map $\Phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ of $\deg = 3$.

Integrability of dET

- Two independent integrals:

$$I_3(x,\epsilon)=\frac{1-\epsilon^2\alpha_2\alpha_3x_1^2}{1-\epsilon^2\alpha_3\alpha_1x_2^2},\quad I_1(x,\epsilon)=\frac{1-\epsilon^2\alpha_3\alpha_1x_2^2}{1-\epsilon^2\alpha_1\alpha_2x_3^2}.$$

- Invariant volume form:

$$\omega=\frac{dx_1\wedge dx_2\wedge dx_3}{\phi(x)},\quad \phi(x)=1-\epsilon^2\alpha_i\alpha_jx_k^2.$$

- bi-Hamiltonian structure found by [M. Petrera, Yu.S.'10].

Geometry of the discrete time Euler top

Space $\mathbb{P}^3$ is foliated by joint level sets of two integrals of dET, each being a spatial elliptic curve – an intersection of two quadrics

$$C_{\lambda\mu} = \mathcal{Q}_\lambda \cap \mathcal{P}_\mu,$$

where

$$\mathcal{Q}_\lambda = \left\{ H_{12}(x,\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2}{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2} = \lambda \right\}$$

is a hyperboloid, while

$$\mathcal{P}_\mu = \left\{ I_3(x,\epsilon) = \frac{1 - \epsilon^2 \alpha_2 \alpha_3 x_1^2}{1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2} = \mu \right\}$$

is a cylinder.

dET is a 3D QRT root

For any $x \in \mathbb{P}^3$, determine λ and μ so that $x \in \mathcal{Q}_\lambda \cap \mathcal{P}_\mu$.

Let ℓ_1, ℓ_2 be two straight line generators of $\mathcal{Q}_\lambda$ through x.

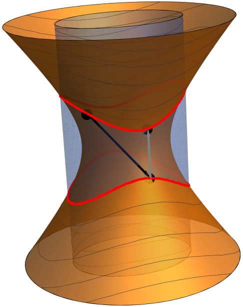
Denote by $i_1(x)$, $i_2(x)$, the second intersection points of ℓ_1, ℓ_2 with $\mathcal{P}_\mu$. This defines two birational involutions $i_1, i_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

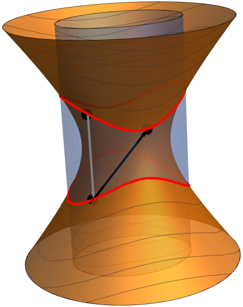
Both $\mathcal{Q}_\lambda$ and $\mathcal{P}_\mu$ are symmetric w.r.t. a linear projective involution $s(x_1, x_2, x_3) = (x_1, x_2, -x_3)$.

Theorem [N. Smeenk' 20]. *The discrete time Euler top can be represented as the 3D QRT root*

$$\Phi = i_1 \circ s = s \circ i_2.$$

dET as a 3D QRT root





dET vs. Cremona inversion

Theorem. [J. Alonso, Yu.S., K. Wei' 21] Discrete time Euler top can be represented as

$$\Phi = M_1 \circ \sigma \circ M_2^{-1},$$

where M_1 and M_2 are linear projective automorphisms of $\mathbb{P}^3$,

$$M_1^{-1} = \begin{pmatrix} b_1 & -b_2 & -b_3 & 1 \\ -b_1 & b_2 & -b_3 & 1 \\ -b_1 & -b_2 & b_3 & 1 \\ b_1 & b_2 & b_3 & 1 \end{pmatrix}, \quad M_2^{-1} = \begin{pmatrix} -b_1 & b_2 & b_3 & 1 \\ b_1 & -b_2 & b_3 & 1 \\ b_1 & b_2 & -b_3 & 1 \\ -b_1 & -b_2 & -b_3 & 1 \end{pmatrix},$$

with $b_i = \epsilon\sqrt{\alpha_j \alpha_k}$, and $\sigma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is the *Cremona inversion*

$$\sigma : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} 1/z_1 \\ 1/z_2 \\ 1/z_3 \\ 1/z_4 \end{bmatrix} = \begin{bmatrix} z_2 z_3 z_4 \\ z_1 z_3 z_4 \\ z_1 z_2 z_4 \\ z_1 z_2 z_3 \end{bmatrix}.$$

Algebraic geometry of Cremona inversion

The critical set and the indeterminacy set:

$$\mathcal{C}(\sigma) = \bigcup_{i=1}^{4} \Pi_i, \qquad \mathcal{I}(\sigma) = \bigcup_{1\le i<j\le 4} \ell_{ij},$$

where $\Pi_i = \{z_i = 0\}$ are the coordinate planes and $\ell_{ij} = \Pi_i \cap \Pi_j$ are lines. Use also the four points

$$e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Singularity confinement patterns:

$$\sigma : \quad \Pi_i \to e_i \to \Pi_i, \quad i = 1, \ldots, 4.$$

Singularity confinement for dET

For $\Phi = M_1 \sigma M_2^{-1}$, we set $A_i := M_1(e_i)$ and $B_i := M_2(e_i)$, then

$$\Phi : \quad M_2(\Pi_i) \to A_i, \quad B_i \to M_1(\Pi_i).$$

Suppose

$$\Phi(A_i) = B_i, \qquad i = 1, \dots, 4,$$

then have the following singularity patterns:

$$\Phi : \quad M_2(\Pi_i) \ \to \ A_i \ \to \ B_i \ \to \ M_1(\Pi_i).$$

The above condition says:

$$(M_1 \sigma M_2^{-1}) M_1 e_i = M_2 e_i \ \Leftrightarrow \ M \sigma M(e_i) = e_i, \quad i = 1, \dots, 4,$$

where $M = M_2^{-1} M_1$. It is satisfied for discrete time Euler top, for which

$$M = M_2^{-1} M_1 \simeq \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

Φ as a 3D QRT root, again

For $\Phi = M \circ \sigma$, the family of quadrics through eight points $A_i = Me_i$, $B_i = e_i$ is two-dimensional, containing two pencils

$$\mathcal{Q}_\lambda = \{z \in \mathbb{P}^3 : Q_0(z) - \lambda Q_1(z) = 0\},$$

$$\mathcal{P}_\mu = \{z \in \mathbb{P}^3 : Q_0(z) - \mu Q_2(z) = 0\},$$

where

$$Q_0(z) = (z_1 + z_3)(z_2 + z_4), \quad Q_1(z) = (z_1 - z_3)(z_2 - z_4),$$

$$Q_2(z) = z_1^2 + z_2^2 - z_3^2 - z_4^2.$$

Base curve of the pencil $\mathcal{Q}_\lambda$ – a skew quadrilateral.

Map Φ is the 3D QRT root defined by $\mathcal{Q}_\lambda$, $\mathcal{P}_\mu$. In particular, it leaves each $\mathcal{Q}_\lambda$ invariant. It is instructive to compute the restriction of Φ to $\mathcal{Q}_\lambda$.

Φ fiberwise

For this, one can parametrize each $\mathcal{Q}_\lambda$ by $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ according to

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x + \lambda^{-1}xy \\ y + 1 \\ x - \lambda^{-1}xy \\ y - 1 \end{bmatrix}$$

(*pencil-adapted coordinates* on $\mathbb{P}^3$). Thus,

$$x = \frac{z_1 + z_3}{z_2 - z_4} = \frac{\lambda(z_1 - z_3)}{z_2 + z_4}, \quad y = \frac{z_2 + z_4}{z_2 - z_4} = \frac{\lambda(z_1 - z_3)}{z_1 + z_3}, \quad \lambda = \frac{Q_0(z)}{Q_1(z)}.$$

In these coordinates:

$$\Phi : \quad \widetilde{x} = y, \quad \widetilde{y} = \frac{y^2 - 1}{x(1 - \lambda^{-2}y^2)}, \quad \widetilde{\lambda} = \lambda.$$

Each $\mathcal{Q}_\lambda$ is invariant, and in pencil-adapted coordinates Φ acts on $\mathcal{Q}_\lambda$ as a λ-dependent 2D QRT root.

Generalized dET

A more general solution of $M \circ \sigma \circ M(e_i) = e_i$, $i = 1, \ldots, 4$:

$$M = M_q = \begin{pmatrix} -1 & q & 1 & q \\ q & -1 & q & 1 \\ 1 & q & -1 & q \\ q & 1 & q & -1 \end{pmatrix}.$$

The corresponding map $\Phi_q = M_q \circ \sigma$:

$$\Phi_q : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{z}_1 \\ \widetilde{z}_2 \\ \widetilde{z}_3 \\ \widetilde{z}_4 \end{bmatrix} = \begin{bmatrix} z_2 z_4 (z_1 - z_3) + q z_1 z_3 (z_2 + z_4) \\ z_1 z_3 (z_2 - z_4) + q z_2 z_4 (z_1 + z_3) \\ z_2 z_4 (z_3 - z_1) + q z_1 z_3 (z_2 + z_4) \\ z_1 z_3 (z_4 - z_2) + q z_2 z_4 (z_1 + z_3) \end{bmatrix}.$$

We have: $\Phi_q = L_q \circ \Phi$, where $L_q = M_q M^{-1}$.

From generalized dET to qP_{III}

Map $\Phi_q = M_q \circ \sigma = L_q \circ \Phi$ has exactly the same singularity confinement patterns as Φ:

$$\Phi_q : \quad \Pi_i \ \to \ A_i \ \to \ B_i \ \to \ M_q(\Pi_i),$$

where $A_i = M_q e_i$, $B_i = e_i$. But:

The family of quadrics through eight points $A_i = M_q e_i$, $B_i = e_i$ is one-dimensional, the pencil $\mathcal{Q}_\lambda$. Φ_q has no rational integrals and maps each $\mathcal{Q}_\lambda$ to $\mathcal{Q}_{q^2\lambda}$. In pencil-adapted coordinates:

$$\Phi_q : \quad \widetilde{x} = y, \quad \widetilde{y} = \frac{y^2 - 1}{x(1 - \lambda^{-2} y^2)}, \quad \widetilde{\lambda} = q^2 \lambda.$$

This is equivalent to qP_{III}:

$$y_{n+1} y_{n-1} = \frac{y_n^2 - 1}{1 - \lambda_n^{-2} y_n^2}, \qquad \lambda_n = \lambda_0 q^{2n}.$$

General scheme

Input data.

1. A pencil $\{C_\mu\}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with the base points $s_1, \ldots, s_8 \in \mathbb{P}^1 \times \mathbb{P}^1$, and the corresponding QRT map $f = i_1 \circ i_2$.
2. One distinguished biquadratic curve C_∞ of the pencil.

Goal.

- Construct a discrete Painlevé equation as a de-autonomization of f along the fiber C_∞.

General scheme

Construction [J. Alonso, Yu.S., K. Wei '24].

1. Let $\mathcal{Q}_0 = \{X_1X_2 - X_3X_4 = 0\}$. Recall that $\mathcal{Q}_0$ is the image of the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$, via

   $$\mathbb{P}^1 \times \mathbb{P}^1 \ni \big([x_1 : x_0], [y_1 : y_0]\big) \mapsto [x_1y_0 : x_0y_1 : x_1y_1 : x_0y_0] \in \mathcal{Q}_0.$$

2. Let $S_1, \ldots, S_8$ be the images of the base points $s_1, \ldots, s_8$ under Segre embedding.

3. To each biquadratic curve

   $$C_\mu : \{a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4x^2 + a_5y^2 + a_6xy + a_7x + a_8y + a_9 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

   there corresponds a quadric

   $$\mathcal{P}_\mu : \{a_1X_3^2 + a_2X_1X_3 + a_3X_2X_3 + a_4X_1^2 + a_5X_2^2 + a_6X_3X_4 + a_7X_1X_4 + a_8X_2X_4 + a_9X_4^2 = 0\} \subset \mathbb{P}^3.$$

   (Actually, C_μ can be identified with $\mathcal{Q}_0 \cap \mathcal{P}_\mu$.)

General scheme

Construction (contunued).

4. Construct the *pencil of quadrics* $\{\mathcal{Q}_\lambda\}$ *in* $\mathbb{P}^3$ *spanned by* $\mathcal{Q}_0$ *and* $\mathcal{P}_\infty$. The base curve of $\{\mathcal{Q}_\lambda\}$ is $\mathcal{Q}_0 \cap \mathcal{P}_\infty$, the image of C_∞ under Segre embedding. Its intersection with the base curve of $\{\mathcal{P}_\mu\}$ consists of $S_1, \ldots, S_8$.

5. Consider *3D QRT involutions* i_1, i_2 on $\mathbb{P}^3$ defined by intersections of generators ℓ_1, ℓ_2 of $\mathcal{Q}_\lambda$ with the quadrics $\mathcal{P}_\mu$. On each quadric $\mathcal{Q}_\lambda$, the map $\Phi = i_1 \circ i_2$ induces a λ-deformation of the original QRT map f.

General scheme

Construction (Painlevé deformation).

6. Find a birational map L on $\mathbb{P}^3$ with the following properties.
   a) L preserves the pencil $\{\mathcal{Q}_\lambda\}$, and maps each $\mathcal{Q}_\lambda$ to $\mathcal{Q}_{\sigma(\lambda)}$, where $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ is a Möbius automorphism fixing the set

$$\mathrm{Sing}(\mathcal{Q}) := \{\lambda \in \mathbb{P}^1 : \ \mathcal{Q}_\lambda \ \text{is} \ \text{degenerate}\}.$$

   b) The maps $L \circ i_1, L \circ i_2$ have the same singularity confinement properties as the QRT involutions i_1, i_2.

   Then the map $\Psi = (L \circ i_1) \circ (L \circ i_2)$ is declared to be a discrete Painlevé equation obtained by the de-autonomization of the QRT map along the fiber C_∞.

Projective classification of pencils of quadrics in $\mathbb{P}^3$

(From: E. Casas-Alvero. *Analytic projective geometry.* EMS, 2014).