Isotropic Geometry and Applications in Geometric Computing

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Timber Gridshell 2023 (Schling et al.)

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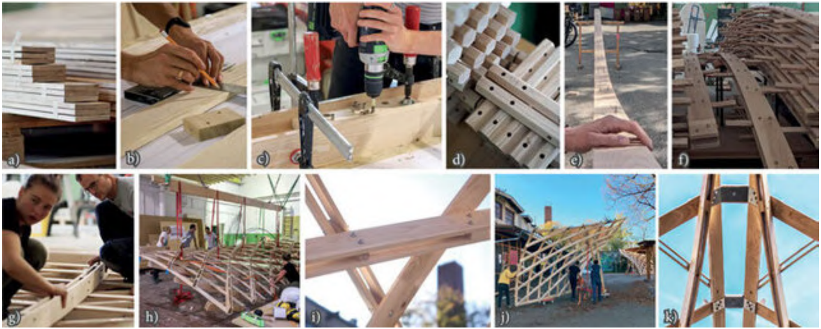
Timber Gridshell 2023 (Schling et al.)



Webs in Reality

Strips tangential (Natterer et al. 2000)

Strips orthogonal (Schling et al. 2022)

Consider a surface S with a parametric representation $s(u, v)$.

- The net of isoparameter curves $u = \text{const}$ and $v = \text{const}$ is called a 2-web on S.
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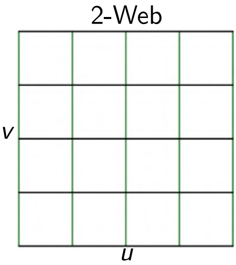

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- The net of isoparameter curves $u =$ const and $v =$ const is called a 2-web on S.
- The net of isoparameter curves $u =$ const and $v =$ const extended by diagonal curves $u + v =$ const is a 3-web (hexagonal) on S.
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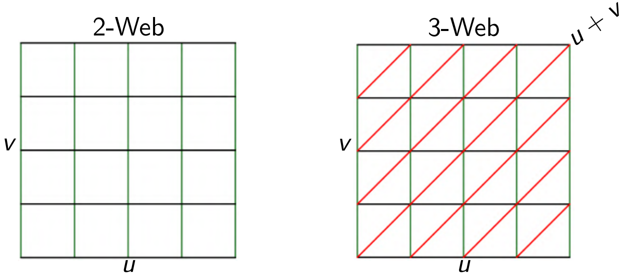

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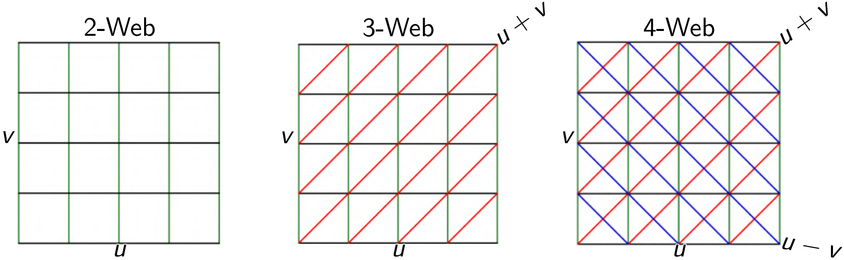
- The net of isoparameter curves $u = \text{const}$ and $v = \text{const}$ is called a 2-web on S.
- The net of isoparameter curves $u = \text{const}$ and $v = \text{const}$ extended by diagonal curve $u + v = \text{const}$ is a (hexagonal) 3-web on S.
- The net of isoparameter curves $u = \text{const}$ and $v = \text{const}$ extended by both diagonal curves $u \pm v = \text{const}$ is called a 4-web on S.



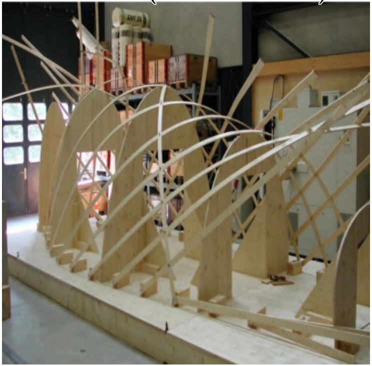
- AA: 2-web formed by 2 families of asymptotic lines on a surface.
  AA Web (Schling et al.)

Example of Webs

- AA: 2-web formed by 2 families of asymptotic lines on a surface.
- GG: 2-web formed by 2 families of geodesic lines on a surface.

AA Web (Schling et al.)

GG Web (EPF Lausanne)



Definition

A 3-web is called a GGG web if each family of curves is geodesic.

Mayrhofer: derives the PDE, but finding an explicit solution is challenging.

Problem: Numerically constructing surfaces with non-constant Gaussian curvature that can be accurately approximated by using the GGG web.

GGG Web (Jorge et al.)

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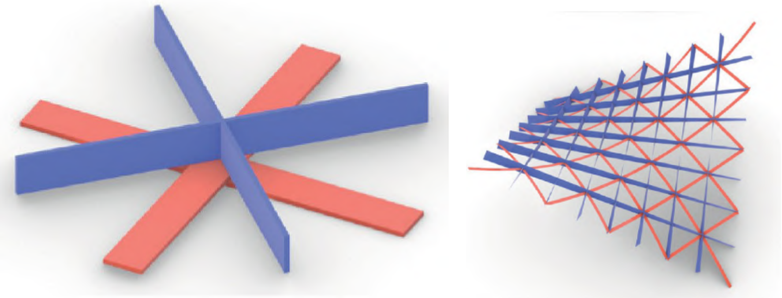
Definition (Schling et al. 2022)

Consider a 4-web on a negatively curved surface such that two families of curves are asymptotic (A), while the other two are geodesics (G). These curves are cyclically organized around each point, following a pattern of A-G-A-G. Such a web is called an AGAG-web.

Problem: Find a non-trivial discrete or smooth AGAG-web.

AGAG structure (Schling et al. 2022)



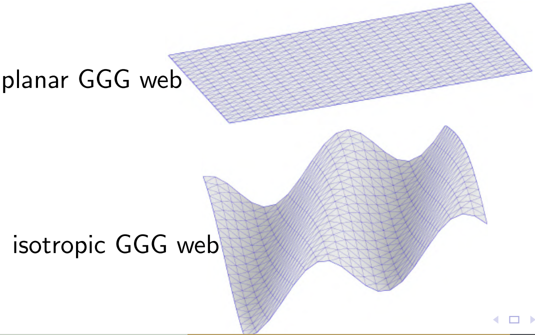
Previous Work

1. Deng, B., Pottmann, H., Wallner, J. (2011).: Optimization of triangle meshes with two geodesic polyline families, and every 4th polyline of the third type is circular and vertical.
2. Schling, E., Wang, H., Hoyer, S., Pottmann, H., (2022).: Workflow for hybrid asymptotic-geodesic webs design.
3. Wang, B., Wang, H., Schling, E., Pottmann, H., (2023).: Designing 3-webs (AAG, AGG, PGG).
4. Pottmann, H., Müller, C., (2023).: Description of all discrete AGAG-webs in isotropic space.

Solution of the Problems

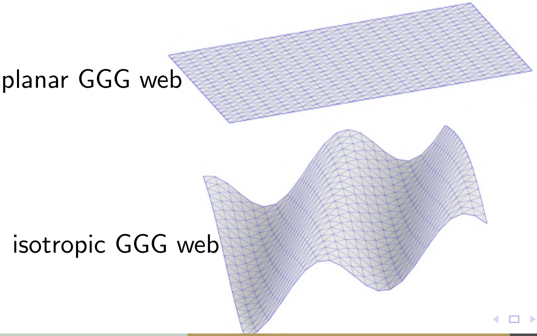
- By quadratic optimization.
- Use isotropic initialization as the initial shape.
- During optimization, use a special dot product that gradually transitions from the isotropic dot product to the Euclidean one.

- An isotropic geodesic is a curve on a surface (without vertical tangents) such that the top view is a straight line (simpler than the Euclidean constraint of orthogonal osculating planes).

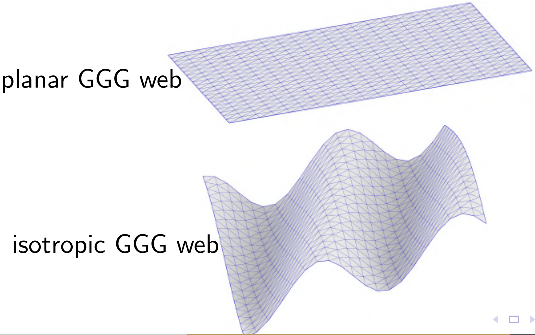


GGG webs in Isotropic Geometry

- An isotropic geodesic is a curve on a surface (without vertical tangents) such that the top view is a straight line (simpler than the Euclidean constraint of orthogonal osculating planes).
- The projection of an isotropic GGG web onto the *xy*-plane is a 3-web of straight lines. These webs consist of the tangents to an algebraic curve of degree 3, which is dual to a cubic curve [Graf and Sauer, 1924].

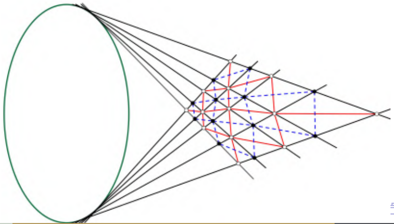


- An isotropic geodesic is a curve on a surface (without vertical tangents) such that the top view is a straight line (simpler than the Euclidean constraint of orthogonal osculating planes).
- The projection of an isotropic GGG web onto the xy-plane is a 3-web of straight lines. These webs consist of the tangents to an algebraic curve of degree 3, which is dual to a cubic curve [Graf and Sauer, 1924].
- We can obtain all isotropic GGG webs by projecting a 3-web of straight lines onto a surface.



Asymptotic curves are a concept of projective geometry and thus are the same in Euclidean and isotropic geometry.
Algorithm to construct a discrete isotopic AGAG (Müller and Pottmann 2023).

- Select two one-parameter families of tangent lines to a conic C.
- Extract a 2-web from the intersection of the tangent lines.
- Draw the two diagonal nets of the 2-web (two planar Koenigs nets).
- Apply the construction by Müller and Pottmann to obtain a discrete isotropic AGAG web.



Method

- Inner product of two vectors $p, q \in R^3$ is

$$\langle p, q \rangle = p_1 q_1 + p_2 q_2 + p_3 q_3.$$

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- The isotropic counterpart is

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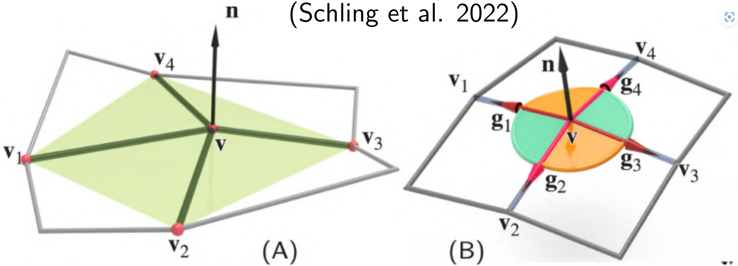
- Inner Product Transformation:

$$\langle p, q \rangle_\varepsilon = p_1 q_1 + p_2 q_2 + \varepsilon p_3 q_3,$$

where ε is gradually increased from $\varepsilon = 0$ to $\varepsilon = 1$.

(Schling et al. 2022)



- Unit normal at the star vertex ($\mathbf{v}$):

$$\mathbf{n}\cdot(\mathbf{v}_3-\mathbf{v}_1)=0,\quad \mathbf{n}\cdot(\mathbf{v}_4-\mathbf{v}_2)=0 \quad \text{and} \quad \|\mathbf{n}\|^2=1.$$

- $\mathbf{v}_1, \mathbf{v}, \mathbf{v}_3$ is part of an asymptotic polyline if:

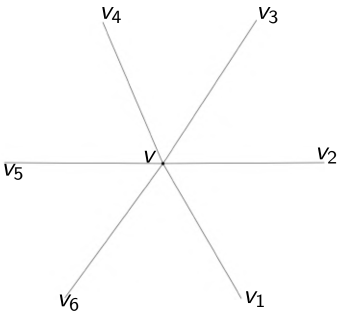
$$\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_1)=0, \quad \text{and} \quad \mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_3)=0.$$

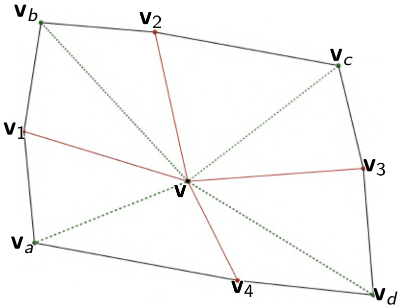
- $\mathbf{v}_2, \mathbf{v}, \mathbf{v}_4$ is part of a geodesic polyline if $\mathbf{v}-\mathbf{v}_2, \mathbf{v}-\mathbf{v}_4, n$ are coplanar:

$$\mathbf{n_g}\cdot(\mathbf{v}-\mathbf{v}_2)=0,\quad \mathbf{n_g}\cdot(\mathbf{v}-\mathbf{v}_4)=0,\quad \mathbf{n}\cdot\mathbf{n_g}=0 \quad \text{and} \quad \|\mathbf{n_g}\|^2=1.$$

Optimization Process

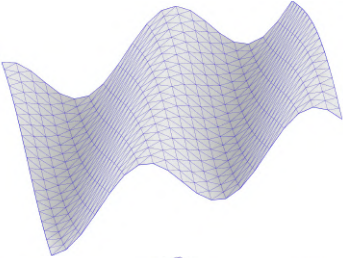
- Optimizing through the special dot product $\langle , \rangle_\varepsilon$ from $\varepsilon = 0$ to $\varepsilon = 1$.
- $\min E_{ggg} = \lambda_{fair} E_{fair} + \lambda_{glide} E_{glide} + \lambda_{geod} E_{geod}$
- $\min E_{AGAG} = \lambda_{fair} E_{fair} + \lambda_{glide} E_{glide} + \lambda_{geod} E_{geod} + \lambda_{asy} E_{asy}$
- The Gauss-Newton method with Levenberg-Marquardt regularization.



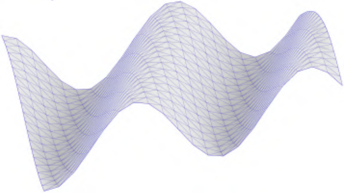


Results

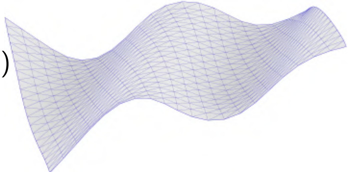
Isotropic GGG ($\epsilon = 0.0$)



$\epsilon = 0.3$

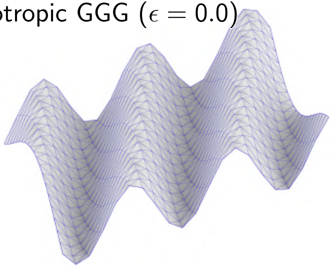


Euclidean GGG ($\epsilon = 1.0$)



Results

Isotropic GGG ($\epsilon = 0.0$)



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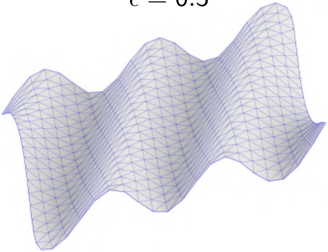
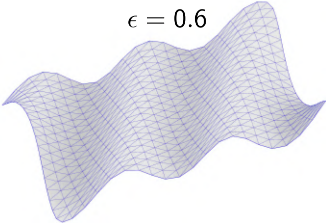
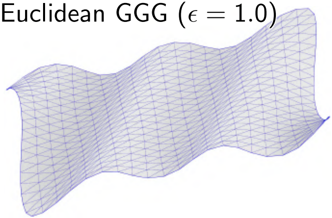


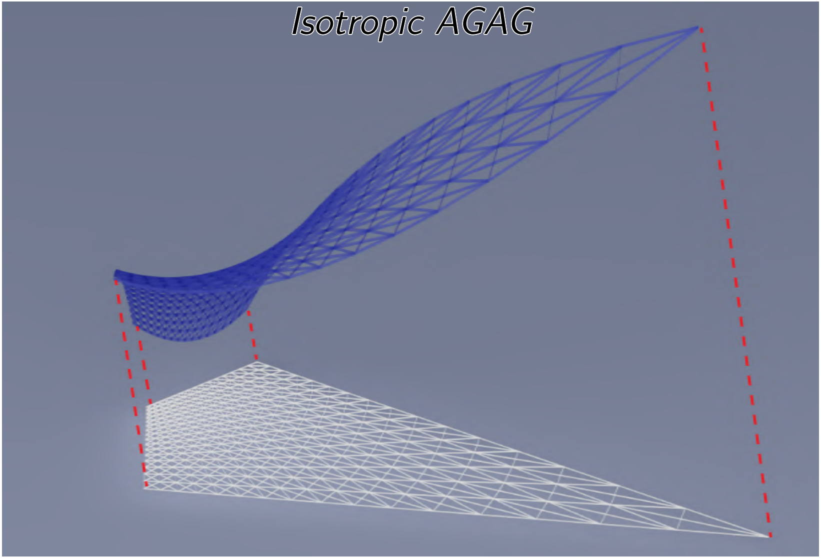
$\epsilon = 0.6$



Euclidean GGG ($\epsilon = 1.0$)



Results



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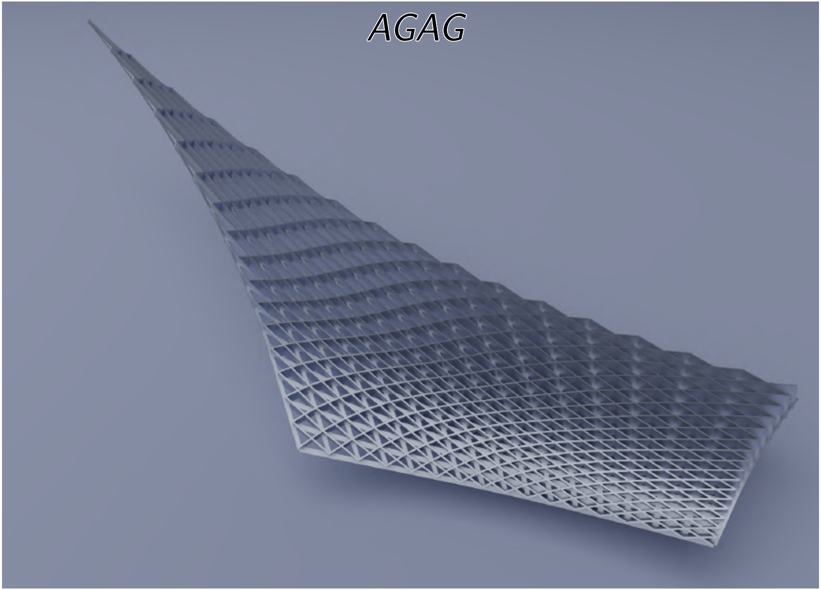

The End:)

(Art Museum in Cagliari, Italy by Zaha Hadid)