Flexible Kokotsakis polyhedra and elliptic functions

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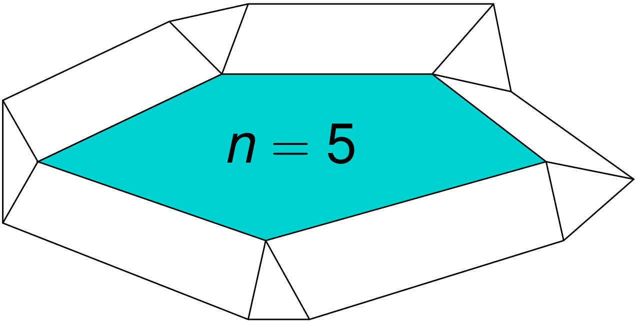
Rigidity and Flexibility of Geometric Structures

Vienna, September 24–28, 2018

Outline

Kokotsakis *n*-polyhedron

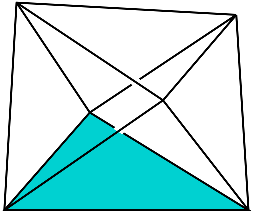
An *n*-gon with quadrilaterals attached to its sides and triangles attached to its vertices.

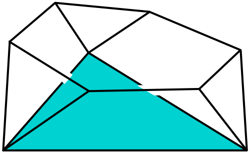


- The inner face stays planar (a plate-and-hinge structure).
- A generic polyhedron of this shape is rigid.

Kokotsakis 3-polyhedra and octahedra

Cut an octahedron in two “halves” $\rightsquigarrow$ two Kokotsakis 3-polyhedra.





Kokotsakis 3-polyhedra and octahedra

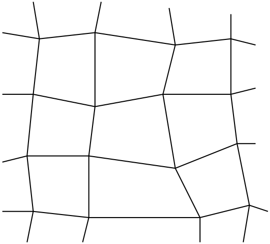
Cut an octahedron in two “halves” $\rightsquigarrow$ two Kokotsakis 3-polyhedra.

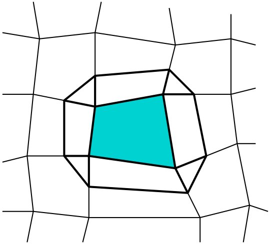
- Bricard’s flexible octahedra provide examples of flexible Kokotsakis 3-polyhedra.
- Conversely, extending the faces of a Kokotsakis 3-polyhedron yields an octahedron (possibly with some vertices at infinity).

Flexibility of generalized octahedra was investigated by Nawratil.

Kokotsakis 4-polyhedra and quad-surfaces

- A quad-surface is a polyhedral surface made of quadrilaterals.
- Regular quad-surface: four quadrilaterals at every vertex.
- The neighborhood of a face in a regular quad-surface is a Kokotsakis 4-polyhedron.



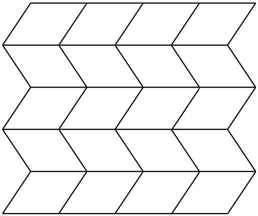


Theorem

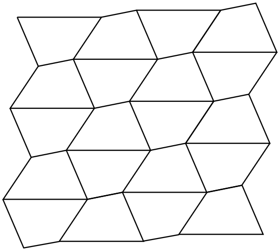
A simply-connected quad-surface is flexible $\Leftrightarrow$ *neighborhoods of all faces are flexible.*

Flexible quad-surfaces: Examples

- Miura-Ori



- Kokotsakis mesh



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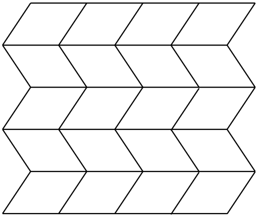

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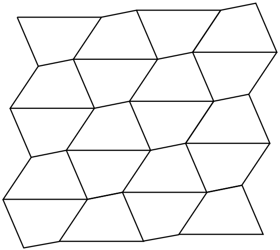
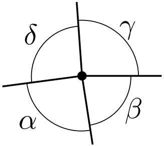


- any mesh where the opposite angles sum up to π: $\alpha + \gamma = \beta + \delta$
- any mesh where the opposite angles are equal (discrete Voss): $\alpha = \gamma$, $\beta = \delta$



A real-life application

History

- Sauer–Graf'31: discrete Voss surfaces, T-surfaces
- Kokotsakis'33: characterization of infinitesimal flexibility, more examples

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- Kokotsakis'33: characterization of infinitesimal flexibility, more examples
- Schief–Bobenko–Hoffmann'08: relation to integrable systems
- Schief: an unpublished preprint discussing an alternative approach
- Karpenkov'10: an algebraic approach proposed
- Stachel–Nawratil'10: spherical linkages, classification of decomposable cases
- I'17: a "complete" classification

Outline

The setup

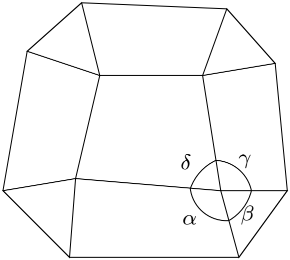
Spherical linkages

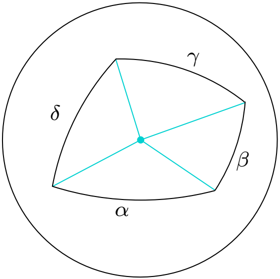
Elliptic functions

Spherical link of a vertex

Take a sphere centered at a vertex of a Kokotsakis polyhedron.

- It intersects the adjacent faces along four arcs of great circles.
- Arc lengths $=$ plane angles of the faces.
- When the polyhedron is deformed, the spherical quadrilateral deforms while preserving its side lengths.

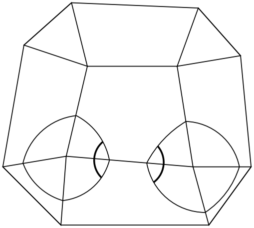


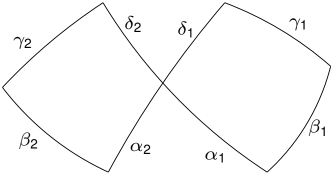


Coupled spherical quadrilaterals

The links of two adjacent vertices are two spherical quadrilaterals with a common angle.

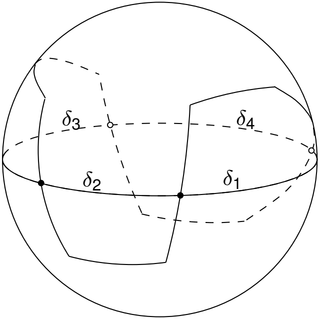
This can be represented by a scissors linkage on the sphere.

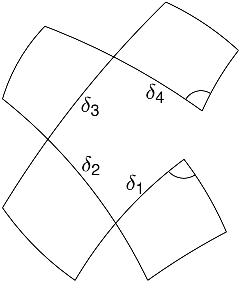




A spherical linkage

The links of all four vertices form a spherical linkage.





The Kokotsakis polyhedron is flexible ⇔ the spherical linkage is flexible (and the marked angles remain equal during the flex).

Spherical tetragonometry

In a triangle, the side lengths determine the angles.

In a quadrilateral, the side lengths determine relations between any pair of angles. (A quadrilateral deforms with one degree of freedom.)

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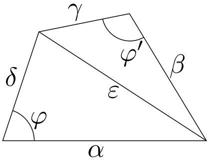

$$\cos\varepsilon = \cos\alpha\cos\delta + \sin\alpha\sin\delta\cos\varphi$$
$$= \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\varphi'$$
$\Rightarrow$ linear relation on $\cos\varphi$ and $\cos\varphi'$

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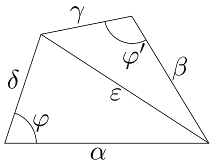

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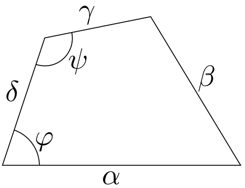
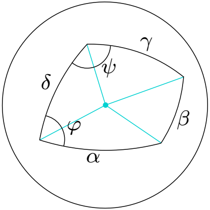


$$\sin\alpha\sin\gamma(\cos\delta\cos\varphi\cos\psi - \sin\varphi\sin\psi)$$
$$-\sin\alpha\cos\gamma\sin\delta\cos\varphi - \cos\alpha\sin\gamma\sin\delta\cos\varphi$$
$$+\cos\beta - \cos\alpha\cos\gamma\cos\delta = 0$$

Polynomial equations

Substitute $z = \cot \frac{\varphi}{2}, w = \cot \frac{\psi}{2}$.

Get a polynomial equation $P(z, w) = 0$.



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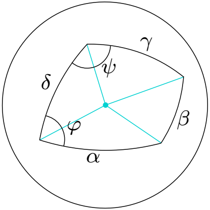

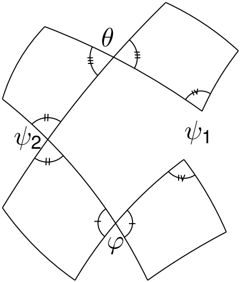

Get a system of polynomial equations:

$$P_4(u, w_2) = 0 \quad P_3(u, w_1) = 0$$
$$P_2(z, w_2) = 0 \quad P_1(z, w_1) = 0$$

Generically, the solution set is finite: the polyhedron is rigid.

Algebraic approach

When does the system

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have a one-parameter set of solutions?

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have a one-parameter set of solutions?

The resultant of P_1 and P_2 as polynomials in z is a polynomial in w_1, w_2. So is the resultant of P_3 and P_4 as polynomials in u.

$$R_1(w_1, w_2) = 0 \quad R_2(w_1, w_2) = 0.$$

The polyhedron is flexible $\Leftrightarrow$ R_1 and R_2 have a common factor.

The reducible case (the common factor of lower degree) was analyzed by Nawratil and Stachel.

Outline

Polynomial equations: a closer look

Equation relating two adjacent angles of a quadrilateral has the form

$$c_{22}z^2w^2 + c_{20}z^2 + c_{02}w^2 + 2c_{11}zw + c_{00} = 0.$$

$$\begin{aligned}
c_{22} &= \sin\frac{\alpha+\beta+\gamma-\delta}{2}\sin\frac{\alpha-\beta+\gamma-\delta}{2}\\
c_{20} &= \sin\frac{\alpha-\beta-\gamma-\delta}{2}\sin\frac{\alpha+\beta-\gamma-\delta}{2}\\
c_{02} &= \sin\frac{\alpha+\beta-\gamma+\delta}{2}\sin\frac{\alpha-\beta-\gamma+\delta}{2}\\
c_{11} &= -\sin\alpha\sin\gamma\\
c_{00} &= \sin\frac{\alpha-\beta+\gamma+\delta}{2}\sin\frac{\alpha+\beta+\gamma+\delta}{2}
\end{aligned}$$

- Conical mesh: $\alpha + \gamma = \beta + \delta$. This implies $c_{22} = 0$.
- Intrinsically flat (origami case): $\alpha + \beta + \gamma + \delta = 2\pi$. Then $c_{00} = 0$.

Conical case: parametrization by trigonometric functions

The solution set of

$$c_{20}z^2 + c_{02}w^2 + 2c_{11}zw + c_{00} = 0$$

can be parametrized as

$$z = p\sin t, \quad w = q\sin(t+\tau), \quad t \in \mathbb{C}.$$

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With every quadruple $(\alpha, \beta, \gamma, \delta)$ that satisfies $\alpha + \gamma = \beta + \delta$ one associates

- two amplitudes p, q (real or purely imaginary);
- a phase shift τ.

General case: elliptic functions

The solution set of

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$$\begin{aligned} & z = p\,\mathrm{sn}(t;k), \quad w = q\,\mathrm{sn}(t+\tau;k), \quad t \in \mathbb{C}, \\ \text{or } & z = p\,\mathrm{cn}(t;k), \quad w = q\,\mathrm{cn}(t+\tau;k). \end{aligned}$$

(Distinction according to the Grashof condition.)

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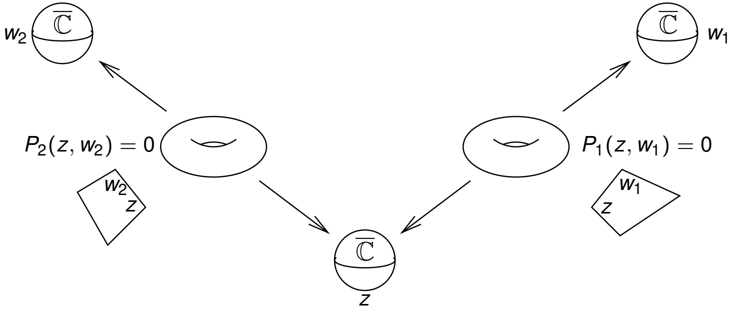
With every generic quadruple $(\alpha, \beta, \gamma, \delta)$ one associates

- a modulus k;
- two amplitudes p, q (real or purely imaginary);
- a phase shift τ.

Riemann surfaces and branched covers



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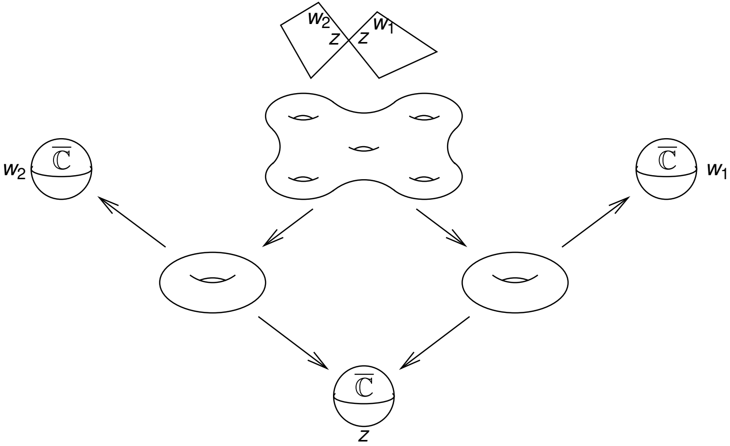
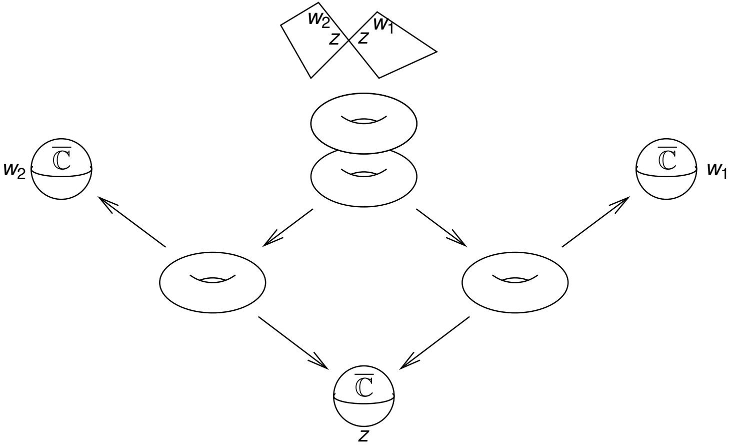


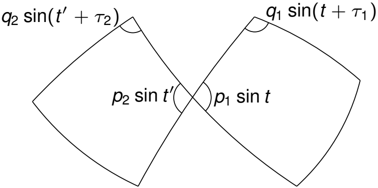
If the branch sets are different, then the configuration space of the coupling is connected.

Riemann surfaces and branched covers



If the branch sets coincide, then the configuration space has two components.

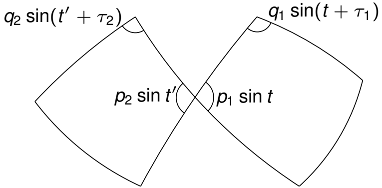
A (good) reducible coupling



Assume $p_1 = p_2 =: p$. Then

$$p_1 \sin t = p_2 \sin t'$$
$$\Leftrightarrow t = t' \text{ or } t' = \pi - t.$$

A (good) reducible coupling



Assume $p_1 = p_2 =: p$. Then

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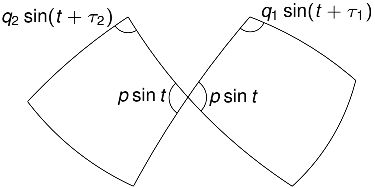
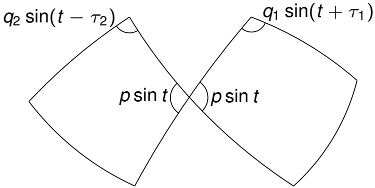
The configuration space of this coupling has two components:

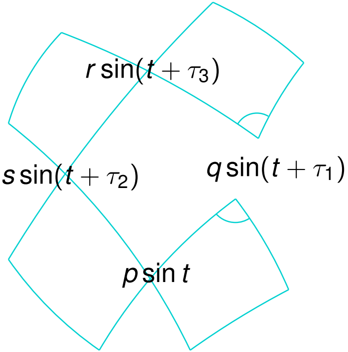




(That is, the resultant $R_1(w_1, w_2)$ factorizes.)

A class of flexible Kokotsakis polyhedra

- All couplings are reducible as above.
- The sum of shifts $= 0 (\mathrm{mod}\ 2\pi)$.



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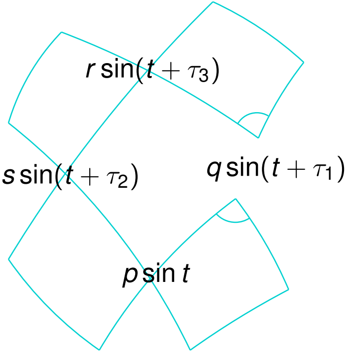
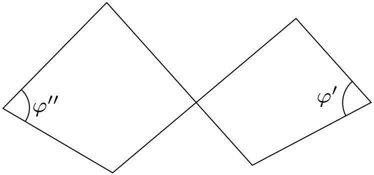


For generic spherical links (whose configurations are parametrized by $\text{sn}(t; k)$ or $\text{cn}(t; k)$) there is one additional condition:

- The elliptic moduli coincide: $k_1 = k_2 = k_3 = k_4$.

Angle condition for reducible couplings



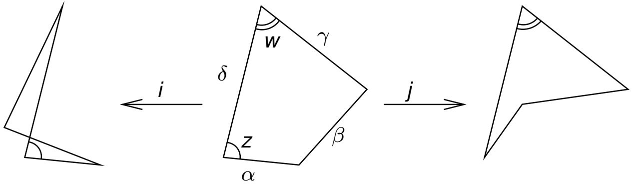
Theorem

In a (good) reducible coupling, the two outermost angles are either equal or complementary:

$$\varphi' = \varphi'' \quad \textit{or} \quad \varphi' = \pi - \varphi''.$$

Compare with the Dixon's angle condition in the Burmester mechanism.

Involutions



Two involutions on the configuration space:

$$i(z, w) = (z, w'), \quad j(z, w) = (z', w).$$

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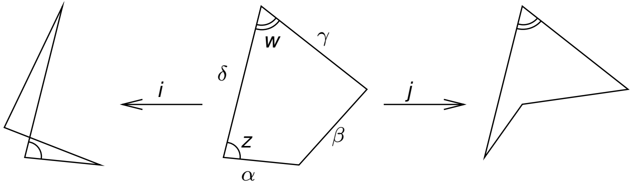
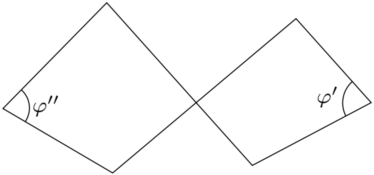


Two involutions on the configuration space:

$$i(z, w) = (z, w'), \quad j(z, w) = (z', w).$$

- The fixed points are branch points of the coordinate projections.
- Fixed points of i correspond to $\varphi' = 0$ or π.

Angle condition for reducible couplings: proof



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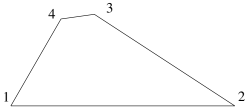
Proof.

- Branch points coincide: $\varphi' \in \{0, \pi\} \Leftrightarrow \varphi'' \in \{0, \pi\}$.
- Linear relation on cosines: $\cos \varphi'' = a \cos \varphi' + b$.

Thus we have $a + b = 1, -a + b = -1$ or $a + b = -1, -a + b = 1$.
Hence $b = 0, a = \pm 1 \Rightarrow \cos \varphi'' = \pm \cos \varphi'$. □

Darboux porism

Take a quadrilateral.



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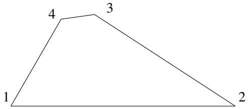
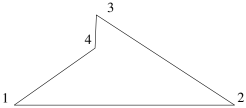


Fold it along the diagonal 13.



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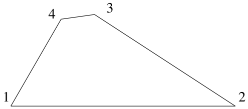

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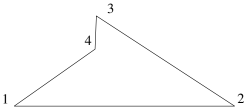
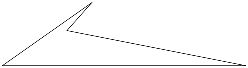


Now fold along 24.



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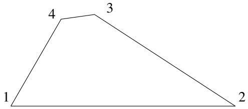

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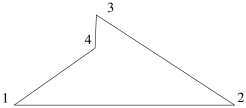

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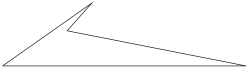
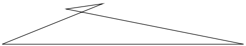
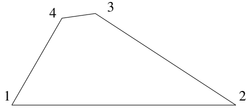


Repeat.

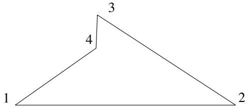


Darboux porism

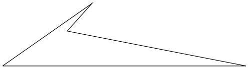
Take a quadrilateral.



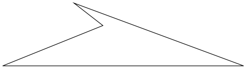
Fold it along the diagonal 13.



Now fold along 24.

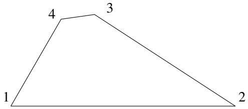


Repeat.

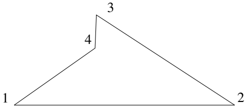


Darboux porism

Take a quadrilateral.



Fold it along the diagonal 13.



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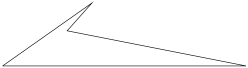
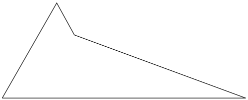
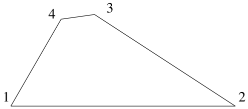


Repeat.



Darboux porism

Take a quadrilateral.



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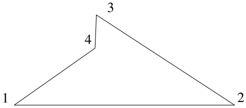

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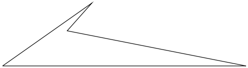
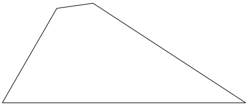
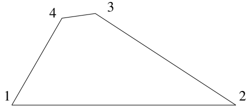


Repeat.

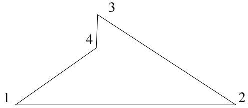


Darboux porism

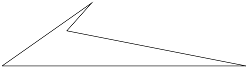
Take a quadrilateral.



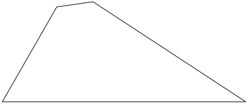
Fold it along the diagonal 13.



Now fold along 24.



Repeat.

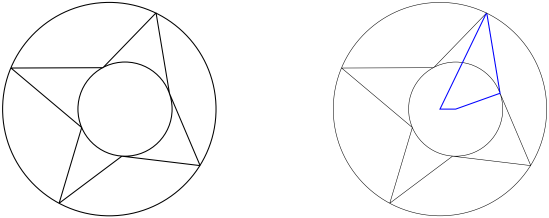


Theorem (Darboux, 1879)

If the folding of a quadrilateral is periodic, then it is periodic for every quadrilateral with the same side lengths.

Bottema porism

Darboux porism is equvalent to Bottema porism.



Theorem (Bottema, 1969)

If an equilateral $2n$-gon can be inscribed into a pair of circles, then infinitely many equilateral $2n$-gons with the same side length can be inscribed into the same pair of circles.

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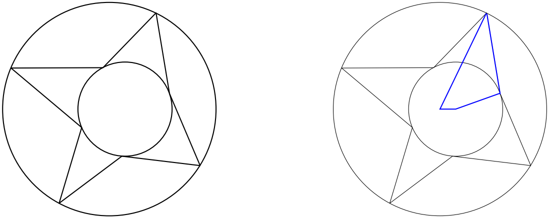


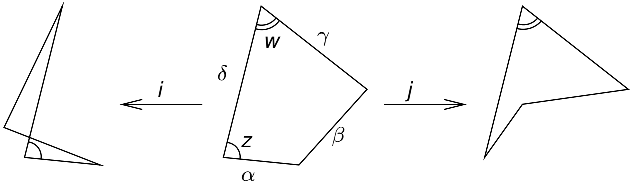
Theorem (Bottema, 1969)

If an equilateral $2n$-gon can be inscribed into a pair of circles, then infinitely many equilateral $2n$-gons with the same side length can be inscribed into the same pair of circles.

- Gives rise to an overconstrained bipartite linkage.
- Can be derived from the Poncelet porism: join the vertices lying on one circle, get an n-gon circumscribed about a conic.

Proof of the Darboux porism

Reformulation: if $i \circ j$ has a fixed point, then it is the identity map.



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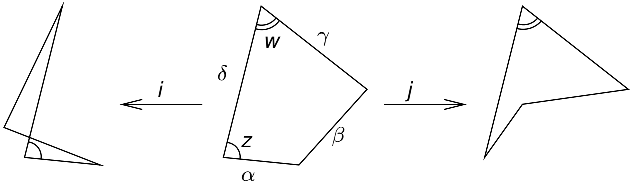

If $z = \mathrm{sn}(t; k)$, $w = \mathrm{sn}(t + \tau; k)$, then in terms of the parameter t

$$i(t) = 2K - t, \quad j(t) = 2K - 2\tau - t.$$

Thus $i(j(t)) = t + 2\tau$.

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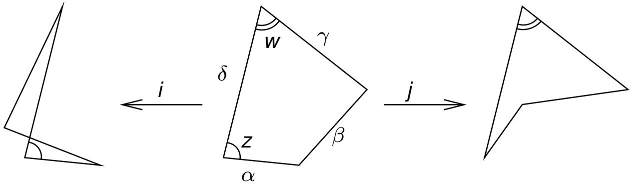

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Thus $i(j(t)) = t + 2\tau$.

A configuration is n-periodic $\Rightarrow$ $2\tau n$ is a period of sn
$\Rightarrow$ any other configuration is n-periodic.