# Kinematic Mapping of SE(4) and the Hypersphere Condition

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# Overview

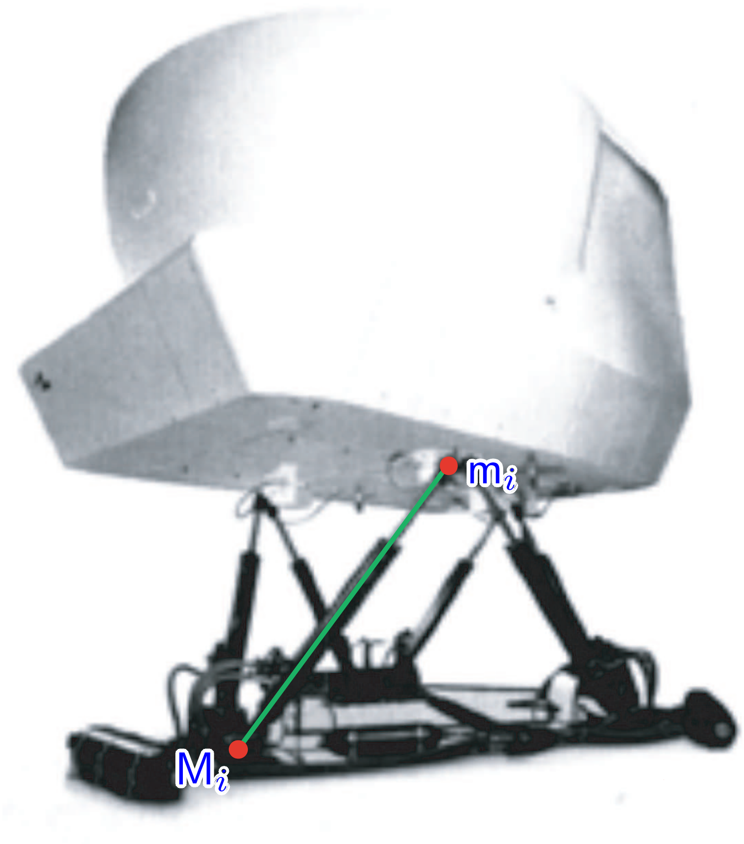
# 1. Motivation

Stewart Gough platforms (SGP) are 6-dof $S_3\underline{P}S_3$ parallel manipulators, as the platform is connected with the base via six $S_3\underline{P}S_3$-legs.

$\underline{P}$ denotes the active prismatic joint.

$S_n$ denotes the passive spherical joint, which admits the group of spherical motions SO(n) of the $n$-dimensional Euclidean space $E^n$.

A SGP is called planar, if the base anchor points $\mathrm{M}_1, \ldots, \mathrm{M}_6$ are coplanar and the platform anchor points $\mathrm{m}_1, \ldots, \mathrm{m}_6$ are coplanar.



# 1. Motivation

Planar SGPs are a lot better understood geometrically than the non-planar ones:

- attachment of additional legs without changing the direct kinematics [1] and singularity set [2],
- self-motions [3] and Duporcq's theorem [4], etc.

We hope to gain a deeper geometric insight into the nature of non-planar SGPs by studying the analogs of planar SGPs in $E^4$, which are so-called hyperplanar 10-dof $S_4\underline{P}S_4$ parallel manipulators.

The basic equation for an algebraic kinematical study of this mechanisms is the so-called hypersphere condition, which means that $\mathrm{m}_i$ is located on a hypersphere centered in the corresponding base anchor points $\mathrm{M}_i$.

For the formulation of this equation, we need a proper kinematic mapping of SE(4).

# 2a. Study Mapping of SE(3)

A kinematic mapping of SE(n) is a bijective mapping between the group of displacements of $E^n$ and a set of points in a certain space. For $n = 3$, this mapping can be constructed by the usage of unit dual quaternions:

**Quaternions:** $\mathfrak{Q} := q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with $q_0, \ldots, q_3 \in \mathbb{R}$ is an element of the skew field of quaternions $\mathbb{H}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the so-called quaternion units.

The conjugated quaternion to $\mathfrak{Q}$ is given by $\widetilde{\mathfrak{Q}} := q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$.

$\mathfrak{Q}$ is called pure quaternion for $q_0 = 0$ and unit quaternion for $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

We embed the points X of $E^3$ with Cartesian coordinates $(x_1, x_2, x_3)$ into the set of pure quaternions by the following mapping:

$$\iota_3 : \mathbb{R}^3 \to \mathbb{H} \quad \text{with} \quad (x_1, x_2, x_3) \mapsto \mathfrak{X} := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

# 2a. Study Mapping of SE(3)

**Dual Quaternions:** An element $\mathfrak{E} + \varepsilon\mathfrak{T}$ of $\mathbb{H} + \varepsilon\mathbb{H}$ is called dual quaternion, where $\varepsilon$ is the dual unit with the property $\varepsilon^2 = 0$.

It is called unit dual quaternion, if $\mathfrak{E}$ is an unit quaternion and following condition holds:

$$e_0 t_0 + e_1 t_1 + e_2 t_2 + e_3 t_3 = 0.$$

The mapping of points X $\in E^3$ to X$' \in E^3$ induced by any element of SE(3), can be written as follows by using $\iota_3$ (e.g. [8]):

$$\mathfrak{X} \mapsto \mathfrak{X}' \quad \text{with} \quad \mathfrak{X}' := \mathfrak{E} \circ \mathfrak{X} \circ \widetilde{\mathfrak{E}} + (\mathfrak{T} \circ \widetilde{\mathfrak{E}} - \mathfrak{E} \circ \widetilde{\mathfrak{T}}), \tag{1}$$

where $\circ$ denotes the well-known quaternion multiplication. Moreover the mapping of Eq. (1) is an element of SE(3) for any unit dual quaternion $\mathfrak{E} + \varepsilon\mathfrak{T}$.

# 2a. Study Mapping of SE(3)

The first summand $\mathfrak{E} \circ \mathfrak{X} \circ \widetilde{\mathfrak{E}}$ of the pure quaternion $\mathfrak{X}'$ is the rotational component, which can be written in vector-representation as $(x_1', x_2', x_3')^T = \mathbf{R}_3(x_1, x_2, x_3)^T$ with

$$\mathbf{R}_3 = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}, \quad (2)$$

where $det\mathbf{R}_3 = (e_0^2 + e_1^2 + e_2^2 + e_3^2)^3 = 1$ holds. As the remaining part of $\mathfrak{X}'$ does not depend on X, it corresponds to a translation $\mathbf{s}_3 := (s_1, s_2, s_3)^T$ with

$$\begin{aligned} s_1 &= 2(e_0t_1 - e_1t_0 + e_2t_3 - e_3t_2), \quad s_2 = 2(e_0t_2 - e_1t_3 - e_2t_0 + e_3t_1), \\ s_3 &= 2(e_0t_3 + e_1t_2 - e_2t_1 - e_3t_0). \end{aligned} \quad (3)$$

# 2a. Study Mapping of SE(3)

As both unit dual quaternions $\pm(\mathfrak{E} + \varepsilon\mathfrak{T})$ correspond to the same Euclidean motion of $E^3$, we consider the homogeneous 8-tuple $(e_0 : \ldots : e_3 : t_0 : \ldots : t_3)$.

These so-called Study parameters can be interpreted as a point of a projective 7-dimensional space $P^7$. Therefore there is a bijection between SE(3) and all real points of $P^7$ located on the so-called Study quadric $\Phi \subset P^7$, which is given by:

$$e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3 = 0,$$

and is sliced along the 3-dimensional generator-space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathfrak{E}$ cannot be normalized.

If the Study mapping is restricted to planar Euclidean displacements within a plane $\alpha \in E^3$, we obtain the so-called Blaschke-Grünwald Mapping of SE(2).

# 2b. Blaschke-Grünwald Mapping of SE(2)

The planar motion group corresponds to a generator-space of the Study quadric $\Phi$ given by $e_2 = e_3 = t_0 = t_1 = 0$ for $\alpha$: $x_1 = 0$ (cf. [8]).

Therefore there is a bijection between SE(2) and all real points $(e_0 : e_1 : t_2 : t_3)$ of $P^3$, with exception of the points located on the line $e_0 = e_1 = 0$.

The vector-representation of planar displacements in dependency of the Blaschke-Grünwald parameters $(e_0 : e_1 : t_2 : t_3)$ can immediately be obtained from Eqs. (2) and (3) and reads as $(x_2', x_3')^T = \mathbf{R}_2(x_2, x_3)^T + \mathbf{s}_2$ with:

$$\mathbf{R}_2 = \begin{pmatrix} e_0^2 - e_1^2 & -2e_0e_1 \\ 2e_0e_1 & e_0^2 - e_1^2 \end{pmatrix}, \qquad \mathbf{s}_2 = \begin{pmatrix} 2(e_0t_2 - e_1t_3), \\ 2(e_0t_3 + e_1t_2). \end{pmatrix},$$

where $det\mathbf{R}_2 = (e_0^2 + e_1^2)^2 = 1$ holds.

# 2c. Klawitter-Hagemann Mapping of SE(4)

Based on Clifford algebras, Klawitter and Hagemann [9] presented an unified concept for constructing kinematic mappings for certain Cayley-Klein geometries.

Especially for $E^2$ and $E^3$, they demonstrated that their approach yields the Blaschke-Grünwald mapping and the Study mapping (see also Selig [14]).

This method maps displacements of SE(4) onto points of $P^{15}$, located in the intersection of nine quadrics, which is additionally sliced along a further quadric.

Due to the large number of homogeneous motion parameters, as well as the resulting set of quadratic constraints, the Klawitter-Hagemann mapping is not suited for performing computational algebraic kinematics in $E^4$.

Therefore we are interested in a simplified kinematic mapping of SE(4).

# 3. New Kinematic Mapping of SE(4)

We embed the points X of $E^4$ with Cartesian coordinates $(x_0, x_1, x_2, x_3)$ into the set of quaternions by the mapping:

$$\iota_4 : \mathbb{R}^4 \to \mathbb{H} \quad \text{with} \quad (x_0, x_1, x_2, x_3) \mapsto \mathfrak{X} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

Moreover we need the quaternion representation theorem for SO(4), which has many fathers (Euler, Cayley, Salmon, Elfrinkhof, Stringham, Bouman; cf. [10]):

**Theorem 1.** The mapping of points $\mathsf{X} \in E^4$ to $\mathsf{X}' \in E^4$ induced by any element of SO(4), can be written as follows (by using $\iota_4$):

$$\mathfrak{X} \mapsto \mathfrak{X}' \quad \text{with} \quad \mathfrak{X}' := \mathfrak{E} \circ \mathfrak{X} \circ \mathfrak{F}, \tag{4}$$

where $\mathfrak{E}$ and $\mathfrak{F}$ is a pair of unit quaternions, which is determined uniquely up to the sign. Moreover the mapping of Eq. (4) is an element of SO(4) for any pair of unit quaternions $\mathfrak{E}$ and $\mathfrak{F}$.

# 3. New Kinematic Mapping of SE(4)

Direct computation shows that the mapping given in Eq. (4) can be written in vector-representation as $(x_0', x_1', x_2', x_3')^T = \mathbf{R}_4(x_0, x_1, x_2, x_3)^T$ with $\mathbf{R}_4 = \mathbf{EF}$ and

$$\mathbf{E} = \begin{pmatrix} e_0 & -e_1 & -e_2 & -e_3 \\ e_1 & e_0 & -e_3 & e_2 \\ e_2 & e_3 & e_0 & -e_1 \\ e_3 & -e_2 & e_1 & e_0 \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f_0 & -f_1 & -f_2 & -f_3 \\ f_1 & f_0 & f_3 & -f_2 \\ f_2 & -f_3 & f_0 & f_1 \\ f_3 & f_2 & -f_1 & f_0 \end{pmatrix},$$

where $det\mathbf{R}_4 = det\mathbf{E}\, det\mathbf{F} = (e_0^2 + e_1^2 + e_2^2 + e_3^2)^2(f_0^2 + f_1^2 + f_2^2 + f_3^2)^2 = 1$ holds.

Moreover due to the free choice of sign in Theorem 1, the decomposition into a left unit quaternion $\mathfrak{E}$ and a right unit quaternion $\mathfrak{F}$ yields a double cover of SO(4).

# 3. New Kinematic Mapping of SE(4)

Therefore we consider again the homogeneous 8-tuple $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$, which can be seen as a point in $P^7$. Hence there is a bijection between SO(4) and all real points of $P^7$, which are located on the quadric $\Psi \subset P^7$ given by

$$(e_0^2 + e_1^2 + e_2^2 + e_3^2) - (f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0, \tag{5}$$

sliced along the 3-dimensional space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathfrak{E}$ cannot be normalized. But this 3-space does not have a real intersection with $\Psi$ and therefore no point of $\Psi$ has to be removed.

Note that Eq. (5) expresses the fact that $\mathfrak{F}$ is also normalized if $\mathfrak{E}$ is.

The extension of this kinematic mapping of SO(4) with respect to translations of $E^4$ can be done as follows:

# 3. New Kinematic Mapping of SE(4)

**Theorem 2.** The mapping of points $\mathsf{X} \in E^4$ to $\mathsf{X}' \in E^4$ induced by any element of SE(4), can be written as follows (by using $\iota_4$):

$$\mathfrak{X} \mapsto \mathfrak{X}' \quad \text{with} \quad \mathfrak{X}' := \mathfrak{E} \circ \mathfrak{X} \circ \mathfrak{F} - 2(\mathfrak{E} \circ \widetilde{\mathfrak{T}}) \ldots\ldots \text{ERRATUM} \tag{6}$$

Moreover the mapping of Eq. (6) is an element of SE(4) for any triple of quaternions $\mathfrak{E}, \mathfrak{F}, \mathfrak{T}$, where $\mathfrak{E}$ and $\mathfrak{F}$ are unit quaternions.

**Proof:** Due to Theorem 1, we only have to show that there is a bijection between the coordinates of the translation vector $\mathbf{s}_4 = (s_0, s_1, s_2, s_3)^T$ and the entries $t_0, \ldots, t_3$ of $\mathfrak{T}$ for a given unit quaternion $\mathfrak{E}$.

On one side, $s_1, s_2, s_3$ equal the expressions given in Eq. (3) and for $s_0$ we get:

$$s_0 = -2(e_0 t_0 + e_1 t_1 + e_2 t_2 + e_3 t_3).$$

# 3. New Kinematic Mapping of SE(4)

On the other side, we have:

$$t_0 = -(e_0 s_0 + e_1 s_1 + e_2 s_2 + e_3 s_3)/2, \quad t_1 = (e_0 s_1 - e_1 s_0 - e_2 s_3 + e_3 s_2)/2,$$
$$t_2 = (e_0 s_2 + e_1 s_3 - e_2 s_0 - e_3 s_1)/2, \quad t_3 = (e_0 s_3 - e_1 s_2 + e_2 s_1 - e_3 s_0)/2,$$

which already proves Theorem 2. □

As both triples of quaternions $\pm(\mathfrak{E}, \mathfrak{F}, \mathfrak{T})$, where $\mathfrak{E}$ and $\mathfrak{F}$ are unit quaternions, correspond to the same Euclidean motion of $E^4$, we consider the homogeneous 12-tuple $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3 : t_0 : \ldots : t_3)$.

These 12 homogeneous motion parameters for $E^4$, which are called the *new parameters* for short, can be interpreted as a point of $P^{11}$.

# 3. New Kinematic Mapping of SE(4)

Therefore there is a bijection between SE(4) and all real points of $P^{11}$ located on the cylinder $\Xi$ over $\Psi$, which is also given by

$$(e_0^2 + e_1^2 + e_2^2 + e_3^2) - (f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0,$$

and is sliced along the 7-dimensional space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathfrak{E}$ cannot be normalized. The real intersection of this 7-space and $\Xi$ equals the 3-dimensional generator-space U of $\Xi$ with:

$$\mathsf{U}: \quad e_0 = e_1 = e_2 = e_3 = f_0 = f_1 = f_2 = f_3 = 0.$$

**Resume:** There is a bijection between elements of SE(4) and real points of $\Xi \setminus \mathsf{U}$.

# 3. New Kinematic Mapping of SE(4)

**Remark:** If we identify $E^3$ with the hyperplane $x_0 = 0$, all points of the 7-dimensional generator-space

$$f_0 = e_0, \quad f_1 = -e_1, \quad f_2 = -e_2, \quad f_3 = -e_3,$$

of $\Xi$, which additionally fulfill the condition that no translation is done in direction of $x_0$ ($\Leftrightarrow s_0 = 0$), map the hyperplane $x_0 = 0$ onto itself.

As the condition $s_0 = 0$ equals the Study condition, the 7-dimensional generator-space of $\Xi$ is the Study parameter space of SE(3). $\diamond$

**Resume:** The Study parameters and subsequently the Blaschke-Grünwald parameters can be obtained from the *new parameters*.

# 4. Hypersphere Condition

The mapping X $\mapsto$ X$'$ implied by an element of SE(n) can be written in vector-form as:

$$\begin{pmatrix} x'_{4-n} \\ \dots \\ x'_3 \end{pmatrix} = \frac{1}{N_n}\left[\mathbf{R}_n \begin{pmatrix} x_{4-n} \\ \dots \\ x_3 \end{pmatrix} + \mathbf{s}_n\right], \qquad (7)$$

for $n = 2, 3, 4$ with $N_2 = e_0^2 + e_1^2$ and $N_3 = N_4 = e_0^2 + e_1^2 + e_2^2 + e_3^2$, respectively, if we neglect the normalizing condition $N_n = 1$. Note that the factor $N_n^{-1}$, which corresponds to the division by 1, is inserted in order to homogenize Eq. (7).

Now we can write the constraint $\Omega_n$ that the point X is located on a hypersphere of $E^n$ with midpoint $(m_{4-n}, \dots, m_3)$ and radius $\rho$ as follows:

$$\Omega_n : \quad (x'_{4-n} - m_{4-n})^2 + \dots + (x'_3 - m_3)^2 - \rho^2 = 0.$$

# 4. Hypersphere Condition

The denominator of $\Omega_n$ cannot vanish due to $N_n \neq 0$ and the nominator is a homogeneous polynomial $P_n$ of degree 4 in the motion parameters.

**n = 2:** $P_2$ factors into $N_2$ and a homogeneous quadratic equation in the Blaschke-Grünwald parameters, which is the so-called circle equation $Q_2$.

**n = 3:** $P_3$ does not behave like $P_2$, but Husty [12] showed that $N_3$ factors out if we add four times the squared Study condition to $P_3$. The remaining homogeneous quadratic equation in the Study parameters is the so-called sphere equation $Q_3$.

**n = 4:** $P_4$ factors into $N_4$ and a homogeneous quadratic equation in the $new$ $parameters$. This is the so-called hypersphere equation $Q_4$.

According to the Remark, we can obtain $Q_3$ from $Q_4$ by setting $m_0 = x_0 = 0$, $f_0 = e_0$, $f_i = -e_i$ for $i = 1, 2, 3$. This also sheds light onto Husty's tricky addition, as it corresponds to the summand $s_0^2$ within the $new$ $parameter$ approach.

# 4. Hypersphere Condition

**Computational Detail:** The hypersphere condition $Q_4$ can be written as follows:

$$
\begin{aligned}
0 =\ & (m_0^2 + m_1^2 + m_2^2 + m_3^2 - \rho^2)N_4 + (x_0^2 + x_1^2 + x_2^2 + x_3^2)(f_0^2 + f_1^2 + f_2^2 + f_3^2) + 4(t_0^2 + t_1^2 + t_2^2 + t_3^2) \\
& + 2m_0\big[2(e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3) - x_0(e_0f_0 - e_1f_1 - e_2f_2 - e_3f_3) + x_1(e_0f_1 + e_1f_0 - e_2f_3 + e_3f_2) \\
& + x_2(e_0f_2 + e_1f_3 + e_2f_0 - e_3f_1) + x_3(e_0f_3 - e_1f_2 + e_2f_1 + e_3f_0)\big] - 4x_0(f_0t_0 - f_1t_1 - f_2t_2 - f_3t_3) \\
& - 2m_1\big[2(e_0t_1 - e_1t_0 + e_2t_3 - e_3t_2) + x_0(e_0f_1 + e_1f_0 + e_2f_3 - e_3f_2) + x_1(e_0f_0 - e_1f_1 + e_2f_2 + e_3f_3) \\
& + x_2(e_0f_3 - e_1f_2 - e_2f_1 - e_3f_0) - x_3(e_0f_2 + e_1f_3 - e_2f_0 + e_3f_1)\big] + 4x_1(f_0t_1 + f_1t_0 + f_2t_3 - f_3t_2) \\
& - 2m_2\big[2(e_0t_2 - e_1t_3 - e_2t_0 + e_3t_1) + x_0(e_0f_2 - e_1f_3 + e_2f_0 + e_3f_1) - x_1(e_0f_3 + e_1f_2 + e_2f_1 - e_3f_0) \\
& + x_2(e_0f_0 + e_1f_1 - e_2f_2 + e_3f_3) + x_3(e_0f_1 - e_1f_0 - e_2f_3 - e_3f_2)\big] + 4x_2(f_0t_2 - f_1t_3 + f_2t_0 + f_3t_1) \\
& - 2m_3\big[2(e_0t_3 + e_1t_2 - e_2t_1 - e_3t_0) + x_0(e_0f_3 + e_1f_2 - e_2f_1 + e_3f_0) + x_1(e_0f_2 - e_1f_3 - e_2f_0 - e_3f_1) \\
& - x_2(e_0f_1 - e_1f_0 + e_2f_3 + e_3f_2) + x_3(e_0f_0 + e_1f_1 + e_2f_2 - e_3f_3)\big] + 4x_3(f_0t_3 + f_1t_2 - f_2t_1 + f_3t_0)
\end{aligned}
$$

Note that the difference of two hypersphere conditions is only linear in $t_0, t_1, t_2, t_3$.

# 5. First Result and Outlook

Based on $Q_4$ it can be proven (cf. presented paper) that singular (infinitesimal movable) poses of 10-dof $S_4\underline{P}S_4$ manipulators have an analogous line-geometric characterization as those of their lower-dimensional counterparts.

**Theorem 3.** A 10-dof $S_4\underline{P}S_4$ manipulator is in a singular configuration $\mathcal{C}$ if and only if the carrier lines of the ten $\underline{P}$-joints belong to a linear complex of lines of $E^4$, i.e. the Grassmann coordinates of the 10 lines are linearly dependent.

- A further kinematic study of (hyperplanar) 10-dof $S_4\underline{P}S_4$ manipulators is dedicated to future research.

- The kinematical analysis of SE(4) in terms of the new parameters (e.g. velocity, acceleration, ...) is in preparation.

# References and Acknowledgements

All references refer to the list of publications given in the presented paper:

Nawratil, G.: Kinematic Mapping of SE(4) and the Hypersphere Condition. Advances in Robot Kinematics (J. Lenarcic, O. Khatib eds.), pages 11–19, Springer, 2014, ISBN 978-3-319-06697-4.

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