# Kinematic interpretation of the Study quadric's ambient space

Georg Nawratil

Institute of Discrete Mathematics and Geometry

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# Introduction: Quaternions $\mathbb{H}$

$$1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k} \ldots \text{quaternionic units}$$

$$\circ \ldots \text{quaternion multiplication}$$

$$\mathfrak{E} := e_0 + e_1\mathfrak{i} + e_2\mathfrak{j} + e_3\mathfrak{k} \ldots \text{quaternion with } e_0, \ldots, e_3 \in \mathbb{R}$$

$$e_0 \ldots \text{scalar part}$$

$$\mathfrak{e} := e_1\mathfrak{i} + e_2\mathfrak{j} + e_3\mathfrak{k} \ldots \text{pure part}$$

$$\widetilde{\mathfrak{E}} := e_0 - \mathfrak{e} \ldots \text{conjugated quaternion to } \mathfrak{E} = e_0 + \mathfrak{e}$$

$$\mathfrak{E} \circ \widetilde{\mathfrak{E}} = 1 \ldots \mathfrak{E} \text{ is a unit-quaternion}$$

We embed the points P of Euclidean 3-space $E^3$ with coordinates $(p_1, p_2, p_3)$ with respect to the Cartesian frame $(\mathsf{O}; x_1, x_2, x_3)$ into the set of pure quaternions by:

$$\iota_3 : \mathbb{R}^3 \to \mathbb{H} \quad \text{with} \quad (p_1, p_2, p_3) \mapsto \mathfrak{p} := p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}.$$

# Introduction: Dual Quaternions $\mathbb{H} + \varepsilon\mathbb{H}$

$\varepsilon$ ... dual unit with $\varepsilon^2 = 0$

$\mathfrak{E} + \varepsilon\mathfrak{T}$ ... dual quaternion with $\mathfrak{T} = t_0 + t_1\mathfrak{i} + t_2\mathfrak{j} + t_3\mathfrak{k}$

$\mathfrak{E} + \varepsilon\mathfrak{T}$ ... dual unit-quaternion iff

- $\mathfrak{E}$ is an unit-quaternion and
- the Study condition holds:

$$e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3 = 0 \quad \Longleftrightarrow \quad \mathfrak{T} \circ \widetilde{\mathfrak{E}} + \mathfrak{E} \circ \widetilde{\mathfrak{T}} = 0$$

The set of dual unit-quaternions is denoted by $\mathbb{E}$. By skipping the so-called Study condition we get a superset $\mathbb{F}$ of $\mathbb{E}$.

**Remark:** $\mathbb{E}$ and $\mathbb{F}$ build a group with respect to the quaternion multiplication. $\diamond$

# Introduction: Study Quadric

Based on the usage of $\mathfrak{E} + \varepsilon\mathfrak{T} \in \mathbb{E}$ the mapping of points P $\in E^3$ induced by any element of SE(3), can be written as:

$$\delta_3 : \mathbb{H} \to \mathbb{H} \quad \text{with} \quad \mathfrak{p} \;\mapsto\; \mathfrak{E} \circ \mathfrak{p} \circ \widetilde{\mathfrak{E}} + (\mathfrak{T} \circ \widetilde{\mathfrak{E}} - \mathfrak{E} \circ \widetilde{\mathfrak{T}}).$$

Moreover this mapping is an element of SE(3) for any $\mathfrak{E}, \mathfrak{T}$ with $\mathfrak{E} + \varepsilon\mathfrak{T} \in \mathbb{E}$.

As both unit dual quaternions $\pm(\mathfrak{E} + \varepsilon\mathfrak{T}) \in \mathbb{E}$ correspond to the same Euclidean motion of $E^3$, we consider the homogeneous 8-tuple $(e_0 : \ldots : e_3 : t_0 : \ldots : t_3)$.

These so-called Study parameters can be interpreted as a point of a projective 7-dimensional space $P^7$. Therefore there is a bijection between SE(3) and all real points of $P^7$ located on the so-called Study quadric $\Phi \subset P^7$, which is sliced along the 3-dimensional generator-space $G: \; e_0 = e_1 = e_2 = e_3 = 0$.

**Remark:** Points of this generator-space are called Pseudosomen by Study [20]. ◇

# Motivation: Line Geometry

It is well known [14] that there exists a bijection between the set $\mathcal{L}$ of lines of the projective 3-space and all real points of the so-called Plücker quadric

$$\Psi: \quad l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0$$

of $P^5$, where the homogeneous 6-tuple $(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12})$ are the Plücker coordinates of the lines. This bijection $\mathcal{L} \to \Psi$ is also known as Klein mapping.

The *extended Klein mapping* identifies each point of $P^5$ with a linear complex of lines in $P^3$. The latter can always be seen as a path-normal complex of an instantaneous screw $\$$ (different from the zero screw), which only varies in speed.

$\Longrightarrow$ Bijection between points of $P^5$ and *homogeneous screw coordinates*
$\$ = (s_{01} : s_{02} : s_{03} : s_{23} : s_{31} : s_{12})$

# Overview

This line/screw-geometric explanation of the complete $P^5$ raises the question for a

**Kinematic interpretation of the Study quadric's ambient space:**

1. **Review: Extended Inverse Kinematic Map $\kappa^{-1}$**
2. **Kinematic Interpretation as a Subgroup of SE(4)**
3. **Straight Lines in the Study Parameter Space**
4. **Application: Interactive Rational Motion Design**
5. **References**

# 1. Review: Extended Inverse Kinematic Map $\kappa^{-1}$

Under this map $\kappa^{-1}$, discussed by Pfurner, Schröcker and Husty (PSH) in [13], any point of $P^7 \setminus G$ is identified with a displacement of SE(3), which corresponds to a point on $\Phi \setminus G$. These two points of $P^7$ are linked by the so-called PSH map [19]:

$$\text{PSH:} \quad \mathbb{F} \to \mathbb{E} \quad \text{with} \quad \mathfrak{E} + \varepsilon\mathfrak{T} \mapsto \mathfrak{E} + \varepsilon\left[\mathfrak{T} - \tfrac{1}{2}\left(\mathfrak{T} \circ \widetilde{\mathfrak{E}} + \mathfrak{E} \circ \widetilde{\mathfrak{T}}\right) \circ \mathfrak{E}\right].$$

**Remark:** Note that the basic principle of $\kappa^{-1}$ was already mentioned by Ge and Purwar [3], who used it inter alia in [15,16]. $\diamond$

We do not want to interpret points of the ambient space $P^7$ of $\Phi$ by means of $\kappa^{-1}$ as this map is not bijective. We overcome this problem by interpreting the points of $P^7$ as displacements of a motion group of the Euclidean 4-space $E^4$.

## 2. Kinematic Interpretation as a Subgroup of SE(4)

We embed points P of $E^4$ with coordinates $(p_0, p_1, p_2, p_3)$ with respect to the Cartesian frame $(\mathsf{O}; x_0, x_1, x_2, x_3)$ into the set of quaternions by the mapping:

$$\iota_4 : \mathbb{R}^4 \to \mathbb{H} \quad \text{with} \quad (p_0, p_1, p_2, p_3) \mapsto \mathfrak{P} := p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = p_0 + \mathfrak{p}.$$

Let us identify $E^3$ with any hyperplane $x_0 = k$ with $k \in \mathbb{R}$. Moreover we consider the subgroup $\mathsf{X}_4$ of displacements of SE(4), which fixes the direction of the $x_0$-axis.

**Theorem 1.** The mapping of points $\mathsf{P} \in E^4$ induced by any element of $\mathsf{X}_4$, can be written as:

$$\delta_4 : \mathbb{H} \to \mathbb{H} \quad \text{with} \quad \mathfrak{P} \mapsto \mathfrak{E} \circ \mathfrak{P} \circ \widetilde{\mathfrak{E}} - 2\mathfrak{E} \circ \widetilde{\mathfrak{T}}.$$

Moreover this mapping is an element of $\mathsf{X}_4$ for any $\mathfrak{E}, \mathfrak{T}$ with $\mathfrak{E} + \varepsilon\mathfrak{T} \in \mathbb{F}$.

# 2. Kinematic Interpretation as a Subgroup of SE(4)

As both dual quaternions $\pm(\mathfrak{E} + \varepsilon\mathfrak{T}) \in \mathbb{F}$ correspond to the same $X_4$-motion of $E^4$, we consider again the homogeneous 8-tuple $(e_0 : \ldots : e_3 : t_0 : \ldots : t_3)$.

$\Longrightarrow$ Bijection between $X_4$ and all real points of $P^7 \setminus G$.

The mapping $\delta_4$ restricted to the *pure part* equals the mapping of $\delta_3$ due to:

$$\delta_4(\mathfrak{P}) = p_0 - 2(e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3) + \delta_3(\mathfrak{p})$$

Displacements of $X_4$ fixing the hyperplanes $x_0 = k$ correspond to points on $\Phi \setminus G$. These points imply SE(3)-displacements in the hyperplanes $x_0 = k$, which we identify with $E^3$. This completes the kinematic interpretation of all points of $P^7 \setminus G$; i.e. the Study parameter space with exception of the "Pseudosomen".

# 2. Kinematic Interpretation as a Subgroup of SE(4)

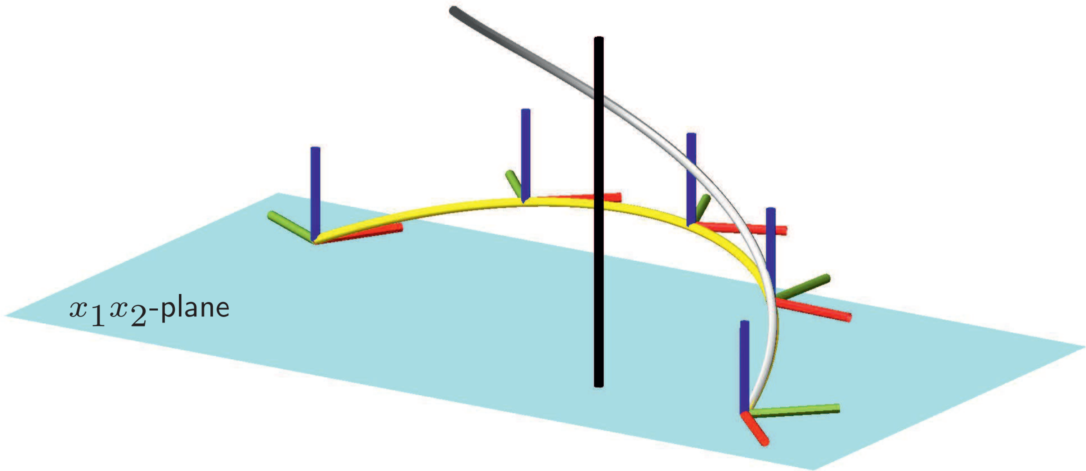
To clarify the meaning of the extended inverse kinematic mapping $\kappa^{-1}$ we compute the action of PSH$(\mathfrak{E} + \varepsilon\mathfrak{T}) \in \mathbb{E}$ on a point P $\in E^4$ yielding:

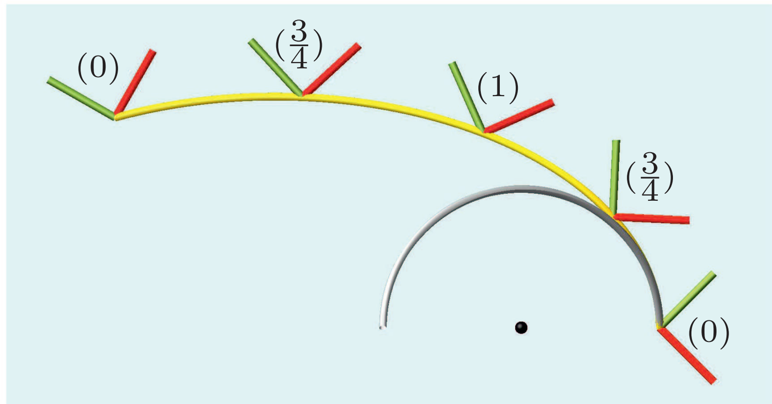
$$\mathfrak{P} \;\mapsto\; \mathfrak{E} \circ \mathfrak{P} \circ \widetilde{\mathfrak{E}} - 2\mathfrak{E} \circ \widetilde{\mathfrak{T}} + (\mathfrak{T} \circ \widetilde{\mathfrak{E}} + \mathfrak{E} \circ \widetilde{\mathfrak{T}}).$$

Therefore the inverse kinematic mapping is nothing else than an orthogonal projection onto the hyperplane $x_0 = k$, which we have identified with $E^3$.

**Lower-dimensional analogue:**
Consider the Schoenflies motion group X with respect to the direction $x_3$ and its subgroup of planar motions SE(2) parallel to the $x_1x_2$-plane. If we apply an orthogonal projection along the $x_3$-direction (analogue of $\kappa^{-1}$) to a X-motion we obtain a planar one.

# 2. Kinematic Interpretation as a Subgroup of SE(4)




The planar motion (right) is obtained as top view of the Schoenflies motion (left). In addition we label the poses of the top view by the $x_3$-coordinate.

The instantaneous screw in the starting pose (left), which is visualized by a gray helix and the black axis in $x_3$-direction, appears in the top view (right) as the instantaneous rotation of the planar motion.

# 3. Straight Lines in the Study Parameter Space

It is known [16,18,19] that straight lines in $P^7$ are in general send by the PSH map to vertical Darboux motions [2,8]. In special cases they correspond to rotations about fixed axes or translations along fixed directions.

To clarify the meaning of straight lines of $P^7$ in terms of $X_4$-motions, we consider Darboux 2-motions studied by Karger [6,7]: In $E^4$ these motions are characterized by the fact that all trajectories are planar (either ellipses or line segments; cf. [7]). We focus on Darboux 2-motions in $E^4$, where all points have circular trajectories.

**Theorem 2.** A circular Darboux 2-motion, which is neither spherical nor a pure translation, has to be composed of a rotation about a fixed plane and a circular translation parallel to this plane.

# 3. Straight Lines in the Study Parameter Space

**Theorem 3.** A straight line $\in P^7 \setminus G$ corresponds to one of the following $X_4$-motions and vice versa:

(i) a translation along a fixed direction,

(ii) a rotation about a fixed plane,

(iii) a circular Darboux 2-motion, which is neither spherical nor a pure translation.

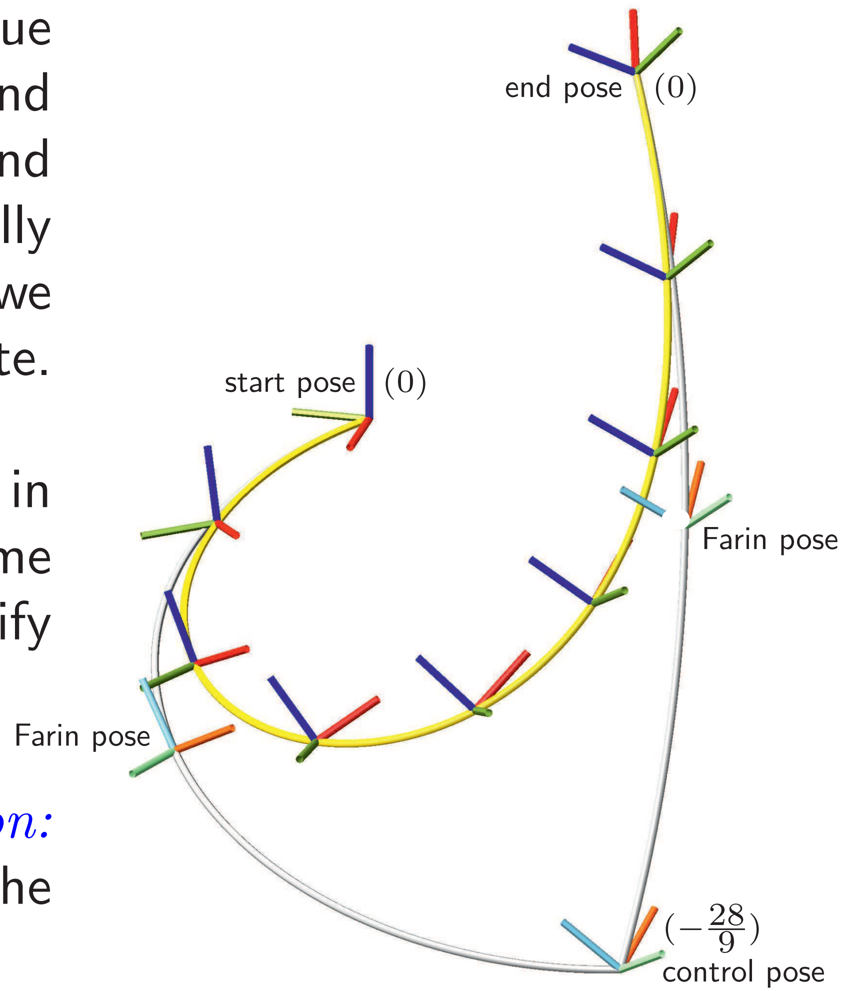
**Remark:** Up to the best knowledge of the author, the existence of these circular Darboux 2-motions has not been mentioned in the literature until now. Moreover they belong to the class of Borel-Bricard motions of $E^4$ (i.e. all points have hyperspherical trajectories). Further examples of this class are given in [21,22]. $\diamond$

# 4. Application: Interactive Rational Motion Design

*Projective de Casteljau construction in $P^7$:* Due to the given interpretation the control points and Farin points of this construction in $P^7$ correspond to poses in $E^4$, which can be projected orthogonally along the $x_0$-direction onto $E^3$. In addition we label the obtained poses of $E^3$ by the $x_0$-coordinate.

As the resulting Farin poses and control poses in $E^3$ are not affinely distorted (as done by some other known methods [5,17]), the user can modify very intuitively the control structure.

*Optimization criterion of the resulting motion:* Minimization of the maximal translation along the $x_0$-direction during the motion.



# 5. References

Finally we referred to

- [12], where the line-symmetry of the circular Darboux 2-motions in $E^4$ and their corresponding basic surfaces are discussed,
- [11], where in analogy to linear complexes of lines some basics on linear complexes of SE(3)-displacements are given.

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All references refer to the list of publications given in the presented paper:


NAWRATIL, G.: Kinematic interpretation of the Study quadric's ambient space. Advances in Robot Kinematics 2018 (J. Lenarcic, V. Parenti-Castelli, Eds.), Springer Proceedings in Advanced Robotics.