Results on Planar Parallel Manipulators with Cylindrical Singularity Surface

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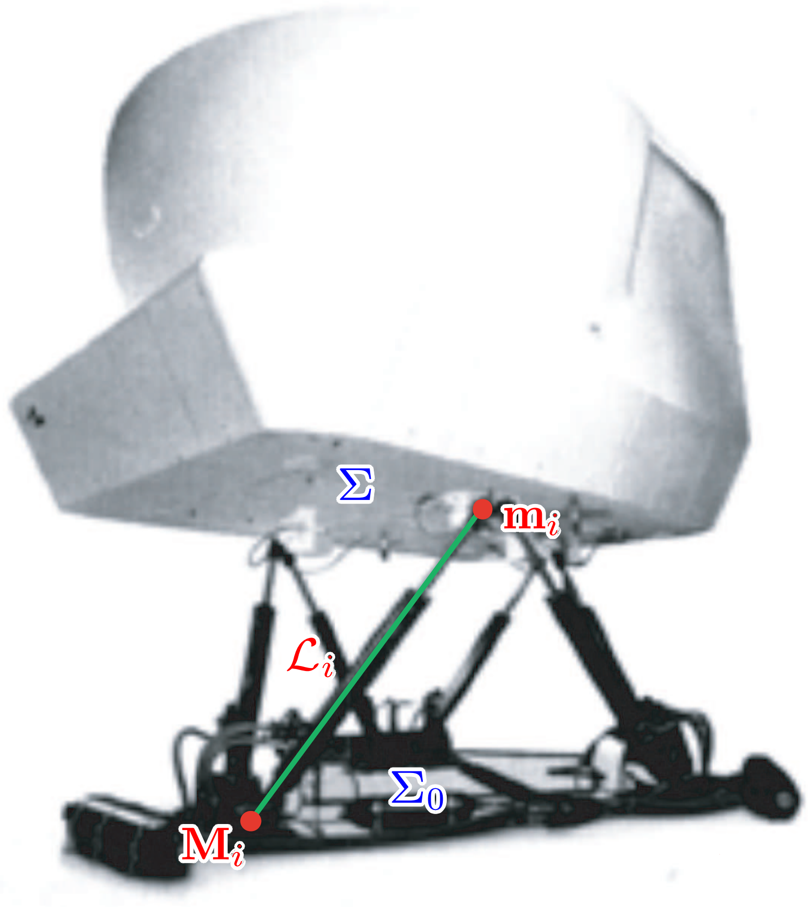
[1] Singular configurations of SGP

The geometry of a Stewart Gough Platform is given by the six base anchor points

$\mathbf{M}_i := (A_i, B_i, C_i)^T$ in the fixed space Σ_0

and by the six platform anchor points

$\mathbf{m}_i := (a_i, b_i, c_i)^T$ in the moving space Σ.



Theorem Merlet [1992]
A SGP is singular iff the carrier lines $\mathcal{L}_i$ of the six legs belong to a linear line complex.

[1] Analytical condition

Plücker coordinates of $\mathcal{L}_i$ can be written as $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - K\mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$

with $\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_1e_3+e_0e_2) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix},$

$\mathbf{t} := (\cos\varphi t_1 - \sin\varphi t_2, \sin\varphi t_1 + \cos\varphi t_2, t_3)^T$ and $K := e_0^2 + e_1^2 + e_2^2 + e_3^2.$

Remark: The group SO_3 is parametrized by Euler Parameters (e_0, e_1, e_2, e_3).

$\mathcal{L}_i$ belong to a linear line complex $\Longleftrightarrow$ $Q := det(\mathbf{Q}) = 0$ with $\mathbf{Q} := \begin{pmatrix} \mathbf{l}_1 & \widehat{\mathbf{l}}_1 \\ \dots & \dots \\ \mathbf{l}_6 & \widehat{\mathbf{l}}_6 \end{pmatrix}$

[2] Preliminary considerations

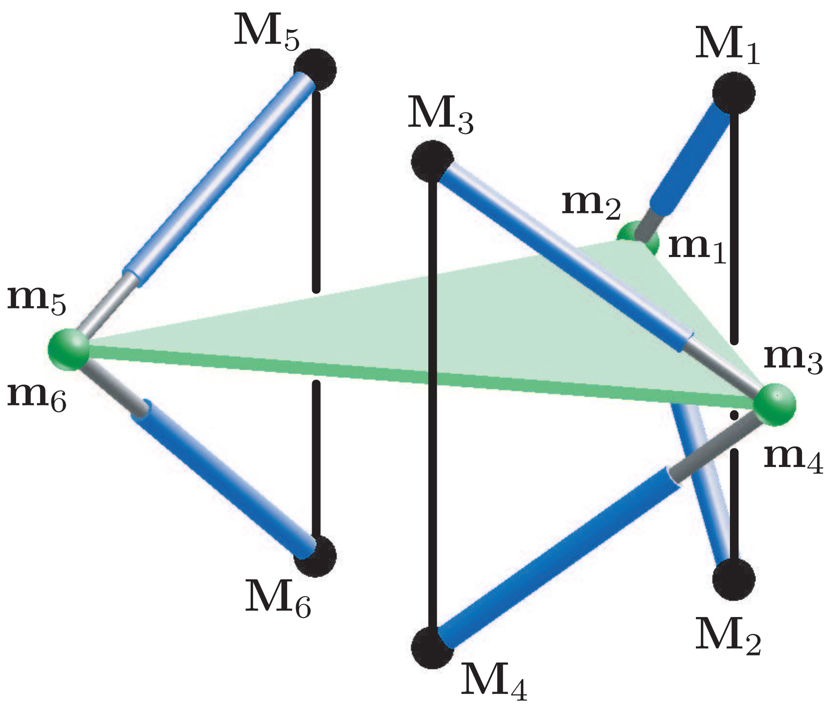
Definition **SGP with Cylindrical Singularity Surface**
The manipulators singularity set is for any orientation of the platform a cylindrical surface with rulings parallel to a given fixed direction p in the space of translations.

The set of SGPs with a cylindrical singularity surface contains the set of architecture singular SGPs. These two sets are distinct due to:

Example [see Figure]

- $\mathbf{m}_1 = \mathbf{m}_2,\ \mathbf{m}_3 = \mathbf{m}_4,\ \mathbf{m}_5 = \mathbf{m}_6$
- $\overline{\mathbf{M}_1\mathbf{M}_2} \parallel \overline{\mathbf{M}_3\mathbf{M}_4} \parallel \overline{\mathbf{M}_5\mathbf{M}_6} \parallel p$

- $\mathbf{M}_1, \ldots, \mathbf{M}_6$ can be coplanar



[2] Preliminary considerations

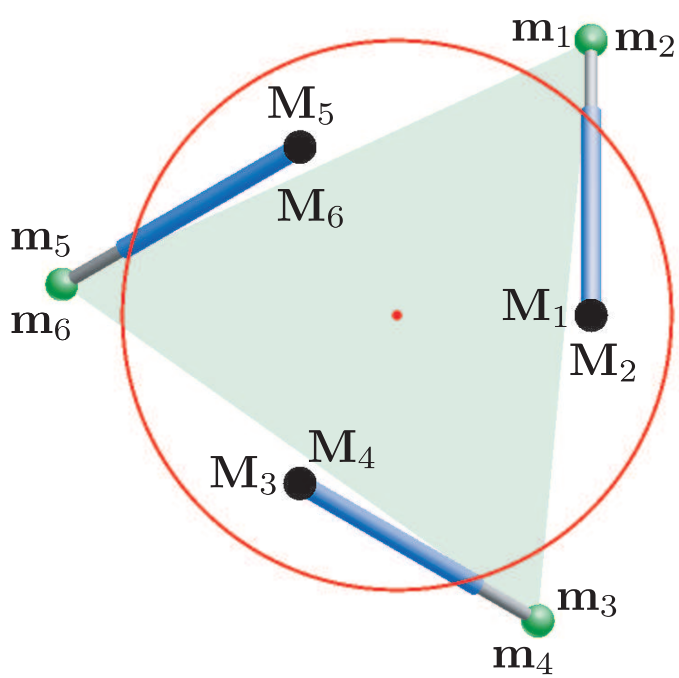
This manipulator is only in a singular configuration iff the three planes $[\mathbf{M}_1, \mathbf{M}_2, \mathbf{m}_1]$, $[\mathbf{M}_3, \mathbf{M}_4, \mathbf{m}_3]$ and $[\mathbf{M}_5, \mathbf{M}_6, \mathbf{m}_5]$ have a common intersection line.

The singularity surface is a quadratic cylinder.

Is this the only SGP with this property?

We distinguish between planar and non-planar SGPs because the structure of architecturally singular SGPs depends on the planarity of the platform and the base; cf. **Karger [2003,2008]**.

In this paper we only deal with planar SGPs.



Projection direction is p.

[3] The Main Theorem

Main Theorem
The set of planar parallel manipulators with no four anchor points on a line which possess a cylindrical singularity surface with rulings parallel to a given fixed direction p for any orientation of the platform equals the set of planar architecture singular manipulators (with no four anchor points on a line).

Idea of the proof

- We choose an Cartesian frame with one axis $t_i \parallel p$.
- Then $Q := det(\mathbf{Q}) = 0$ must be independent of t_i for all $e_0, \ldots, e_3, t_j, t_k$.
- The analytical proof is based on the resulting equations.

[4] Preparatory work for the proof

[A] Choose of Cartesian frames in the fixed space and the moving space

- As we consider only SGPs with planar platform we set $c_i = 0$ for $i = 1, \ldots, 6$.
- We set up the planar base in a more general position as

$$C_1 = 0, \quad C_i = \left[C_2(B_3A_i - A_3B_i) + A_2C_3B_i\right]/(A_2B_3) \quad \text{for} \quad i = 4, 5, 6.$$

- **Lemma of Karger [2003]**
For planar parallel manipulators with no four points on a line we can assume

$$A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0 \qquad \text{and}$$

$$A_2B_3B_4B_5a_2(a_4 - a_3)coll(3, 4, 5) \neq 0 \qquad \text{with}$$

$$coll(i, j, k) := a_i(b_j - b_k) + a_j(b_k - b_i) + a_k(b_i - b_j).$$

[4] Preparatory work for the proof

[B] Algebraic characterization of the subset of architecture singular SGPs

We perform the same elementary row operations with the matrix $\mathbf{Q}$ as described by **Karger [2003]**. Then the last row of $\mathbf{Q}$ is of the form

$$(r_{11}K_1 + r_{12}A_2K_2, r_{21}K_1 + r_{22}A_2K_2, r_{31}K_1 + r_{32}A_2K_2, r_{21}C_2K_3 + r_{22}C_2K_4, r_{31}A_2K_3 + r_{32}A_2K_4 - r_{11}C_2K_3 - r_{12}C_2K_4, -r_{21}A_2K_3 - r_{22}A_2K_4)D^{-1}$$

with $D := A_2B_3B_4B_5coll(3,4,5)$ and r_{ij} the entries of the rotary matrix $\mathbf{R}$.

Theorem of Karger [2003]
$K_1 = K_2 = K_3 = K_4 = 0$ are the four conditions which are satisfied iff a planar parallel manipulator with no four points on a line is architecturally singular.

[5] Sketch of the proof

I) Base is not parallel to p

(i) Base is orthogonal to p

(ii) Base is not orthogonal to p

II) Base is parallel to p

(i) $\mathrm{M}_1\mathrm{M}_2$ is parallel to p

(ii) $\mathrm{M}_1\mathrm{M}_2$ is not parallel to p

(a) $\mathrm{M}_1\mathrm{M}_2$ is orthogonal to p

(b) $\mathrm{M}_1\mathrm{M}_2$ is not orthogonal to p

I) Base is not parallel to p

(i) Base is orthogonal to p $(C_2 = C_3 = 0)$

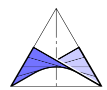
The proof of this case is hidden in the proof of the **Theorem of Karger [2003]**. Karger sets $t_1 = t_2 = 0$ and eliminates t_3 from Q. He proves in four steps $(\mathbf{a}), \ldots, (\mathbf{d})$ that the resulting equations can only vanish for $K_1 = \ldots = K_4 = 0$.

(ii) Base is not orthogonal to p

We start such as Karger by setting $t_1 = t_2 = 0$. Now Q can be written as

$$Q = A_2^2(r_{11}r_{22} - r_{12}r_{21})Q_3t_3^3 + A_2B_3Q_2t_3^2 + Q_1t_3 + Q_0.$$

With the coefficients Q_1, Q_2, Q_3 the steps $(\mathbf{a})$ and $(\mathbf{b})$ can be done one by one. The steps $(\mathbf{c})$ and $(\mathbf{d})$ are different and therefore given in **Nawratil [2008,A]**.



II) Base is parallel to p $(C_2 = C_3 = 0)$

We eliminate t_1 from Q. We denote the coefficients of $t_1^i t_2^j t_3^k$ from Q by Q^{ijk}.

(i) $\mathrm{M}_1\mathrm{M}_2$ is parallel to p $(\varphi = 0)$

From Q^{101} we can factor out K and from Q^{100} we can even factor out K^2.

Finally, we denote the coefficient of $e_0^a e_1^b e_2^c e_3^d$ of Q^{ijk} by P_{abcd}^{ijk} and compute:

$$P_{4110}^{101} - P_{1401}^{101} - P_{1041}^{101} + P_{0114}^{101} = K_1 B_3 B_4 B_5 coll(3,4,5) \qquad \Longrightarrow K_1 = 0$$

$$P_{0222}^{101} + P_{2022}^{101} - P_{2202}^{101} - P_{2220}^{101} = K_2 A_2 B_3 B_4 B_5 coll(3,4,5) \qquad \Longrightarrow K_2 = 0$$

$$P_{3120}^{100} - P_{2031}^{100} - P_{1302}^{100} + P_{0213}^{100} = K_3 a_2 B_3 B_4 B_5 coll(3,4,5) \qquad \Longrightarrow K_3 = 0$$

$$P_{3210}^{100} - P_{2301}^{100} - P_{1032}^{100} + P_{0123}^{100} = K_4 a_2 B_3 B_4 B_5 coll(3,4,5) \qquad \Longrightarrow K_4 = 0$$

Remark: This is the shortest possible analytical proof of the Theorem of Karger.

II) Base is parallel to p $(C_2 = C_3 = 0)$

(ii) $\mathrm{M}_1\mathrm{M}_2$ is not parallel to p

(a) $\mathrm{M}_1\mathrm{M}_2$ is orthogonal to p $(\varphi = \pi/2)$

From Q^{ijk} $(i > 0)$ we can factor out K. From Q^{100} we can even factor out K^2. We factor out $(e_0 e_1 - e_2 e_3)$ of Q^{2jk} and compute the following 15 polynomials:

$P_1[18] := P^{111}_{2200}$ $\quad P_2[42] := P^{201}_{1010}$ $\quad P_3[12] := P^{110}_{3300}$ $\quad P_4[42] := P^{200}_{2020}$ $\quad P_5[72] := P^{200}_{2110}$

$P_6[36] := P^{100}_{3210} - P^{100}_{2301} - P^{100}_{1032} + P^{100}_{0123}$ $\quad P_7[42] := P^{101}_{4110} - P^{101}_{1401} - P^{101}_{1041} + P^{101}_{0114}$

$P_8[30] := P^{101}_{4200} + P^{101}_{2400} + P^{101}_{0042} + P^{101}_{0024}$ $\quad P_9[30] := P^{101}_{4200} + P^{101}_{2400} - P^{101}_{0042} - P^{101}_{0024}$

$P_{10}[18] := P^{111}_{2110} - P^{111}_{1201}$ $\quad P_{11}[42] := P^{101}_{3111} + P^{101}_{1311}$ $\quad P_{12}[36] := P^{110}_{3210} - P^{110}_{2301}$

$P_{13}[24] := P^{110}_{3120} - P^{110}_{2031}$ $\quad P_{14}[12] := P^{100}_{3300} + P^{100}_{0033}$ $\quad P_{15}[24] := P^{100}_{2121} - P^{100}_{1212}$

(ii) $\mathrm{M_1M_2}$ is not parallel to p

(b) $\mathrm{M_1M_2}$ is not orthogonal to p $\quad (\sin\varphi\cos\varphi \neq 0)$

In this case we compute the following 20 polynomials:

$$P_1[12] := P^{100}_{3300} + P^{100}_{0033} \qquad P_2[36] := P^{100}_{3030} - P^{100}_{0303} \qquad P_3[78] := P^{101}_{0402} - P^{101}_{2040}$$

$$P_4[66] := P^{101}_{4020} - P^{101}_{2040} \qquad P_5[30] := P^{101}_{4200} + P^{101}_{0042} \qquad P_6[66] := P^{101}_{4020} + P^{101}_{0402}$$

$$P_7[36] := P^{101}_{4200} - P^{101}_{0024} \qquad P_8[42] := P^{101}_{0042} - P^{101}_{0024} \qquad P_9[18] := P^{102}_{3100} + P^{102}_{1300}$$

$$P_{10}[18] := P^{102}_{2011} - P^{102}_{1120} \qquad P_{11}[108] := P^{101}_{3111} - P^{101}_{1311} \qquad P_{12}[102] := P^{101}_{4110} - P^{101}_{1401}$$

$$P_{13}[24] := P^{100}_{3210} - P^{100}_{0123} - P^{100}_{2301} + P^{100}_{1032} \qquad P_{14}[42] := P^{100}_{3210} + P^{100}_{0123} + P^{100}_{2301} + P^{100}_{1032}$$

$$P_{15}[48] := P^{100}_{3210} + P^{100}_{0123} - P^{100}_{2301} - P^{100}_{1032} \qquad P_{16}[36] := P^{100}_{3120} - P^{100}_{0213} - P^{100}_{2031} + P^{100}_{1302}$$

$$P_{17}[66] := P^{101}_{4110} + P^{101}_{1401} + P^{101}_{1041} + P^{101}_{0114} \qquad P_{18}[54] := P^{101}_{4110} - P^{101}_{1401} + P^{101}_{1041} - P^{101}_{0114}$$

$$P_{19}[48] := P^{101}_{3201} - P^{101}_{2310} - P^{101}_{0132} + P^{101}_{1023} \qquad P_{20}[150] := P^{101}_{3201} + P^{101}_{2310} - P^{101}_{0132} - P^{101}_{1023}$$

ad (ii) (a,b) In both cases we proceed as given in Nawratil [2008,B]

- $K_2 = 0$: All 15 resp. 20 polynomials can only vanish for architecture singular manipulators, i.e. $K_1 = K_3 = K_4 = 0 \quad \Longrightarrow \quad K_2 \neq 0$

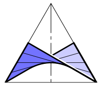
- $K_2 \neq 0$: All coefficients of t_1 can only vanish for the following two solutions:

$$S_1: \quad A_i = B_i \cot\varphi,\ A_j = B_j \cot\varphi,\ A_k = A_2 + B_k \cot\varphi,$$
$$b_k = 0,\ a_2 = a_k,\ a_i = K_1 b_i/(K_2 A_2), a_j = K_1 b_j/(K_2 A_2),$$
$$K_3 = 0 \quad \text{and} \quad K_4 = 0 \quad (\star)$$

$$S_2: \quad A_i = A_2 + B_i \cot\varphi,\ A_j = A_2 + B_j \cot\varphi,\ A_k = B_k \cot\varphi,$$
$$a_i = a_2 + b_i K_3/K_4,\ a_j = a_2 + b_j K_3/K_4,\ a_k = b_k = 0,$$
$$A_2 K_2 + K_4 = 0 \quad \text{and} \quad K_1 + K_3 = 0 \quad (\star\star)$$

for $i, j, k \in \{3, 4, 5\}$ and $i \neq j \neq k \neq i$ without contradicting

$$A_2 B_3 B_4 B_5 a_2 (a_4 - a_3) coll(3, 4, 5) \neq 0.$$



The close of the proof

We show that both solutions S_i imply contradictions for the choice of $\mathbf{M}_6$ and $\mathbf{m}_6$:
If we set $A_2 = 1$ and replace K_i in $(\star)$ and $(\star\star)$ by the explicit expressions we get:

$(\star)$ $\quad K_3 = (A_6 - B_6 \cot \varphi)(a_k - a_6) \qquad K_4 = (A_6 - B_6 \cot \varphi) b_6$

- $a_6 = a_k,\ b_6 = 0 \Longrightarrow K_2 = 0$
- $A_6 = B_6 \cot \varphi \quad \Longrightarrow \mathbf{M}_1, \mathbf{M}_i, \mathbf{M}_j, \mathbf{M}_6$ are collinear

$(\star\star)$ $\quad K_1 + K_3 = (1 - A_6 + B_6 \cot \varphi) a_6 \qquad K_2 + K_4 = (1 - A_6 + B_6 \cot \varphi) b_6$

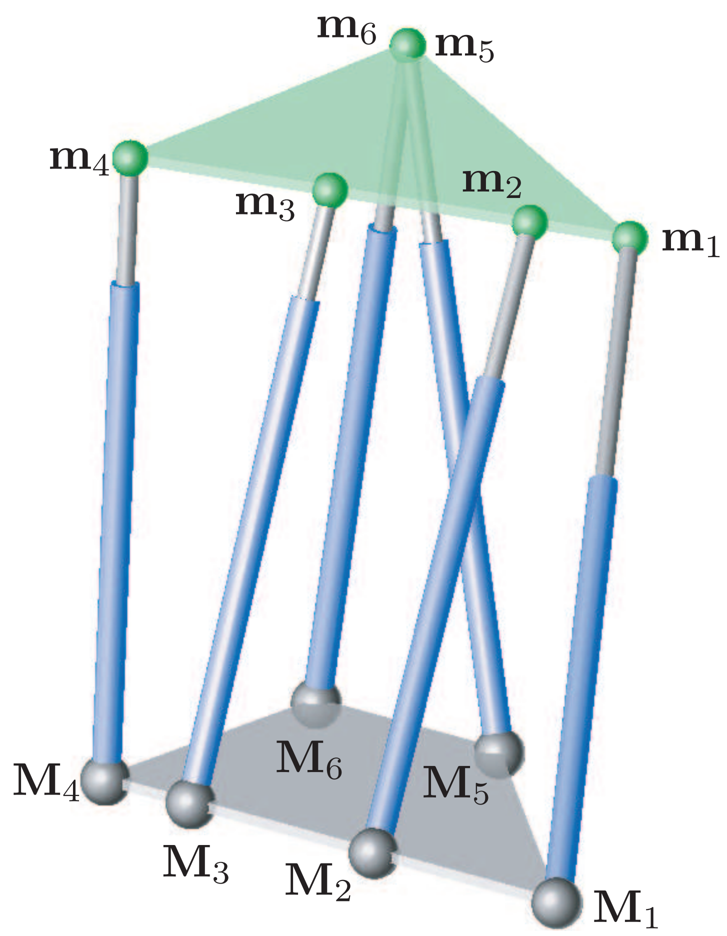
- $a_6 = 0$ and $b_6 = 0 \Longrightarrow K_2 = 0$
- $A_6 = 1 + B_6 \cot \varphi \Longrightarrow \mathbf{M}_2, \mathbf{M}_i, \mathbf{M}_j, \mathbf{M}_6$ are collinear $\square$

[6] A further example

S_1 and S_2 imply a further example for a planar SGP with cylindrical singularity surface. For the computation see **Nawratil [2008,A]**.

- $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ are collinear,
- $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ are collinear,
- $\overline{\mathbf{M}_5\mathbf{M}_6} \parallel \overline{\mathbf{M}_1\mathbf{M}_2} \parallel p$,
- and $\mathbf{m}_5 = \mathbf{m}_6$.

SGP is in a singular position iff $\mathbf{m}_5 = \mathbf{m}_6$ lies in the base or $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are coplanar $\Longrightarrow$ singularity surface splits into two planes



[7] Remark

Theorem Röschel and Mick [1998]
Planar SGPs are architecturally singular iff $\{\mathbf{M}_i, \mathbf{m}_i\}$ for $(i = 1, \ldots, 6)$ are four-fold conjugate pairs of points with respect to a 3-dimensional linear manifold of correlations or one of the two sets $\{\mathbf{M}_i\}$ and $\{\mathbf{m}_i\}$ is situated on a line.

Therefore the given main theorem can be reformulated as follows:

Main Theorem
Planar SGPs with no four points on a line and a cylindrical singularity surface must consist of four-fold conjugate pairs of anchor points with respect to a 3-dimensional linear manifold of correlations.

It would be nice to have a geometric proof for the main theorem similar to the one presented by **Röschel and Mick [1998]**.

[7] Conclusion

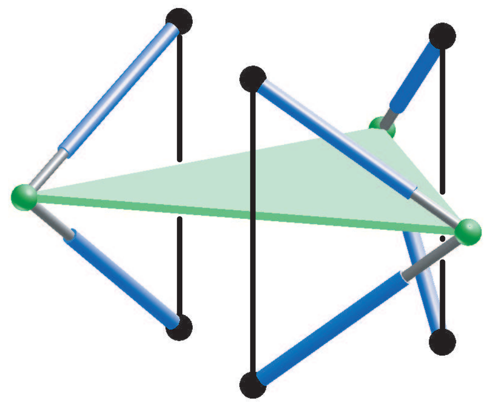
- We proved that there do not exist non-architecturally singular planar SGPs and no four anchor points collinear which possess a cylindrical singularity surface.

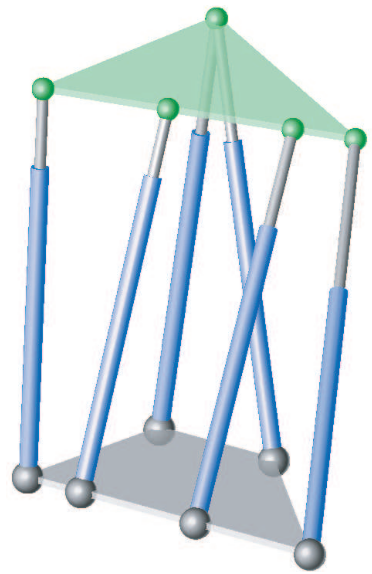
- We gave the shortest possible analytical proof for the **Theorem of Karger**.

- Moreover, we presented two examples of planar manipulators with cylindrical singularity surface.

- A complete list of planar SGPs with a cylindrical singularity surface is in preparation.

  Nawratil, G., All Planar Parallel Manipulators with Cylindrical Singularity Surface, in preparation.





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