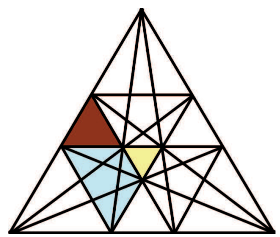
Self-motions of planar projective Stewart Gough platforms

Georg Nawratil

Institute of Geometry, TU Dresden, Germany

Research was supported by FWF (I408-N13)



Advances in Robot Kinematics, June 24-28 2012, Innsbruck, Austria

Austrian Science Fund FWF

Table of contents

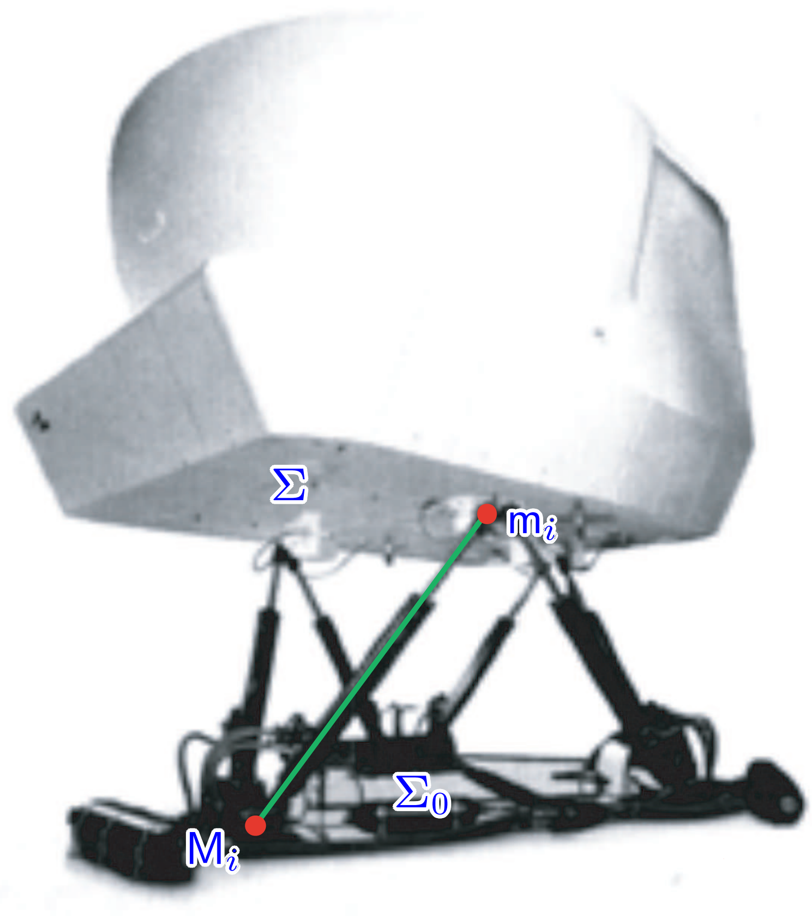
[1] What is a self-motion of a SGP?

The geometry of a SGP is given by the six base anchor points $\mathsf{M}_i \in \Sigma_0$ and by the six platform points $\mathsf{m}_i \in \Sigma$.

A SGP is called planar, if $\mathsf{M}_1, \ldots, \mathsf{M}_6$ are coplanar and $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are coplanar. The carrier planes are denoted by π_{M} resp. π_{m}.

M_i and m_i are connected with a S<u>P</u>S leg.

If all <u>P</u>-joints are locked, a SGP is in general rigid. But under particular conditions, the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion.



[1] Planar projective SGPs

Definition 1
A planar SGP is called projective if M_i and m_i are related by a non-singular projectivity κ; i.e. $\mathsf{m}_i\kappa = \mathsf{M}_i$ for $i = 1, \ldots, 6$.

Remark: If κ is singular, all base anchor points would collapse into a line or a point, which yields trivial cases of architecturally singular SGPs. ◇

Due to the results of Chasles [8], Karger [4,9], Röschel and Mick [6], a planar projective SGP is architecturally singular if and only if one set of anchor points is located on a conic section.

As it is well known that architecturally singular SGPs possess self-motions (over $\mathbb{C}$) in each pose, we are only interested in non-architecturally singular planar projective SGPs with self-motions.

[2] Basic results

Theorem 1 (Proof was given by Karger [9])
A singular configuration of non-architecturally singular planar projective SGP does not depend on the distribution of the anchor points in the platform and the base, but only on the mutual position of the planes π_M and π_m and on the correspondence between them. The configuration is singular if and only if either one of the legs can be replaced by a leg of zero length or two legs can be replaced by aligned legs.

Lemma 1 (Proof is given in the presented paper, pages 28–29)
One can attach a two-parametric set of additional legs to planar projective SGPs without changing the forward kinematics and singularity surface.

Remark: Due to Lemma 1, it is clear why a singular configuration does not depend on the distribution of the anchor points in π_M and π_m (cf. Theorem 1). ◇

$[2]$ Basic results

s denotes the line of intersection of π_{M} and π_{m} in the projective extension of the Euclidean 3-space.

Definition 2
A self-motion of a non-architecturally singular planar projective SGP is called elliptic, if in each pose of this motion s exists with $\mathsf{s} = \mathsf{s}\kappa$ and where the projectivity from s onto itself is elliptic.

Theorem 2
A self-motion of non-architecturally singular planar projective SGPs can only be:

1. a spherical self-motion with rotation center $\mathsf{m}\kappa = \mathsf{m}$,
2. a Schönflies self-motion, where the direction of the rotation axis is parallel to the planes π_{M} and π_{m},
3. an elliptic self-motion.

[2] Basic results

Proof of Theorem 2

As in any pose of a self-motion of a planar projective SGP, the manipulator has to be in a singular configuration, we can apply Theorem 1. Therefore, the manipulator is singular if and only if:

a. π_M and π_m coincide ($\Rightarrow \exists$ real fixed point $\Rightarrow$ case 1 or 2),

b. $S = S\kappa$ (real fixed point $\Rightarrow$ case 1 or 2) holds, where S is the intersection point of s and sκ,

c. $s = s\kappa$. If the restriction of κ to s is hyperbolic or parabolic, we also get at least one real fixed point ($\Rightarrow$ case 1 or 2). The elliptic projectivity yields case 3. $\square$

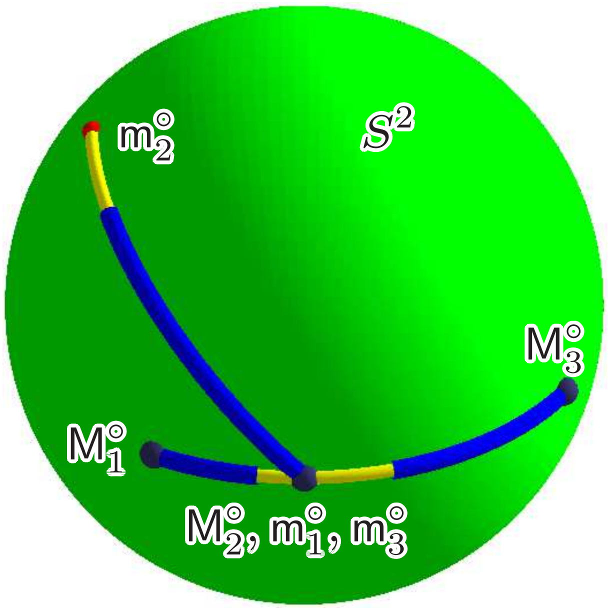
Remark: Due to Theorem 2, only the three listed types of motions are candidates for self-motions of non-architecturally singular planar projective SGPs. $\diamond$

[3] Spherical self-motions

If a planar projective SGP has a spherical self-motion about $\mathsf{m}\kappa = \mathsf{m}$, the spherical image of this manipulator with respect to the unit sphere S^2 centered in $\mathsf{m}\kappa = \mathsf{m}$ has to have a self-motion as well.

Therefore the problem reduces to the determination of non-degenerated spherical 3-dof RPR manipulators with self-motions, where the base points M_1°, M_2°, M_3° and the platform points m_1°, m_2°, m_3° are located on great circles.

Due to Nawratil [14], there exists only the illustrated solution (after relabeling of anchor points and interchange of platform and base).

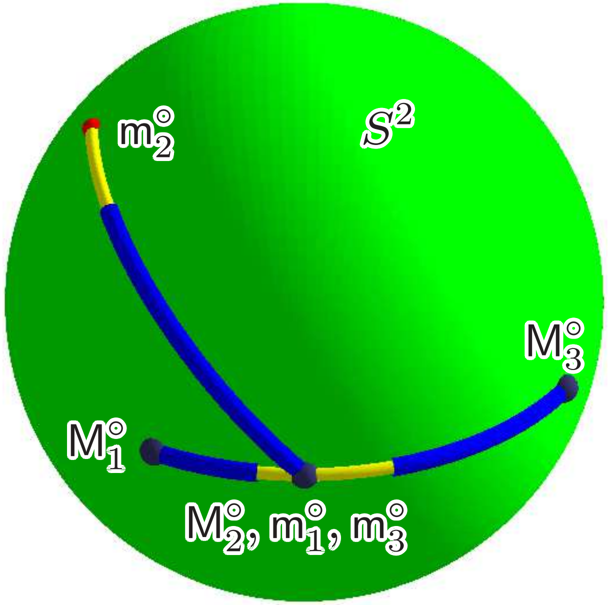


[3] Spherical self-motions

This 3-dof R$\underline{\text{P}}$R manipulator has a pure rotational self-motion around the axis $a := [m\kappa = m, m_1^\circ = m_3^\circ = M_2^\circ]$.

Therefore, we can only add an additional leg $\overline{m_4^\circ M_4^\circ}$ without restricting the self-motion if $m_4^\circ = m_1^\circ$ or $M_4^\circ = M_2^\circ$ holds.

Therefore, κ has to map all platform anchor points $\notin$ a on points of a $\Rightarrow$ κ is singular $\Rightarrow$

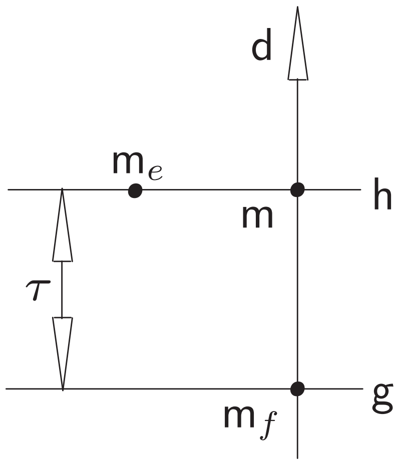


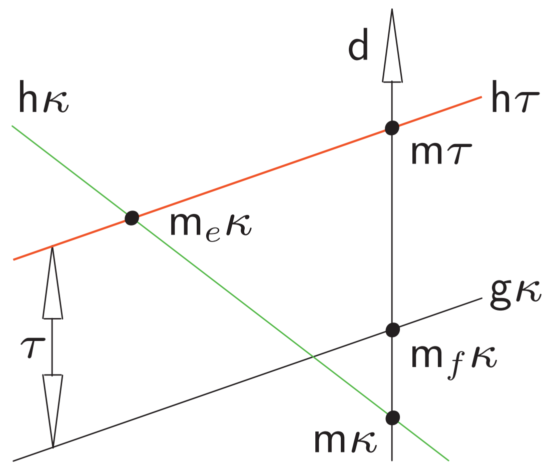
Theorem 3
Non-architecturally singular planar projective SGPs do not have spherical self-motions with rotation center $m\kappa = m$.

[4] Schönflies self-motions

The Schönflies motion group consists of all translations combined with all rotations about a fixed direction d, which in our case is parallel to π_{M} and π_{m}.

It is well known (e.g. Husty and Karger [15]), that platform points being on lines parallel to d have congruent trajectories in a Schönflies motion. Therefore, every leg can be translated in direction d without changing this motion.





$[4]$ **Case $\mathsf{h}\kappa \neq \mathsf{h}\tau$**

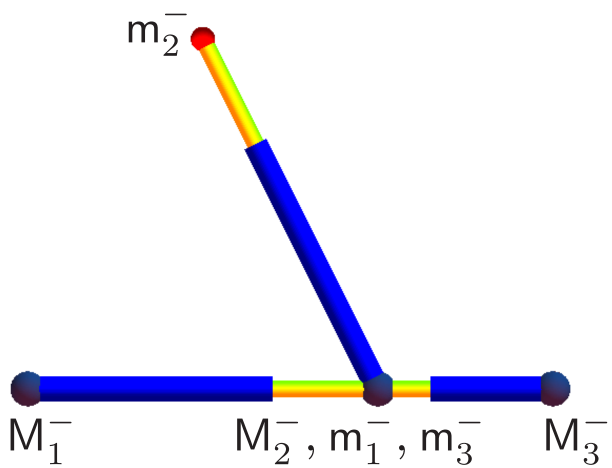
Now, every point $\mathsf{m} \in \mathsf{h}$ (with exception of m_e) can only rotate about the line $[\mathsf{m}\tau, \mathsf{m}\kappa] \parallel \mathsf{d}$. Therefore, the platform cannot move in direction d during the self-motion and the problem reduces to the following planar one:

Determine all non-degenerated 3-dof R<u>P</u>R manipulators with self-motions, where the platform points m_1^-, m_2^-, m_3^- and base points M_1^-, M_2^-, M_3^- are collinear.

It is well known, that only two solutions exist:

- <u>Planar analogue of the spherical self-motion:</u>

  The same arguments as in the spherical case yield again a contradiction.

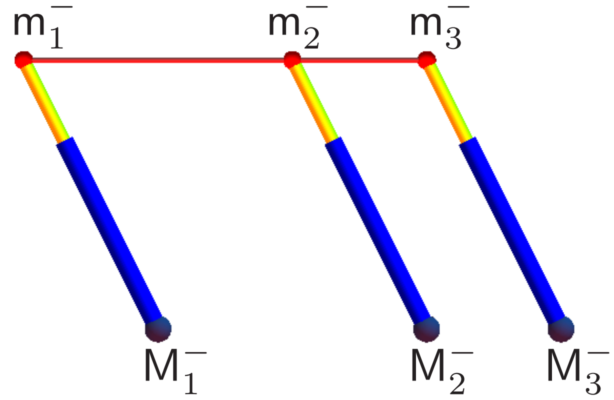


Austrian Science Fund FWF

[4] Case h$\kappa \neq$ hτ

- Circular translation:

  If we choose the y-axis of the moving and the fixed frame in direction of d, the matrix $\mathbf{P}$ of the projectivity κ can be written as:



$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ p_{21} & 1 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \quad \text{with} \quad p_{33} \in \mathbb{R} \setminus \{0,1\} \quad \text{and} \quad p_{21}, p_{31}, p_{32} \in \mathbb{R}. \qquad (\star)$$

As ideal points are mapped onto ideal points, κ is an affinity.

Remark: For the proof of the matrix $\mathbf{P}$ see pages 31–32 of the presented paper. ◇

$[4]$ Case h$\kappa =$ hτ

For this case, it can also be proven (cf. page 32 of the presented paper) that κ has to be an affinity with the following matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & 1 \end{pmatrix} \quad \text{with} \quad p_{22} \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad p_{21}, p_{31}, p_{32} \in \mathbb{R}. \qquad (\circ)$$

Theorem 4
A non-architecturally singular planar projective SGP can only have a Schönflies self-motion with the direction d of the rotation axis parallel to π_{M} and π_{m}, if it belongs to the subset of planar affine SGPs.
Moreover, if we choose the y-axis of the moving and the fixed frame in direction of d, the affinity κ has to be of the form given in $(\star)$ or $(\circ)$.

$[5]$ Planar affine SGPs with self-motions

Theorem 5
Assume a non-architecturally singular planar affine SGP is determined by $\mathbf{M}_i = \mathbf{a} + \mathbf{A}\mathbf{m}_i$. Then this manipulator has a self-motion if and only if the singular values s_1 and s_2 of $\mathbf{A}$ with $0 < s_1 \leq s_2$ fulfill $s_1 \leq 1 \leq s_2$.

Proof of Theorem 5
First of all, we prove that planar affine SGPs cannot have elliptic self-motions: If $\mathsf{s} = \mathsf{s}\kappa$ is not the ideal line, then at least the ideal point of $\mathsf{s} = \mathsf{s}\kappa$ is a fixed point. Therefore, $\mathsf{s} = \mathsf{s}\kappa$ has to be the ideal line during the whole elliptic self-motion. Hence, the self-motion is a Schönflies motion with d orthogonal to $\pi_{\mathsf{M}} \parallel \pi_{\mathsf{m}}$.

As all points of π_{m} have to run on spherical paths, this Schönflies motion can only be the Borel Bricard motion due to Husty and Karger [15]. Therefore, the corresponding points of π_{m} and π_{M} have to be related by an inversion ($\neq$ projectivity).

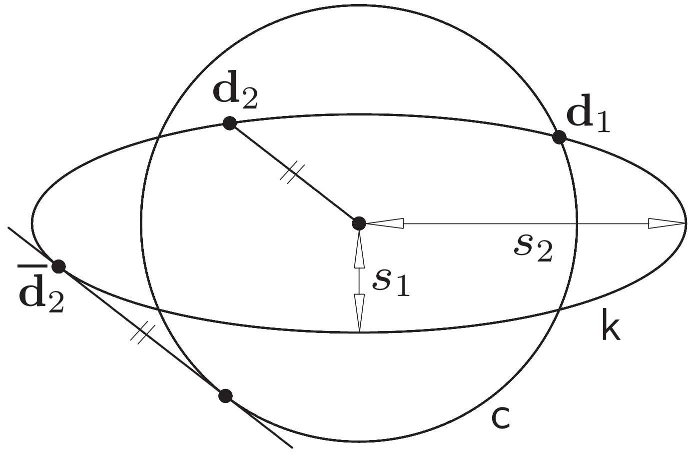
[5] Continuing the proof of Theorem 5

Therefore, planar affine SGPs with self-motions have to be of type (∘) or (⋆). We consider the image of the unit vectors $\mathbf{c} = (\cos\varphi, \sin\varphi) \in \pi_{\mathsf{m}}$ for $\varphi \in [0, 2\pi]$. Clearly, the tie points of the vectors $\mathbf{Ac}$ are located on an ellipse k.

Now, it can easily be seen (cf. page 33 of the presented paper) that the necessary and sufficient condition for an affinity of type:

(∘) is that k and c have a common point,

(⋆) is that k and c have a common tangent.



Clearly, we only get real common points and tangents of k and the unit circle c if the singular values $0 < s_1 \leq s_2$ of $\mathbf{A}$ fulfill $s_1 \leq 1 \leq s_2$. □

[5] Remarks on self-motions of planar affine SGPs

- All self-motions of planar affine SGPs are pure translations, and the self-motion is two-dimensional only if the platform and the base are congruent and all legs have equal length.

  This can easily be proven by direct computations (cf. Nawratil [14]), but this result was also already known to Karger [9].

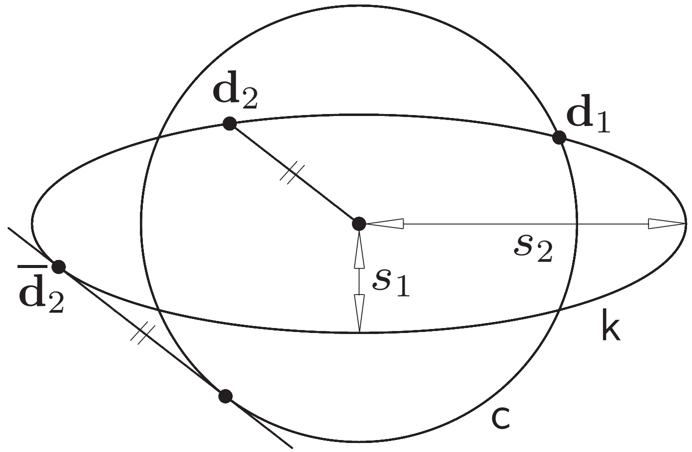
- Theorem 5 also implies the result of Karger [10], that non-architecturally singular planar equiform SGPs cannot have self-motions, as in this case $s_1 = s_2 \neq 1$ holds.

- All planar affine SGPs given in $(\star)$ and $(\circ)$ are Schönflies-singular manipulators due to item (3) and item (2), respectively, of Theorem 3 given by Nawratil [16].

[6] Conclusion and future research

We proved that non-architecturally singular planar projective SGPs have either:

- elliptic self-motions or
- pure translational self-motions.



The latter are the only self-motions of planar affine SGPs. For these manipulators, we also presented a geometric characterization in Theorem 5.

It remains open whether elliptic self-motions even exist, as no examples are known.

Remark: An answer to this question was recently found (paper is in preparation).

[7] References and acknowledgements

All references refer to the list of publications given in the presented paper:

Nawratil, G.: Self-motions of planar projective Stewart Gough platforms.
Latest Advances in Robot Kinematics (J. Lenarcic, M. Husty eds.), pp. 27–34, Springer, 2012, ISBN 978-94-007-4619-0.

Acknowledgements
This research is supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.