Self-motions of parallel manipulators associated with flexible octahedra

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$[1a]$ Bricard octahedra

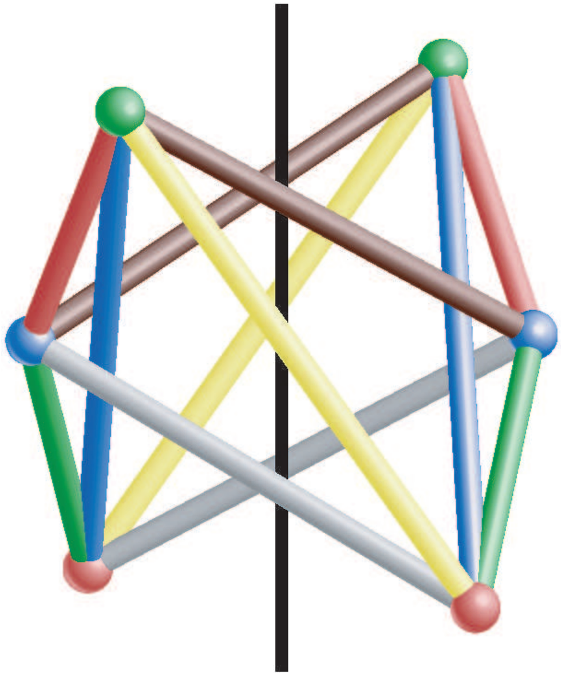
An octahedron is called flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e. every face remains congruent to itself during the flex.

All flexible octahedra in the Euclidean 3-space E^3, where no two faces coincide permanently during the flex, were firstly determined by Bricard [7].

There are 3 types of these so-called Bricard octahedra:

Bricard octahedra of type I

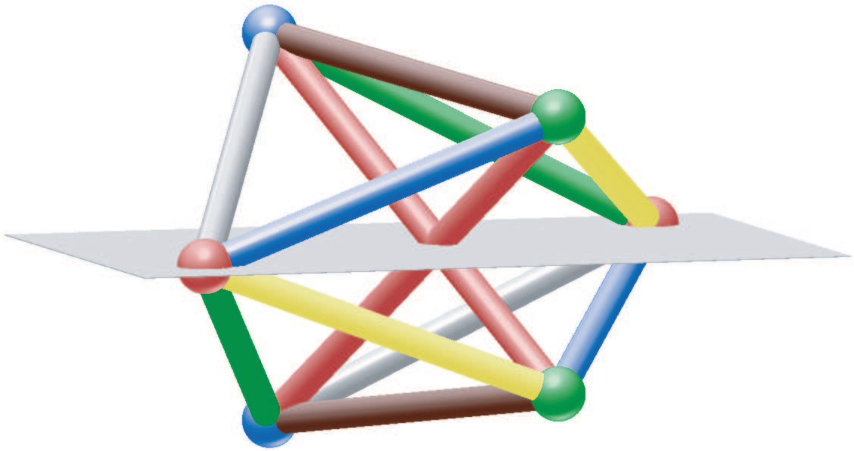
All three pairs of opposite vertices are symmetric with respect to a line.



$[1a]$ Bricard octahedra

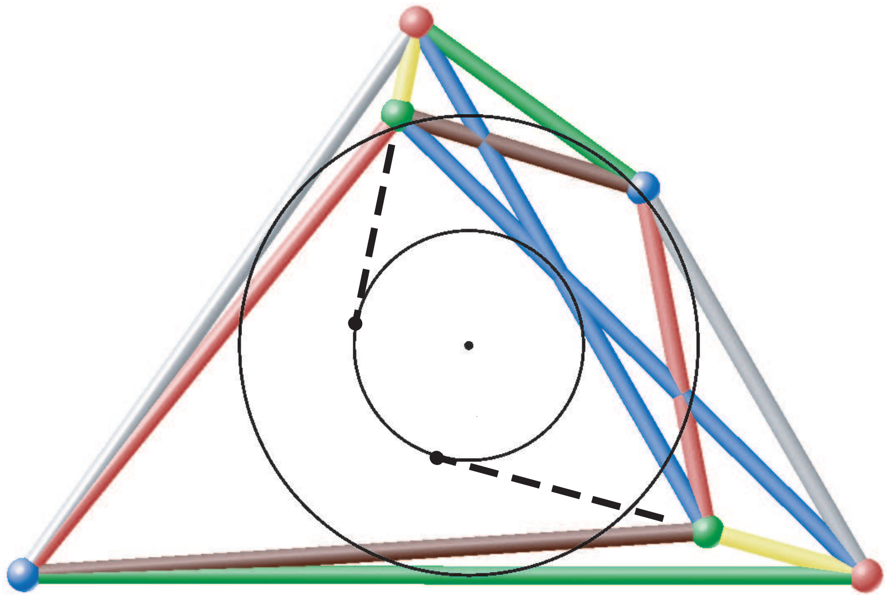
Bricard octahedra of type II

Two pairs of opposite vertices are symmetric with respect to a plane through the remaining two vertices.



Bricard octahedra of type III

These octahedra possess two flat poses and can be constructed as follows [8]:



$[1b]$ Stewart Gough Platform

The geometry of a planar SGP is given by the six base anchor points M_i with

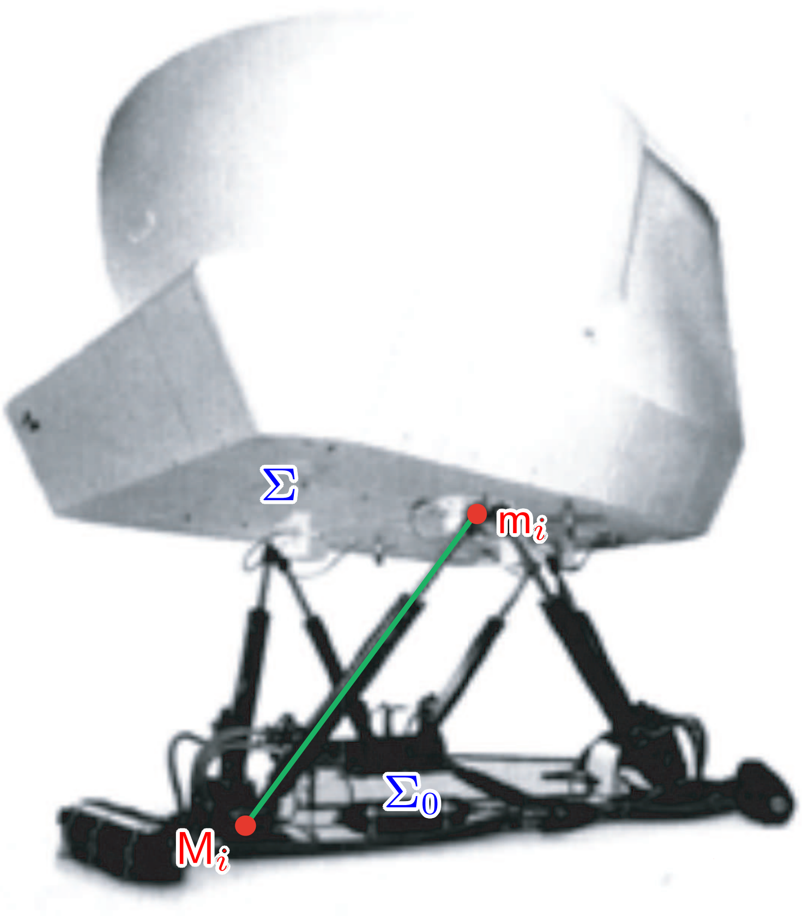
$\mathbf{M}_i := (A_i, B_i, 0)^T$ in the fixed space Σ_0,

and by the six platform points m_i with

$\mathbf{m}_i := (a_i, b_i, 0)^T$ in the moving space Σ.

M_i and m_i are connected with a S<u>P</u>S leg.

Theorem [A]
A SGP is singular (infinitesimal flexible) if and only if the carrier lines of the six S<u>P</u>S legs belong to a linear line complex.

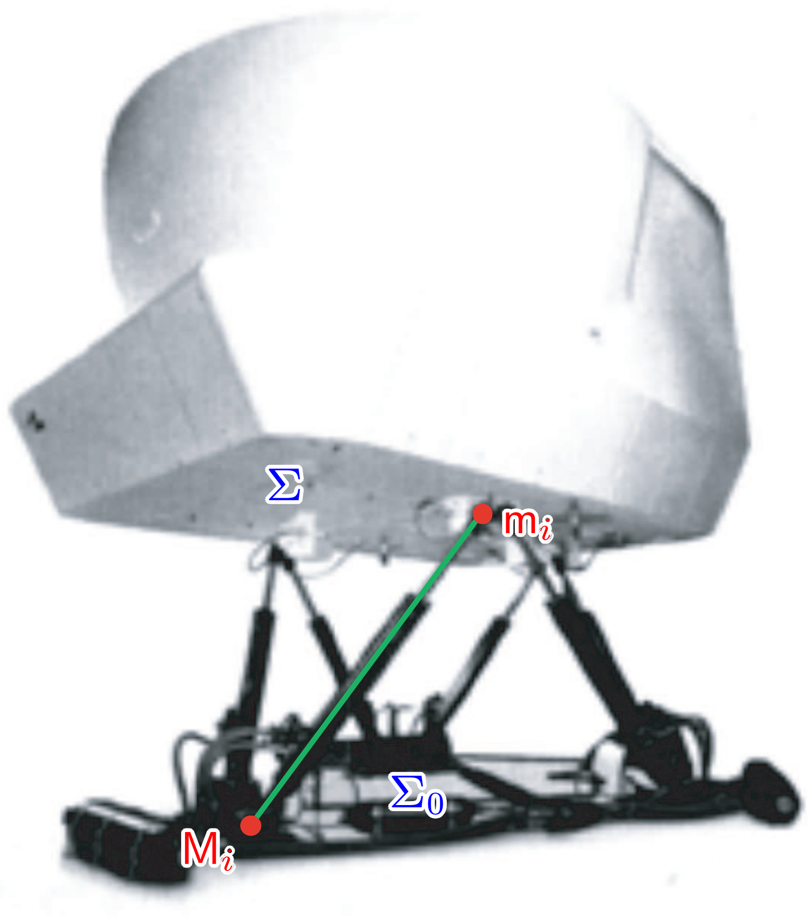


$[1c]$ Self-motions and the Borel Bricard problem

If all P-joints are locked, a SGP is in general rigid. But, in some special cases the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion.

Note that in each pose of the self-motion, the SGP has to be singular. Moreover, all self-motions of SGPs are solutions to the famous Borel Bricard problem [3–6].

Borel Bricard problem (still unsolved)
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.



$[1d]$ Architecturally singular SGPs

Manipulators which are singular in every possible configuration, are called architecturally singular [16].

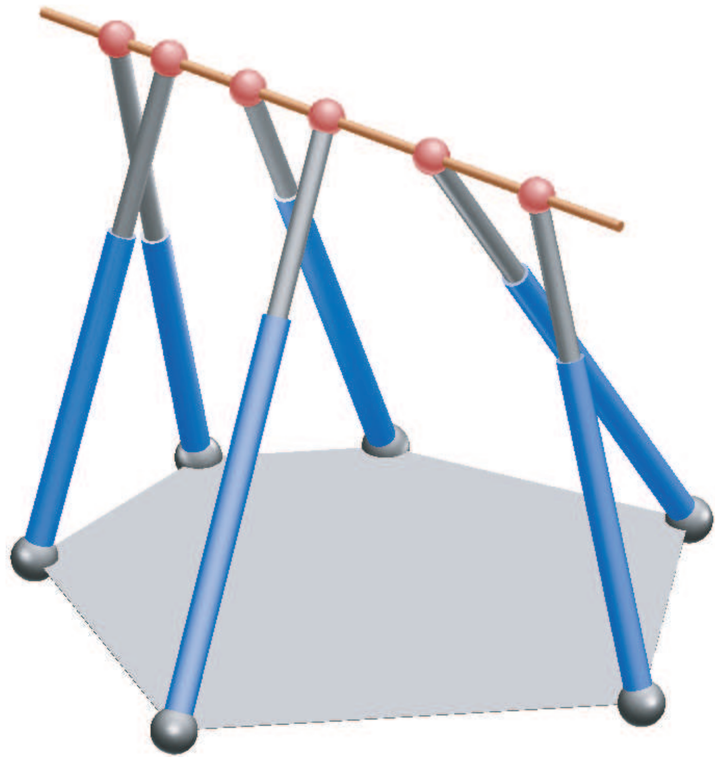
Architecturally singular SGPs are well studied:

⋆ For the planar case see [17–20],

⋆ For the non-planar case see [21–23].

It is well known, that architecturally singular SGPs possess self-motions in each pose.

Therefore we are only interested in self-motions of non-architecturally singular SGPs.



$[2]$ Parallel manipulators of TSSM type

A TSSM consists of a platform Σ, which is connected via three S<u>P</u>R legs with the base Σ_0, where the axes r_i of the R-joints are coplanar.

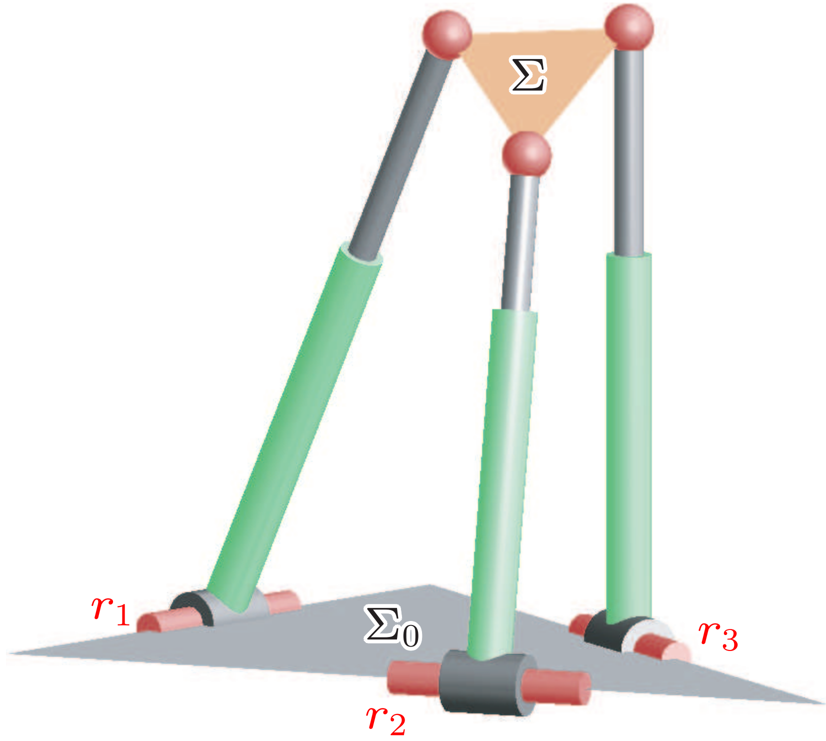
In the following we give a complete classification of non-architecturally singular TSSM designs with self-motions. For this, we distinguish four subcases of TSSMs:

$[a]$ TSSM with intersecting axes

$[b]$ TSSM with copunctal axes

$[c]$ TSSM with 2 coinciding axes

$[d]$ TSSM with 2 parallel axes

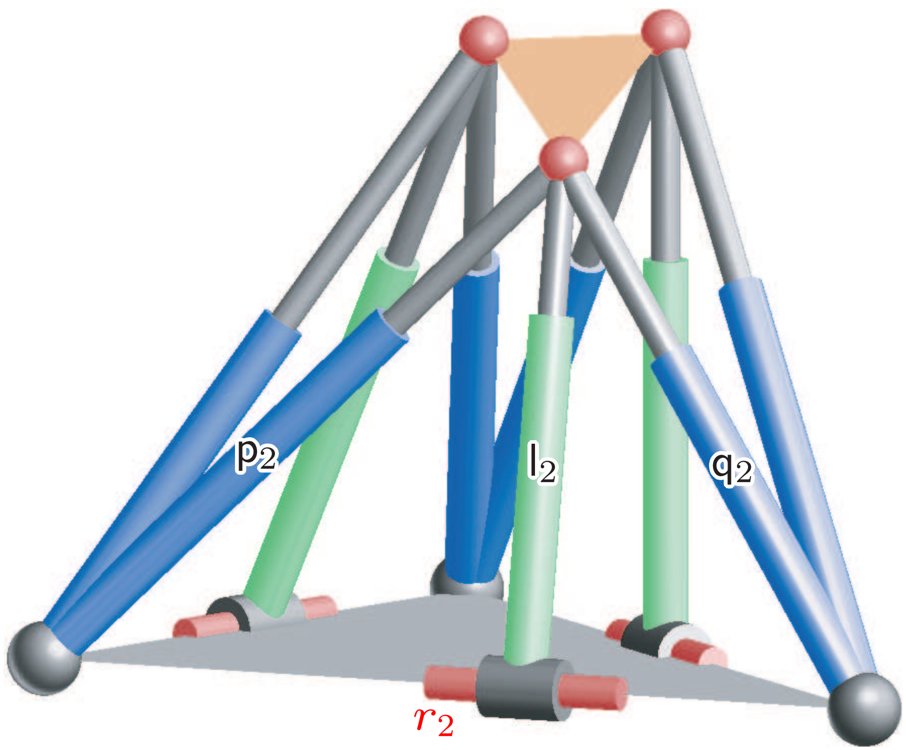


$[2a]$ TSSMs with intersecting axes

As we can replace each S$\underline{\text{P}}$R leg l_i by two S$\underline{\text{P}}$S legs p_i and q_i, the determination of TSSM self-motions can be traced back to those of planar 6-3 SGP.

Moreover, by applying Δ-transforms [B] the planar 6-3 SGP can be transformed into an octahedral manipulator.

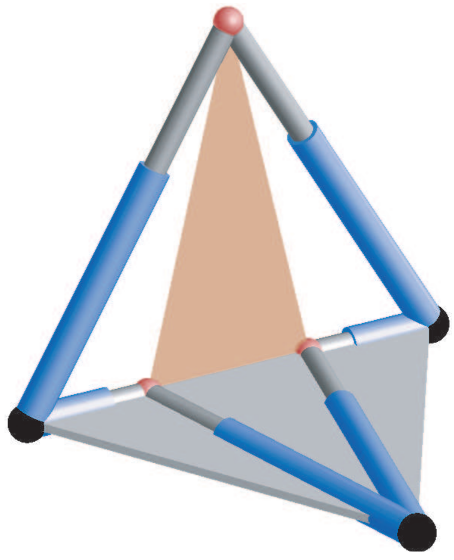
Therefore TSSMs with self-motions correspond to the three types of Bricard octahedra, if we assume that no two faces coincide permanently during the flex.



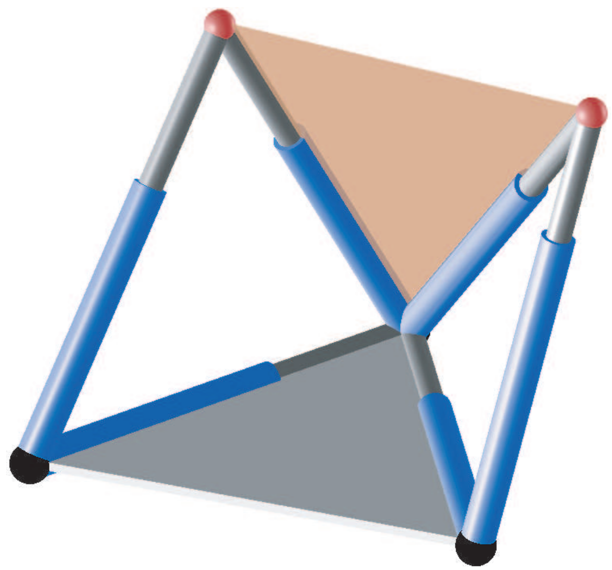
$[2a]$ TSSMs with intersecting axes

Without this assumption we get two more types of self-motions [C], which are also known as *butterfly motion* and *spherical four-bar motion* [D].

Butterfly motion



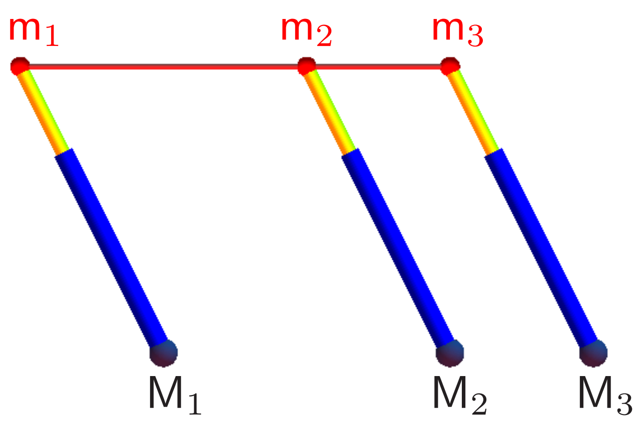
Spherical four-bar motion



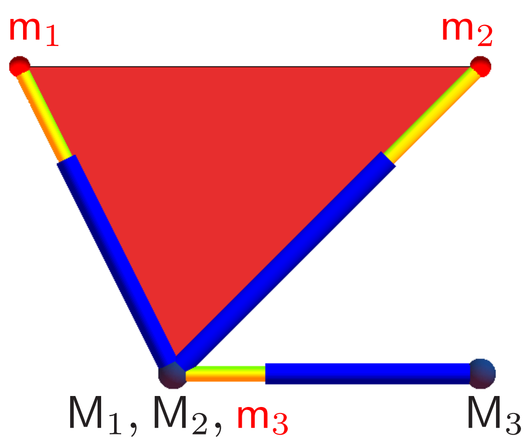
$[2b]$ TSSMs with 3 parallel axes

In this case the problem reduces to a planar one, as these manipulators possess a cylindrical singularity surface [E]. The self-motions correspond to those of the planar 3-dof R<u>P</u>R manipulator with three collinear base anchor points:

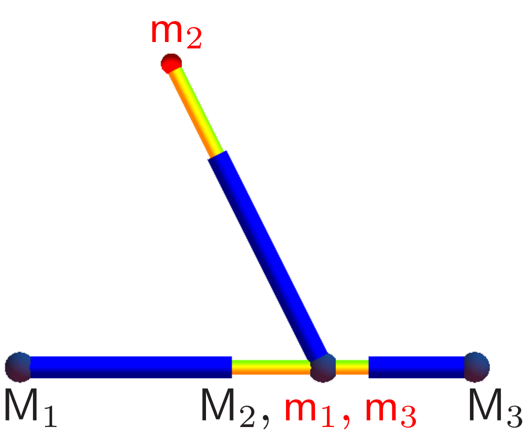
Circular translation



Pure rotation 1



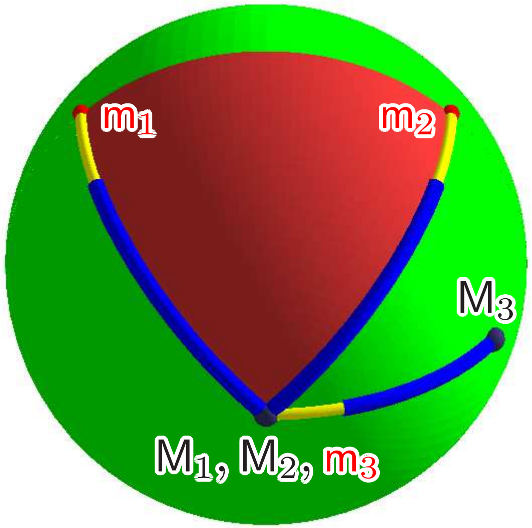
Pure rotation 2



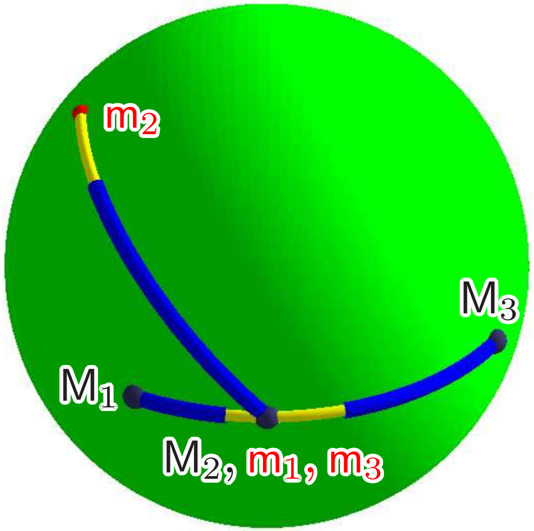
$[2b]$ TSSMs with copunctal axes

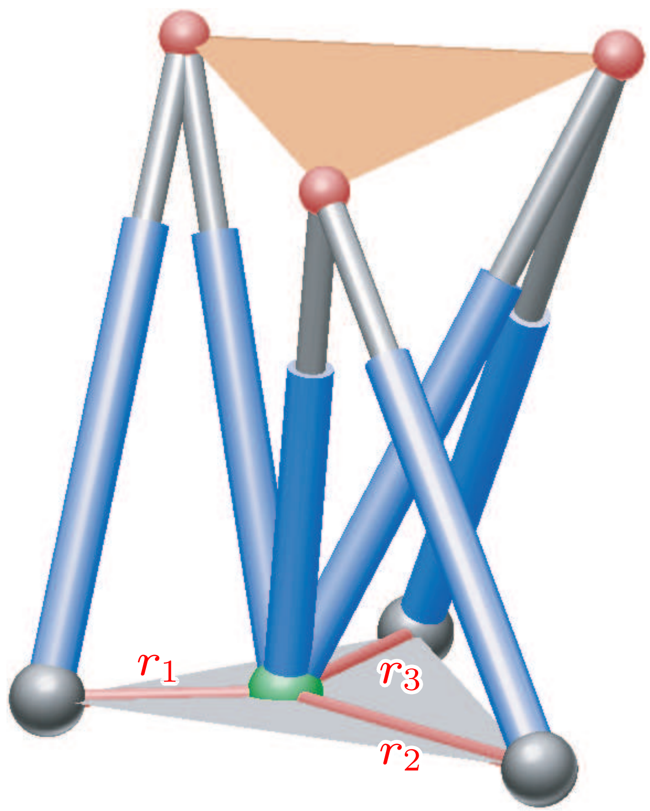
In this case the problem reduces to a spherical one. The self-motions correspond to those of the spherical 3-dof R<u>P</u>R manipulator with three collinear base anchor points:

Pure rotation 1



Pure rotation 2





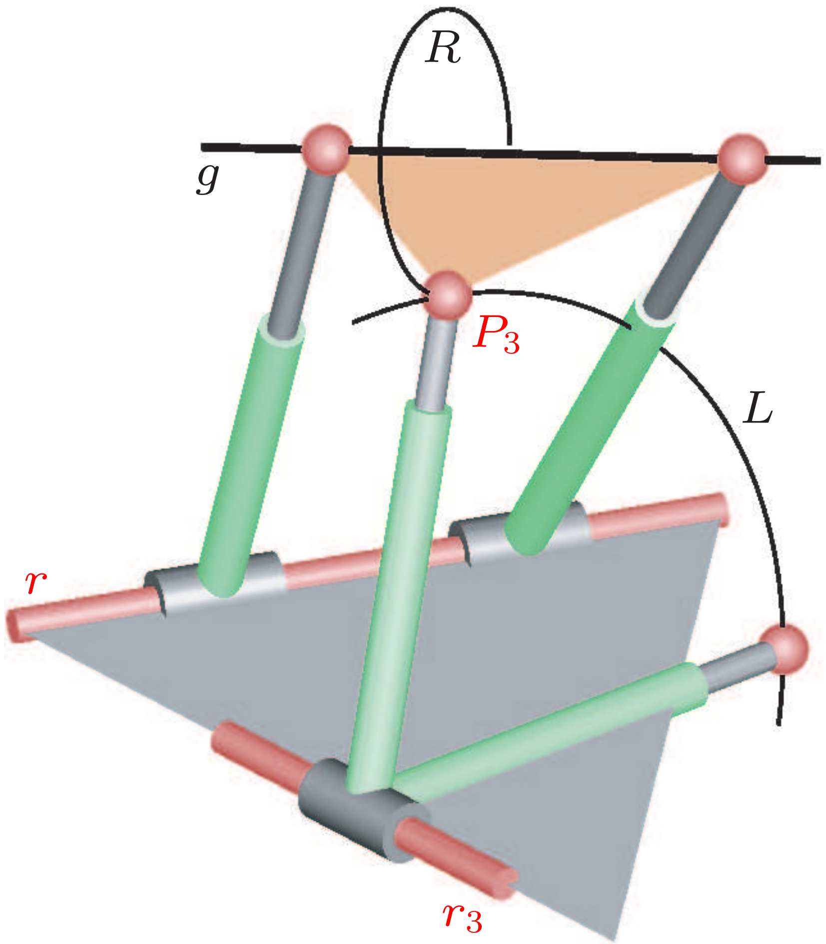
$[2c]$ TSSMs with 2 coinciding axes

If we disconnect the third leg from the platform, the point P_3 describes a so-called fourth order cyclide of revolution Φ.

This surface is generated by the rotation of the circle R about $r := r_1 = r_2$.

Now there exists a self-motion if the circle L (or a segment of it) is located on Φ.

Therefore the problem reduces to the determination of all circle sections on Φ.



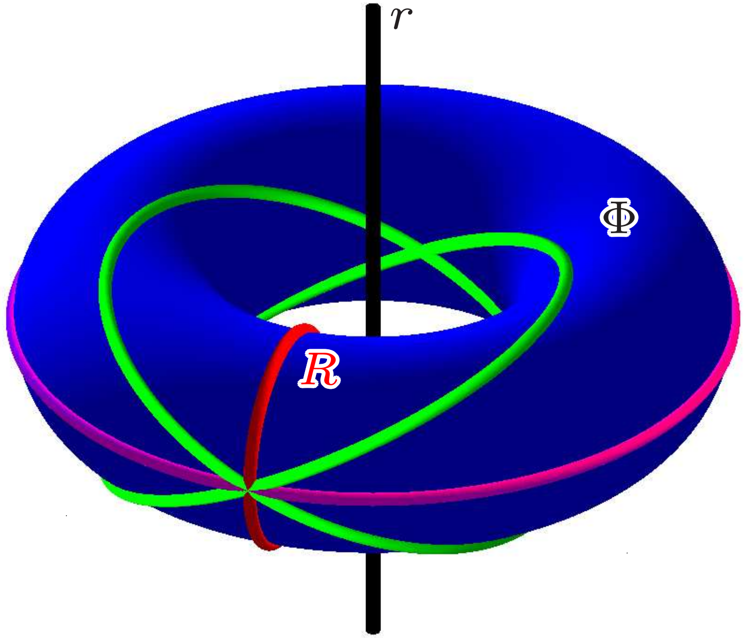
$[2c]$ TSSMs with 2 coinciding axes

Case A) If g and r are parallel, then Φ is a part of a plane. If they intersect, then Φ is a part of a sphere. In these cases the determination of circles is trivial. Self-motions correspond to planar resp. spherical four-bar motions.

Case B) g and r are skew and R lies in a meridian plane: Φ is a torus, which carries meridian circles and R-circles.

In the case of a ring torus we get two more sets generated by the *Villarceau circles*.

As non of the circle axes intersect r, we only get a rotational self-motion for $P_3 \in r$.



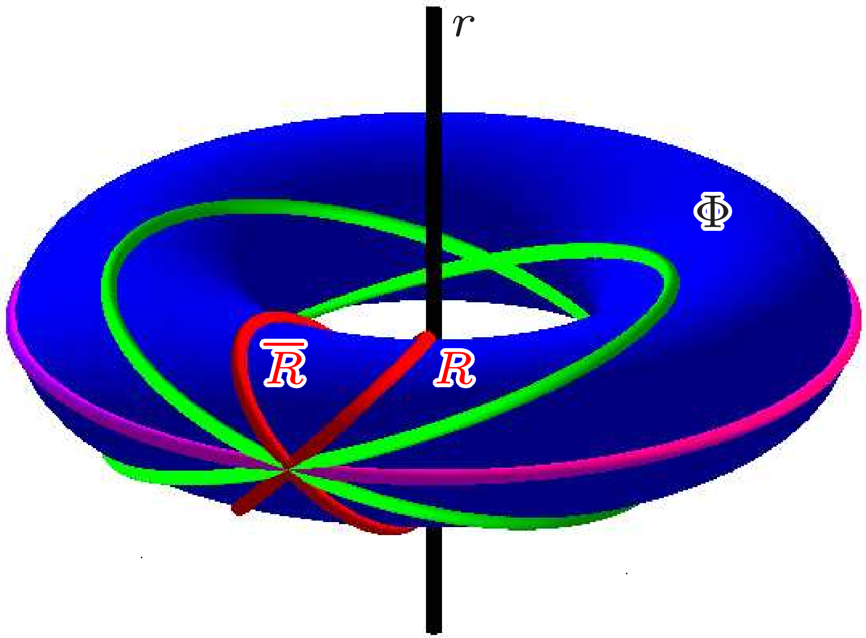
$[2c]$ TSSMs with 2 coinciding axes

Case C) g and r are skew and R is not located in a meridian plane: If we reflect R on a meridian plane we get $\overline{R}$.

Φ has at least three sets of circles (meridian circles, R-circles, $\overline{R}$-circles).

Only in the case where R and r have no point in common, there exist two further sets, which can be constructed similarly to the *Villarceau circles* [F].

As non of the circle axes intersect r, we only get a rotational self-motion for $P_3 \in r$.

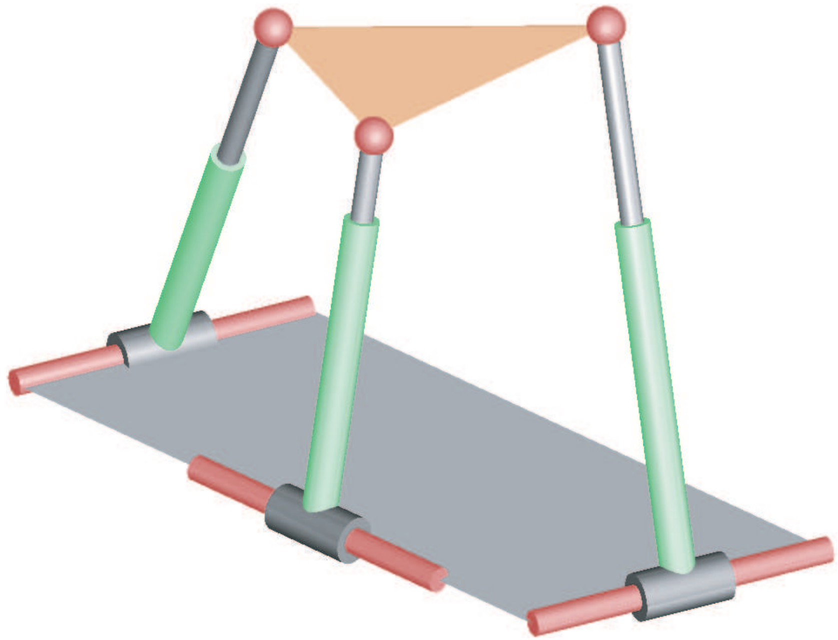


$[2d]$ TSSMs with 2 parallel axes

Similar considerations as in item $[2a]$ yield that self-motions of these TSSMs correspond with flexible octahedra where one vertex is an ideal point (assumed that no two faces coincide permanently during the flex).

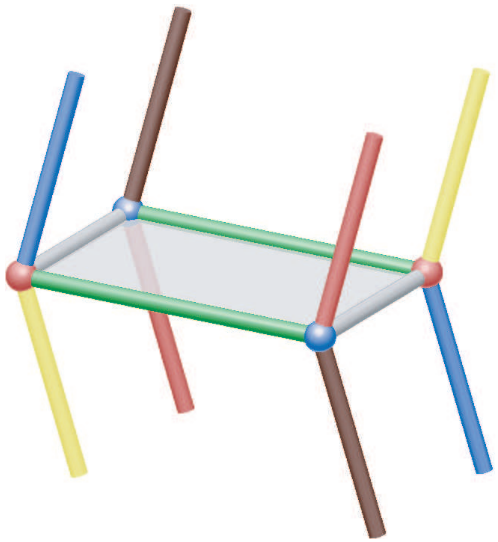
We determined all flexible octahedra with one vertex at infinity in [14]:

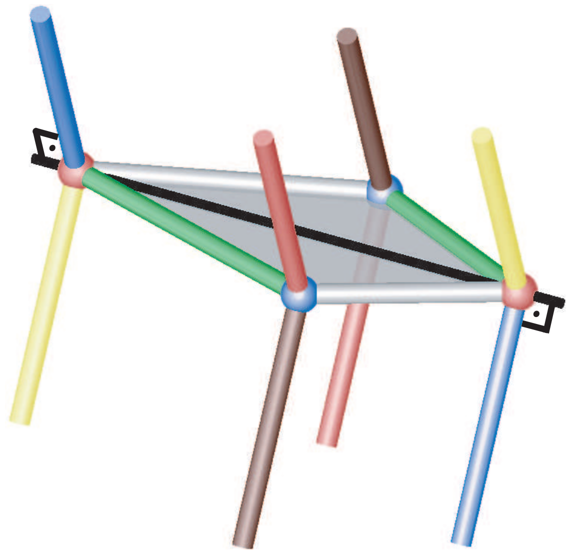
- ⋆ Flexible octahedra of type II, where one vertex located in the plane of symmetry is an ideal point.
- ⋆ Flexible octahedra of type III, with one vertex at infinity. Construction is similar to the type III Bricard octahedron [8].



$[2d]$ TSSMs with 2 parallel axes

It is not obvious that these flexible octahedra are the only ones where one vertex is an ideal point, as there could even exist flexible octahedra, which do not have flexible counterparts with six finite vertices [15].





For the special case of coinciding faces during the flex, we get again the *butterfly motion* and the *spherical four-bar motion*.

[2] Self-motions of TSSMs

Recapitulation

Self-motions of TSSMs can only be:

⋆ circular translations,

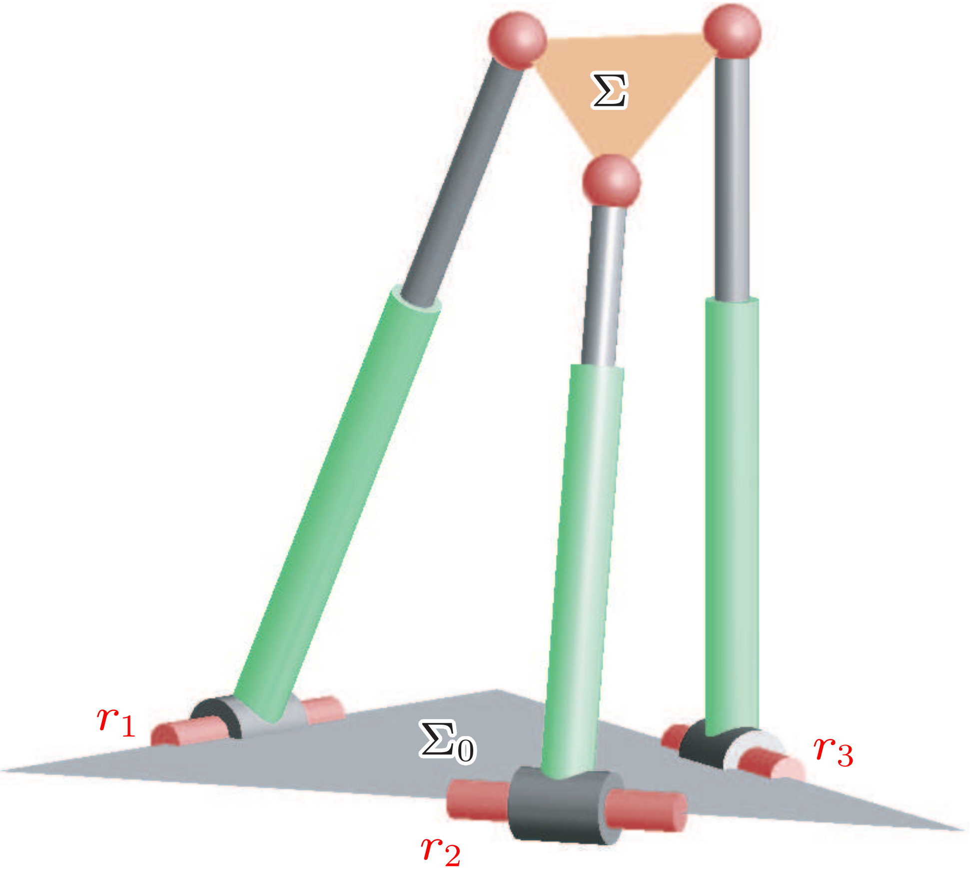
⋆ pure rotations,

⋆ planar four-bar motions,

⋆ spherical four-bar motions,

⋆ self-motions of Bricard octahedra,

⋆ self-motions of flexible octahedra with one vertex at infinity.



[3] Review on SGPs with self-motions

Beside the presented self-motions, only a few more are known:

- Husty and Zsombor-Murray [24]: SGP with Schönflies self-motion
- Zsombor-Murray et al. [25]: SGP with 2-parametric line-symmetric self-motion (see also Krames [26])
- Husty and Karger [27] proved that the list of Schönflies Borel Bricard motions given by Borel [3] is complete
- Karger and Husty [28]: Self-motions of the original SGP
- Karger [29,30] presented a method for designing planar SGPs with self-motions of the type $e_0 = 0$, where e_0 denotes an Euler parameter

$[3a]$ Redundant SGPs

According to Husty [1], the "sphere constraint" that m_i is located on a sphere with center M_i and radius R_i can be expressed by a homogeneous quadratic equation Λ_i in the Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$.

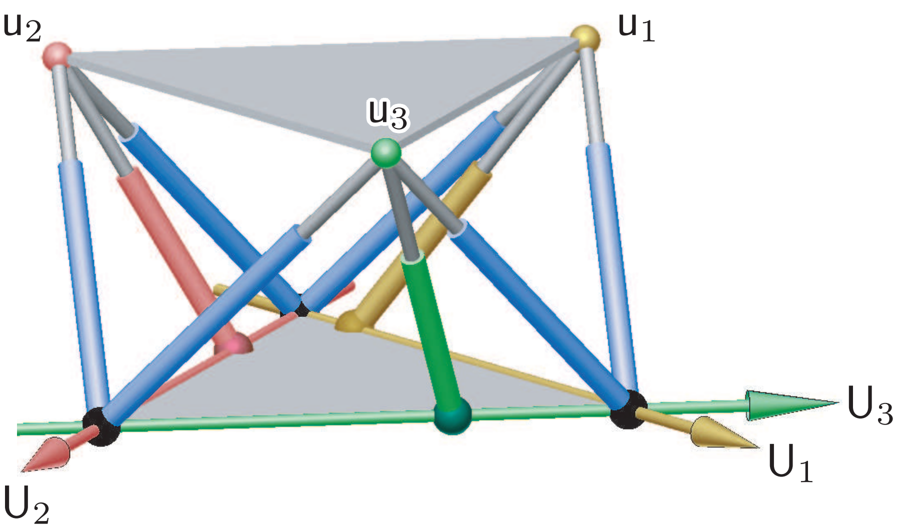
Therefore the direct kinematic problem corresponds to the solution of the system $\Lambda_1, \ldots, \Lambda_6, \Psi$ where $\Psi : \sum_{i=0}^{3} e_i f_i = 0$ is the equation of the Study quadric.

If a planar SGP is not architecturally singular, then at least a one-parametric set of legs $\lambda_1 \Lambda_1 + \ldots + \lambda_6 \Lambda_6$ can be added without changing the direct kinematics [G,H].

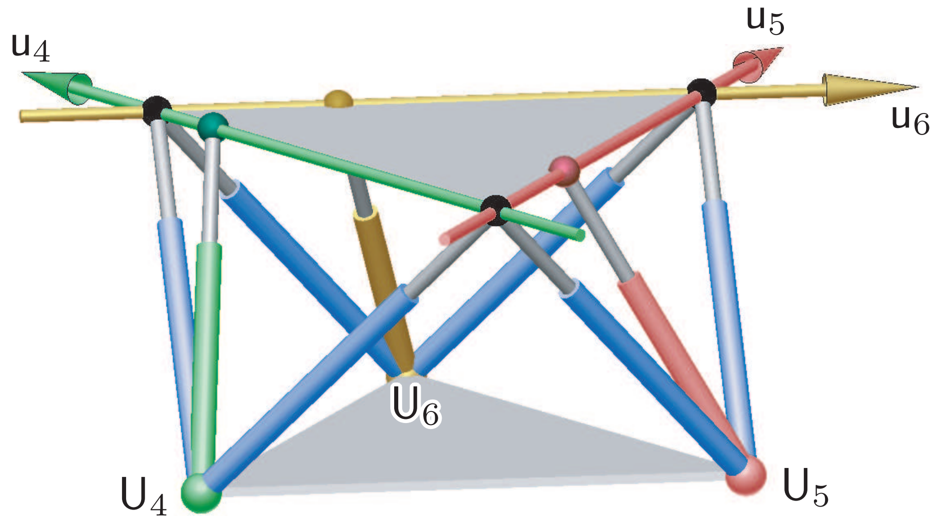
As the solvability condition of the underlying linear system of equations (Eq. (30) of [H]) is equivalent with the criterion given in Eq. (12) of [I], also the singularity surface of the SGP does not change by adding legs of this one-parametric set.

$[3a]$ Redundant SGPs

Moreover, it was shown [G,H] that in general the base anchor points M_i as well as the corresponding platform anchor points m_i are located on planar cubic curves C and c, which can also split up.



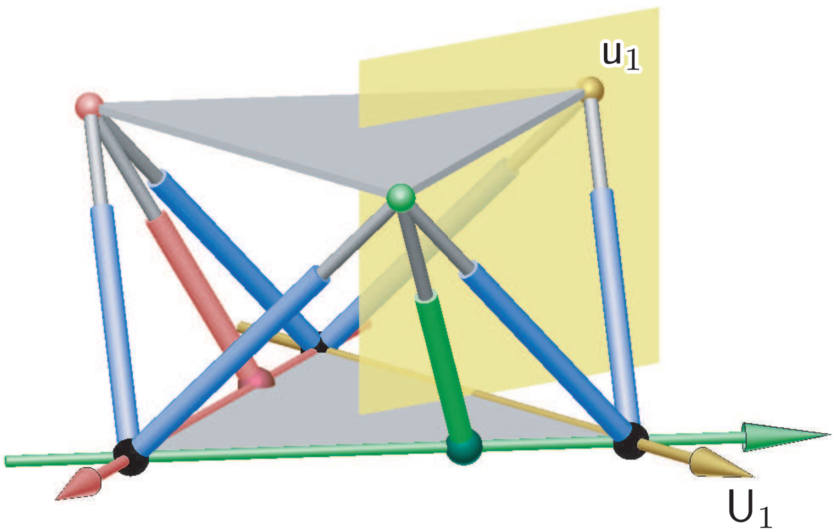
Cubic C of the octahedral SGP



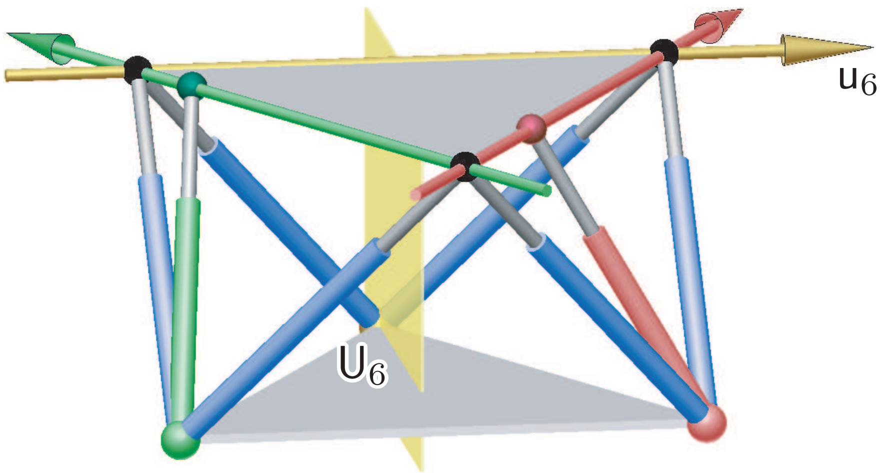
Cubic c of the octahedral SGP

$[3b]$ Darboux and Mannheim motion

The Darboux constraint that u_i moves in a plane $\in \Sigma_0$ orthogonal to the direction of the ideal point U_i is a homogeneous quadratic equation Ω_i in the Study parameters ($i = 1, 2, 3$).



The Mannheim constraint that a plane of Σ orthogonal to u_j slides through the point $\mathsf{U}_j \in \Sigma_0$ is a homogeneous quadratic equation Π_j in the Study parameters ($j = 4, 5, 6$).



$[3c]$ Self-motions implied by Bricard octahedra I

As the points U_i and u_i are corresponding points of the cubics C and c we get:

$$\Omega_i = \sum_{k=1}^{6} \lambda_{i,k}\Lambda_k \quad \text{and} \quad \Pi_j = \sum_{k=1}^{6} \lambda_{j,k}\Lambda_k \quad \text{for} \quad i = 1,2,3 \quad \text{and} \quad j = 4,5,6.$$

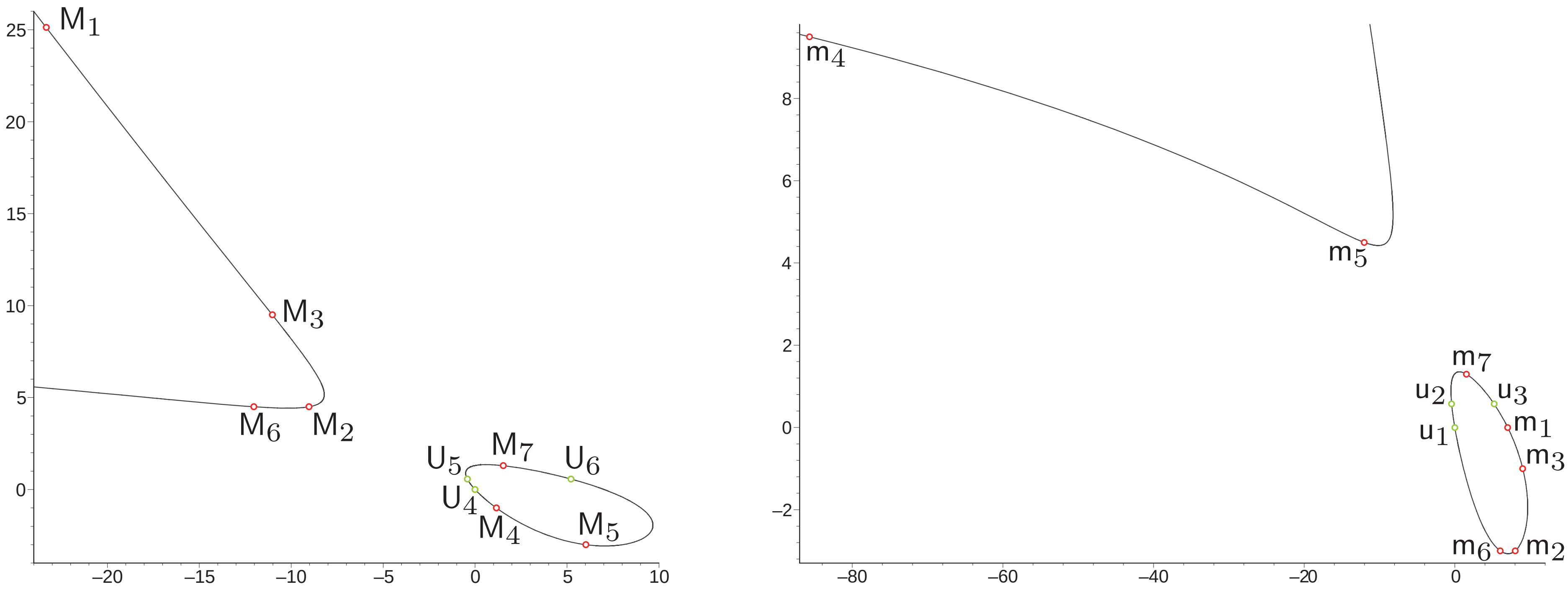
It can easily be seen [31], that the system $\Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5, \Pi_6$ is redundant $\Longrightarrow$ manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is architecturally singular.

Moreover, if the underlying SGP is a Bricard octahedron of type I, then $\mathsf{u}_1, \ldots, \mathsf{U}_6$ has even a two-parametric self-motion (type II DM self-motion [31,32]).

By adding an arbitrary leg Λ to $\Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5$ we get an one-parametric self-motion. Further legs are determined by:

$$\lambda\Lambda + \sum_{i=1}^{3} \nu_i\Omega_i + \sum_{j=4}^{5} \mu_j\Pi_j.$$

$[3c]$ **Example**



Remark

Note that all self-motions implied by Bricard octahedra of type I are line-symmetric motions. Moreover these self-motions can even be parametrized [31].

[4] References

For [1-32] see the extended abstract. The remaining references [A-I] are as follows:

[A] Merlet, J-P.: Singular configurations of parallel manipulators and Grassmann geometry, International Journal of Robotics Research **8**(5) 45–56 (1989)

[B] Borras, J., Thomas, F., Torras, C.: On Δ-Transforms, IEEE Transactions on Robotics **25**(6) 1225–1236 (2009)

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[F] Krames, J.: Über Drehzykliden vierter Ordnung, Monatshefte für Mathematik **80** 45–60 (1975)

[G] Husty, M., Mielczarek, S., Hiller, M.: A redundant spatial Stewart-Gough platform with a maximal forward kinematics solution set, Advances in Robot Kinematics: Theory and Applications (J. Lenarcic, F. Thomas eds.), 147–154, Springer (2002)

[H] Mielczarek, S., Husty, M.L., Hiller, M.: Designing a redundant Stewart-Gough platform with a maximal forward kinematics solution set, Proceedings of the International Symposium of Multibody Simulation and Mechatronics, Mexico City, Mexico (2002)

[I] Borras, J., Thomas, F., Torras, C.: Singularity-invariant leg rearrangements in doubly-planar Stewart-Gough platforms, Proceedings of Robotics Science and Systems, Zaragoza, Spain (2010)