Generalizing the Control Number for 6-dof UCU Hexapods with classic or eccentric U-joints

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***Abstract*— In this paper, we present a novel index, called the Generalized Control Number ($GCTN$), which evaluates the closeness of a given non-singular configuration of a hexapod with UCU-legs to the next singularity. The $GCTN$ is invariant with respect to similarity transformations (choice of units) and under Euclidean motions of the reference frame (choice of fixed frame). Moreover, this index indicates the closeness to all types of singularities (end-effector and leg singularities) simultaneously and it has a clear geometric/kinematic meaning.**

I. INTRODUCTION

Given is a parallel manipulator with six degrees of freedom (dofs), where the fixed base is denoted by Σ_0 and the moving platform by Σ, on which the end-effector EE is installed. Moreover, Σ_0 is connected with Σ by six UCU-legs, where U denotes an universal joint and C a cylindrical one.

It is well known, that a C-joint has two dofs, where one is a translation along the cylinder axis c and the other a rotation around c. For the hexapods under consideration, only the translation along c can be controlled actively; the rotational component is passive. Therefore, the C-joint can be replaced by a composition of an active prismatic joint (P-joint) along c and a passive rotational joint (R-joint) with rotary axis c.

A U-joint also has two dofs, as it can also be seen as a serial $2R$-chain, with orthogonal axes u_1 and u_2. If these axes intersect each other, we have the classic U-joint and if this is not the case, we get a so-called *eccentric* one (cf. [3], [4]). For the UCU-legs, both types of U-joints are allowed, which are in all cases passive joints of the manipulator.

Remark 1. *According to [3], [4] the advantages of eccentric U-joints are, that they have a significantly larger pivoting range, which results in an extension of the manipulators workspace. At the same time, the joints can be produced cheaper and they can be designed more compact and stiffer, which additionally improves the accuracy.* $\diamond$

Moreover, we assume that the connection of each U-joint with a C-joint fulfills the following two design constraints:

- the lines u_2, c and n are copunctal,
- and c intersects u_2 orthogonally,

where n denotes the common normal of u_1 and u_2 and where u_2 denotes the axis of the U-joint, which is linked with the C-joint (cf. Fig. 1, left). This assumptions keep the kinematic structure of the UCU-legs simple enough for practical application (cf. [4]).

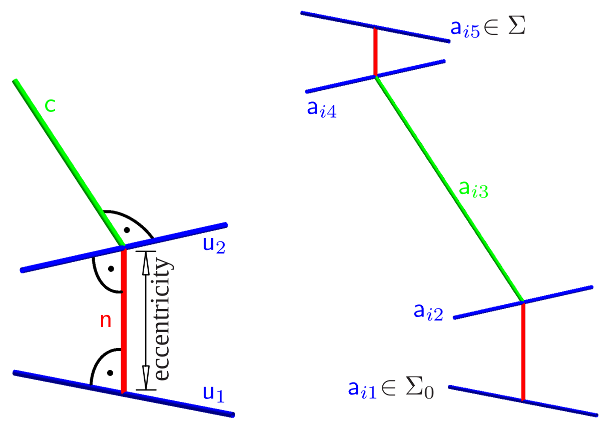

Fig. 1. Left: Connection of an eccentric U-joint and a C-joint. We get a classic U-joint, if the eccentricity equals zero. Right: Schematic sketch of the serial RRPRRR-chain, which corresponds with the i^{th} UCU-leg.

Summed up, each leg connecting Σ_0 with Σ can also be seen as a serial RRPRRR-chain, where the P-joint is active and the five R-joints are passive. We denote the j^{th} rotation axis of the i^{th} leg by a_{ij} for $i = 1, \ldots, 6$ and $j = 1, \ldots, 5$ (cf. Fig. 1, right).

Based on this notation, we first study the instantaneous kinematic of the hexapod with UCU-legs in Section II, where the different types of singularities of this manipulator are listed as well. In Section III, we make preliminary considerations on an index, which evaluates the closeness of a given non-singular configuration to the next singularity. Moreover, we discuss already existing performance indices from this point of view and repeat the so-called Control Number for Stewart Gough manipulators in more detail. Based on this, we generalize the Control Number for the hexapods under consideration in Section IV. We close the paper by demonstrating the validity of this index on the basis of a concrete example, which is given in Section V.

II. INSTANTANEOUS KINEMATICS

We use the dual vector calculus for the representation of screws and lines (cf. page 154 of [17]). Therefore, the rotation axis a_{ij} is given by

$$\underline{\mathbf{a}}_{ij} = \mathbf{a}_{ij} + \varepsilon\widehat{\mathbf{a}}_{ij}, \tag{1}$$

where $\mathbf{a}_{ij}$ is the unit vector (column vector) along the rotation axis with respect to the fixed frame. $\widehat{\mathbf{a}}_{ij}$ is the so-

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called moment vector, which is given by $\mathbf{x}_{ij} \times \mathbf{a}_{ij}$, where $\mathbf{x}_{ij}$ is the coordinate vector (column vector) of an arbitrary point $\mathsf{X}_{ij} \in \mathsf{a}_{ij}$ with respect to the fixed frame. Further it should be noted, that ε is the dual unit, which has the property $\varepsilon^2 = 0$.

The screw for the prismatic joint of the i^{th} leg is given by

$$\underline{\mathbf{t}}_i = \mathbf{o} + \varepsilon\widehat{\mathbf{t}}_i, \tag{2}$$

where $\mathbf{o}$ denotes the zero vector and $\widehat{\mathbf{t}}_i$ the unit vector in direction of the translation with respect to the fixed frame. Therefore, in our case $\widehat{\mathbf{t}}_i$ equals $\mathbf{a}_{i3}$.

A. Jacobian matrix $\mathbf{J}_i$ of the i^{th} leg

As every leg can be seen as a serial RRPRRR-robot, the 6×6 Jacobian matrix $\mathbf{J}_i$ of the i^{th} leg can be written as (cf. [7]):

$$\mathbf{J}_i = \begin{pmatrix} \mathbf{a}_{i1} & \mathbf{a}_{i2} & \mathbf{a}_{i3} & \mathbf{a}_{i4} & \mathbf{a}_{i5} & \mathbf{o} \\ \widehat{\mathbf{a}}_{i1} & \widehat{\mathbf{a}}_{i2} & \widehat{\mathbf{a}}_{i3} & \widehat{\mathbf{a}}_{i4} & \widehat{\mathbf{a}}_{i5} & \widehat{\mathbf{t}}_i \end{pmatrix}. \tag{3}$$

Therefore, the instantaneous screw $\underline{\mathbf{q}} = \mathbf{q} + \varepsilon\widehat{\mathbf{q}}$ of Σ with respect to Σ_0 can be computed as

$$\begin{pmatrix} \mathbf{q} \\ \widehat{\mathbf{q}} \end{pmatrix} = \mathbf{J}_i \begin{pmatrix} \omega_{i1} \\ \vdots \\ \omega_{i5} \\ \tau_i \end{pmatrix}, \tag{4}$$

where ω_{ij} denotes the angular velocity of the j^{th} R-joint and τ_i the translatory velocity of the P-joint of the i^{th} leg.

The spear coordinates $(\mathbf{p}^T, \widehat{\mathbf{p}}^T)$ of the axis p (= normalized Plücker coordinates of p, cf. page 155 of [17]) of the screw $\underline{\mathbf{q}}$ can be computed according to

$$\mathbf{p} = \frac{1}{\omega}\,\mathbf{q}, \quad \widehat{\mathbf{p}} = \frac{1}{\omega}\left(\widehat{\mathbf{q}} - \frac{\omega\widehat{\omega}}{\omega^2}\,\mathbf{q}\right), \tag{5}$$

for $\omega = \|\mathbf{q}\| \neq 0$ and $\omega\widehat{\omega} = \mathbf{q}\widehat{\mathbf{q}}$. The screw parameter h is given by $h := \widehat{\omega}/\omega$, where $\widehat{\omega}$ is the translatory velocity and ω the angular velocity of the screw $\underline{\mathbf{q}}$.

If $\omega = \|\mathbf{q}\| = 0$ holds, then $\underline{\mathbf{q}}$ is an instantaneous translation along the direction $\widehat{\mathbf{q}}$. In this case, the axis is the ideal line of any plane orthogonal to $\widehat{\mathbf{q}}$.

B. Jacobian matrix $\mathbf{J}$ of the EE

If we assume that $rk(\mathbf{J}_i) = 6$ for $i = 1, \ldots, 6$, then Eq. (4) can be rewritten as

$$\mathbf{J}_i^{-1} \begin{pmatrix} \mathbf{q} \\ \widehat{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \omega_{i1} \\ \vdots \\ \omega_{i5} \\ \tau_i \end{pmatrix}. \tag{6}$$

By denoting the sixth row of $\mathbf{J}_i^{-1}$ by $(\widehat{\mathbf{j}}_i, \mathbf{j}_i)$, the 6×6 Jacobian matrix $\mathbf{J}$ of the platform can be written as

$$\mathbf{J} = \begin{pmatrix} \widehat{\mathbf{j}}_1 & \mathbf{j}_1 \\ \vdots & \vdots \\ \widehat{\mathbf{j}}_6 & \mathbf{j}_6 \end{pmatrix}. \tag{7}$$

Moreover, it should be noted that the instantaneous screw $\underline{\mathbf{j}}_i := \mathbf{j}_i^T + \varepsilon\widehat{\mathbf{j}}_i^T$ equals an instantaneous rotation around the carrier line of the i^{th} P-joint. Therefore, $(\mathbf{j}_i, \widehat{\mathbf{j}}_i)$ are the spear coordinates $(\mathbf{a}_{i3}^T, \widehat{\mathbf{a}}_{i3}^T)$ of the axis a_{i3} (cf. proof of the later given Theorem 1).

Note, that $\mathbf{J}$ transforms the instantaneous screw of the platform into the translatory velocity of the active joints, i.e.

$$\mathbf{J}\begin{pmatrix} \mathbf{q} \\ \widehat{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_6 \end{pmatrix}. \tag{8}$$

C. Types of singularities

In the following, we distinguish different types of singularities:

1) $rk(\mathbf{J}_i) < 6$: This is a so-called leg singularity. Geometrically, this means that the five rotary axes and the axis of the translation (ideal line) belong to a so-called *linear line complex* (cf. Section 3 of [17]). In this case, there exist angular velocities ω_{ij} and a translatory velocity τ_i that

$$\tau_i\underline{\mathbf{t}}_i + \sum_{j=1}^{5}\omega_{ij}\underline{\mathbf{a}}_{ij} = \underline{\mathbf{o}} \tag{9}$$

holds, where $\underline{\mathbf{o}} = \mathbf{o} + \varepsilon\mathbf{o}$ denotes the zero screw. We distinguish two cases:

   a) $\tau_i \neq 0$: In this case, the translatory velocity of the i^{th} active joint cannot be transmitted onto the EE, as the velocity ratio

   $$(\tau_1 : \ldots : \tau_i : \ldots : \tau_6) = (0 : \ldots : 1 : \ldots : 0) \tag{10}$$

   causes an instantaneous standstill of Σ; i.e. $\underline{\mathbf{q}} = \underline{\mathbf{o}}$.

   b) $\tau_i = 0$: Now, there is an infinitesimal redundant mobility of the leg itself (but not of Σ). In the worst case, this can result in a self-motion of the leg.

   Finally, it should be mentioned, that in a leg singularity the leg loses $6 - rk(\mathbf{J}_i)$ dofs. If an infinitesimal screw is applied to the platform, which belongs to the set of lost dofs, then this can yield a breaking of the leg.

2) $rk(\mathbf{J}) < 6$: This is a so-called EE singularity. Due to the observation of Subsection II-B, this singularity can also be interpreted by means of line geometry as follows: The hexapod is in an EE singularity, if and only if, the carrier lines of the prismatic legs belong to a linear line complex. In this case, there exists at least a screw $\underline{\mathbf{q}} \neq \underline{\mathbf{o}}$ that

$$\mathbf{J}\begin{pmatrix} \mathbf{q} \\ \widehat{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \mathbf{o} \\ \mathbf{o} \end{pmatrix} \tag{11}$$

holds. Therefore, in an EE singularity, the platform is infinitesimal movable while all active joints are locked. Finally it should be noted, that in the worst case, this singularity can result in a self-motion of Σ.

Remark 2. *This singularity study also shows, that the hexapods under consideration only have line-based singularities, even though the last three joints of each leg are not equivalent with a spherical joint (S-joint), if an eccentric U-joint is used at Σ. Therefore, these are more general parallel manipulators with line-based singularities, than those characterized in Section 4 of [2].* $\diamond$

III. PERFORMANCE INDEX

In future applications, it is planned that the hexapod's motion is controlled directly by ordinary skilled workers and not by highly-qualified academics, e.g. wheel loaders will be coupled by hexapods with different EEs (dredger bucket, stacker forks, snowplough, gripper, ...).[1] Therefore, there is an interest in an index, which gives the operator a feedback about the closeness of a given non-singular hexapod-configuration to the next singular one.

As it is well known, that there does not exist a distance metric in the pure mathematical sense, if rotational and translatory dofs are involved (which is the case for a 6-dof hexapod), we are looking for a performance index PI, which assigns to each configuration $\mathcal{C}$ a scalar $PI(\mathcal{C}) \in \mathbb{R}$ obeying the following six properties:

1) $PI(\mathcal{C}) \geq 0$ for all $\mathcal{C}$ of the configuration space,
2) $PI(\mathcal{C}) = 0$ if and only if $\mathcal{C}$ is singular,
3) $PI(\mathcal{C})$ is invariant under Euclidean motions of the reference frame,
4) $PI(\mathcal{C})$ is invariant under similarities,
5) $PI(\mathcal{C})$ has a geometric/kinematic meaning,
6) $PI(\mathcal{C})$ is computable in real-time.

A further challenge for the definition of the requested index is, that it has to evaluate the closeness to different types of singularities simultaneously, as separated computations of the closeness to EE singularities and leg singularities (for each leg) go at the expense of the computation time (cf. demand 6), and one is confronted with the problem of combining the obtained values to a single meaningful closeness index (cf. demand 5). But exactly this clear geometric/kinematic meaning is of importance for identifying a critical value, which indicates that a given configuration is too close to a singularity for guaranteeing a save performance of the hexapod.

As the set of singular poses of a manipulator is solely determined by its geometry, a performance index, which makes demands to evaluate the closeness to the next singularity, should only depend on geometric/kinematic properties of the inspected non-singular pose. Therefore, such a performance index must not depend on the EE. As a consequence, all known condition number indices (either based on the characteristic point [21], operation ellipsoid [14], [15] or velocity of three EE points [8]) as well as the local singularity transmission index [11], which depends on the choice of the application point, are out of question.

Moreover, the requested index must not depend on non-kinematic parameters as mass or stiffness, which exclude also the indices presented in [1], [6], [18]. In the following we discuss the remaining EE independent performance indices, which are known to the author, in more detail:

[1]Cf. research project *"MOBIMA – Arbeitsausrüstungen mit parallelkinematischen Strukturen für mobile Arbeitsmaschinen"* funded by the German Ministry of Education and Research. For more details please see: `http://tu-dresden.de/die_tu_dresden/fakultaeten/fakultaet_maschinenwesen/iwm/forschung/2012_mobima`

A. Manipulability [20]

A drawback of the manipulability $M(\mathcal{C})$ is, that it is not invariant under similarity transformations and therefore it depends on the choice of units (cf. [12]). To overcome this problem, some authors use the following relation as index:

$$M^\star := \frac{M(\mathcal{C})}{M(\mathcal{C}_{max})}, \tag{12}$$

with $\mathcal{C}_{max}$ denoting the configuration of the hexapod, where the manipulability is maximal ($\Rightarrow M^\star \in [0,1]$). But the computation of $\mathcal{C}_{max}$ is a highly non-linear task and was only done for some special manipulators of Stewart Gough (SG) type[2] (cf. [9], [10]). Moreover, only in some special cases $M(\mathcal{C}_{max})$ can be interpreted geometrically as the volume, spanned by the framework (cf. [9]).

B. Best fitting linear line complex [16], [19]

As our studied manipulator only has line-based singularities (cf. Remark 2), also this index has to be taken into consideration, which is again not invariant under similarities. But one can solve this problem as for the case of the manipulability.

The much bigger problem is, that this index does not consider singular linear line complexes, where the axis is an ideal line (cf. page 166 of [17]). In order to close this gap, the authors of [16] proposed the computation of a second index. But it is not clear how these two indices should be combined to a single geometric/kinematic meaningful value, evaluating the closeness to the next linear line complex.

Beside the already mentioned drawbacks of the manipulability and the method of the best fitting linear line complex, these two indices cannot master the challenge formulated in the paragraph below the six demands.

C. Control Number [13], [14], [15]

As pointed out by the author in [13], [14], [15], the Control Number CTN fulfills all six demands. Therefore, this index is best suited for measuring the closeness to the next singularity in the author's mind, but until now the CTN is only defined for SG manipulators. As we want to generalize the CTN for the hexapods under consideration within the next section, we repeat its basic idea and definition in the following two paragraphs:

As in each pose, the SPS-leg allows a rotational self-motion around the line spanned by the centers of the S-joints, we are only interested in an index, which evaluates the closeness to EE singularities. Note that EE singularities of SG manipulators have the same geometric interpretation as the one given in item 2 of Subsection II-C. Therefore, SG manipulators are also infinitesimal movable in EE singularities, which means that there exists an infinitesimal motion of Σ while all actuators are locked. As a consequence, the velocity of Σ can be arbitrarily large (even infinity), and therefore the posture is uncontrollable. In practice,

[2]These are hexapods with six SPS-legs, where both S-joints are passive and the P-joint is active.

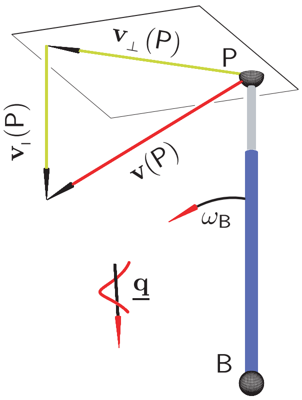

Fig. 2. Definition of the angular velocity ω_B of the spherical base joint (with center B): $\mathbf{v}(\mathsf{P})$ denotes the velocity of the platform point P (= center of S-joint) with respect to the instantaneous screw $\underline{\mathbf{q}}$. Moreover, $\mathbf{v}_|(\mathsf{P})$ (resp. $\mathbf{v}_\perp(\mathsf{P})$) is the component of $\mathbf{v}(\mathsf{P})$ along (resp. orthogonal to) the carrier line of the P-joint. Then, ω_B is defined as the length of $\mathbf{v}_\perp(\mathsf{P})$ divided through the leg length. Note, that the definition of ω_P can be done analogously with respect to the inverse motion $-\underline{\mathbf{q}}$.

configurations must be avoided, where minor (or even zero) variations of the leg lengths have uncontrollable large effects on the instantaneous displacement of Σ. The question is, which measurable parameter of the SG manipulator indicates the circumstance of uncontrollability in a natural way and has a geometric/kinematic meaning for the manipulator.

The answer to this question are the angular velocities of the S-joints (cf. Fig. 2). We computed the maximum λ_{max} and the minimum λ_{min} of the sum of the squared angular velocities of the passive joints under the normalizing condition that the sum of the squared translatory velocities of the active joints equals 1. Then, the CTN is defined as:

$$CTN := \sqrt{\frac{\lambda_{min}}{\lambda_{max}}} \in [0,1]. \tag{13}$$

Remark 3. *For a more detailed review of the indices discussed in Subsection III-A, III-B and III-C for parallel manipulators of SG type, please see [15]. Moreover, these indices are also compared within Subsection 6.4 of [15].* $\diamond$

IV. Generalized Control Number

The base for the definition of a generalized version of the CTN is the following theorem:

Theorem 1. *A leg singularity of type (a) with $rk(\mathbf{J}_i) = 5$ cannot exist.*

Proof: As all rotary axes $\mathsf{a}_{i1}, \ldots, \mathsf{a}_{i5}$ intersect (or are even identical with) the carrier line of the i^{th} P-joint (= line a_{i3}), the linear line complex spanned by the axes $\mathsf{a}_{i1}, \ldots, \mathsf{a}_{i5}$ equals the path normal complex of the instantaneous screw $\underline{\mathbf{j}}_i$ (cf. page 164 of [17]). This linear line complex is a singular one and uniquely determined, if $\underline{\mathbf{a}}_{i1}, \ldots, \underline{\mathbf{a}}_{i5}$ are linearly independent ($\Rightarrow rk(\mathbf{J}_i) = 5$).

Therefore, a leg singularity of type (a) with $rk(\mathbf{J}_i) = 5$ exists, if and only if, the axis t of $\underline{\mathbf{t}}_i$ intersects the carrier line of the P-joint. But this can never happen, as t is the ideal line of the plane orthogonal to a_{i3}. $\square$

A consequence of this theorem is, that a leg singularity of type (a) can only occur if $rk(\mathbf{J}_i) < 5$ holds, but this implies the existence of a leg singularity of type (b). Therefore, our performance index only has to indicate EE singularities and leg singularities of type (b). The common characteristic property of these two singularities is, that there exists an infinitesimal mobility while all active joints are fixed. Therefore, the so-called Generalized Control Number $GCTN$ can be used as index. The definition and computation of the $GCTN$ is given as follows:

We calculate the extreme values of the objective function (sum of the squared angular velocities of the passive joints)

$$\zeta: \sum_{i=1}^{6}\sum_{j=1}^{5}\omega_{ij}^2 \tag{14}$$

under the normalizing condition (sum of the squared translatory velocities of the active joints)

$$\nu: \sum_{i=1}^{6}\tau_i^2 - 1 = 0. \tag{15}$$

Under consideration of Eq. (6), these two equations can be expressed in dependency of $\underline{\mathbf{q}}$. As the resulting equations are quadratic functions in $\underline{\mathbf{q}}$, they can be written as:

$$\zeta(\underline{\mathbf{q}}): (\mathbf{q}^T, \widehat{\mathbf{q}}^T)\mathbf{Z}\begin{pmatrix}\mathbf{q}\\ \widehat{\mathbf{q}}\end{pmatrix}, \tag{16}$$

and

$$\nu(\underline{\mathbf{q}}): (\mathbf{q}^T, \widehat{\mathbf{q}}^T)\mathbf{N}\begin{pmatrix}\mathbf{q}\\ \widehat{\mathbf{q}}\end{pmatrix} - 1 = 0, \tag{17}$$

where $\mathbf{Z}$ and $\mathbf{N}$ are 6×6 matrices, with $\mathbf{N} = \mathbf{J}^T\mathbf{J}$.

We solve the optimization problem by introducing a Lagrange multiplier λ (cf. [5]). Then, the approach simplifies under consideration of

$$\nabla\zeta(\underline{\mathbf{q}}) = 2\,\mathbf{Z}\begin{pmatrix}\mathbf{q}\\ \widehat{\mathbf{q}}\end{pmatrix}, \quad \nabla\nu(\underline{\mathbf{q}}) = 2\,\mathbf{N}\begin{pmatrix}\mathbf{q}\\ \widehat{\mathbf{q}}\end{pmatrix}, \tag{18}$$

to the general eigenvalue problem

$$(\mathbf{Z} - \lambda\,\mathbf{N})\begin{pmatrix}\mathbf{q}\\ \widehat{\mathbf{q}}\end{pmatrix} = \begin{pmatrix}\mathbf{o}\\ \mathbf{o}\end{pmatrix}. \tag{19}$$

This system of linear equations only has a non-trivial solution, if the determinant of the matrix $\mathbf{Z} - \lambda\,\mathbf{N}$ vanishes. Each solution λ_i (general eigenvalue of $\mathbf{Z}$ with respect to $\mathbf{N}$) of the resulting characteristic polynomial of degree 6 in λ corresponds with a general eigenvector $\mathbf{e}_i$. Due to Eqs. (17) and (19) we get

$$\zeta(\mathbf{e}_i) = \lambda_i, \tag{20}$$

which implies that the greatest λ_+ and smallest λ_- general eigenvalue equal the requested extrema.

Theorem 2. *The $GCTN$, which is given by*

$$GCTN := \sqrt{\frac{\lambda_-}{\lambda_+}} \in [0,1], \tag{21}$$

fulfills all six stated requirements.

Proof: Due to the definition of the index, all demands, with exception of the second one, are trivially fulfilled. Therefore, we only comment on demand 2: The value of λ_+ equals ∞, if and only if, the manipulator is in an EE singularity or leg singularity of type (b), as only in these configurations an instantaneous self-mobility of the manipulator exists, while all active actuators are locked (cf. Section II-C).

Hence, it remains to check the case $\lambda_- = 0$: In this case, all passive joints have an instantaneous standstill. As a consequence, an instantaneous change of the EE's orientation is not possible and therefore only a pure translation can be performed at this moment. A pure translation can only be done if all six legs are parallel to each other, but this already implies $rk(\mathbf{J}) \leq 3$, as the six carrier lines of the P-joints belong to a bundle of lines (cf. page 142 of [17]). □

Remark 4. *Note, that the $GCTN$ also masters the challenge formulated in the paragraph below the six demands. It should be mentioned that, similar to the CTN (cf. [14]), the $GCTN$ can also be used for parallel manipulators with more than six legs, i.e. redundant hexapods with UCU-legs.* ◇

If a given configuration of the hexapod is indicated by a small $GCTN$-value to be close to a singularity, then the corresponding eigenvector $\mathbf{e}_+$ of λ_+ contains the following extra information:

The joint ratio $\mathbf{r}^+ := (\tau_1^+ : \ldots : \tau_6^+)$, computed from $\underline{\mathbf{q}}_+ := \mathbf{q}_+ + \varepsilon\widehat{\mathbf{q}}_+$ with $(\mathbf{q}_+^T, \widehat{\mathbf{q}}_+^T) := \mathbf{e}_+^T$ by Eq. (8), corresponds with the most uncontrollable motion of the hexapod, as small variations of the prismatic joints have large effects on the instantaneous transformation of the whole manipulator (EE singularity) or of a substructure (leg singularity). Therefore, the joint ratio $\mathbf{r}^+$ should be avoided.

This joint ratio $\mathbf{r}^+$ can also be used for evaluating the quality of an arbitrary instantaneous joint ratio $\mathbf{r} := (\tau_1 : \ldots : \tau_6)$ by computing the angle ρ enclosed by the one-dimensional subspaces $\mathbf{r}$ and $\mathbf{r}^+$:

$$\rho := \arccos \frac{\pm\mathbf{r}\cdot\mathbf{r}^+}{\|\mathbf{r}\|\|\mathbf{r}^+\|}, \tag{22}$$

where the sign $\pm$ has to be chosen that $\rho \in [0, \pi/2]$ holds.

Moreover, we can even detect whether the given configuration is close to either an EE singularity or a leg singularity by computing

$$\mu_i(\underline{\mathbf{q}}_+) := \sum_{j=1}^{5} \omega_{ij}^2 \tag{23}$$

for $i = 1, \ldots, 6$. If $\mu_i(\underline{\mathbf{q}}_+)$ is not far away from λ_+, then the manipulator is in the neighborhood of a leg singularity of the i^{th} leg, as $\mu_1(\underline{\mathbf{q}}_+) + \ldots + \mu_6(\underline{\mathbf{q}}_+) = \lambda_+$ holds. Otherwise, we are close to an EE singularity.

Remark 5. *Note, that the $GCTN$ can also be used to optimize the kinematic design of the hexapods under consideration, as this was done for SG manipulators with respect to the CTN in [14], [15]. From this perspective, the hexapod should be isotropic in the central configuration $\mathcal{C}_\odot$ of the workspace, i.e. $GCTN(\mathcal{C}_\odot) = 1$. The topic of isotropy is dedicated to future research.* ◇

V. EXAMPLE

Due to the simplicity of the inverse kinematics, we study a hexapod, where both U-joints of all UCU-legs are classic ones. Moreover, we assume that the centers B_i and P_i of the base U-joint and platform U-joint, respectively, of the i^{th} leg are located on semi-regular hexagons with a circumcircle of radius 1. Without loss of generality, we can choose a Cartesian coordinate system in Σ_0, that B_i and P_i have the following coordinate vectors $\mathbf{b}_i$ and $\mathbf{p}_i$, respectively:

$$\mathbf{b}_i = (\cos\alpha_i, \sin\alpha_i, 0)^T, \quad \mathbf{p}_i = (\cos\beta_i, \sin\beta_i, d)^T,$$

with

$$\begin{aligned} \alpha_1 &= \beta_2 - \frac{\pi}{3} = -\alpha, & \alpha_2 &= \beta_1 + \frac{\pi}{3} = \alpha, \\ \alpha_3 &= \beta_4 - \frac{\pi}{3} = \frac{2\pi}{3} - \alpha, & \alpha_4 &= \beta_3 + \frac{\pi}{3} = \frac{2\pi}{3} + \alpha, \\ \alpha_5 &= \beta_6 - \frac{\pi}{3} = \frac{4\pi}{3} - \alpha, & \alpha_6 &= \beta_5 + \frac{\pi}{3} = \frac{4\pi}{3} + \alpha. \end{aligned}$$

Moreover, we set the design parameter α equal to $\pi/12$ and the configuration parameter d equal to 1 in order to get a presentable graphical illustration of the hexapod's central configuration $\mathcal{C}_\odot$. In addition, we can still select the direction of the first rotational axis a_{i1} through B_i as well as the direction of the last rotational axis a_{i5} through P_i. They are chosen in a way, that they contain the center of the corresponding circumcircle (cf. Fig. 3).

By rotating the platform of $\mathcal{C}_\odot$ around the line g spanned by the centers of the two circumcircles about the angle $\pi/2$, we get into the EE singularity illustrated in Fig. 4. The value of the $GCTN$, in dependency of the rotation angle $\delta \in [0, \pi/2]$, is displayed in Fig. 6.

By rotating the platform of $\mathcal{C}_\odot$ around g about the angle $\pi/6$, we get into the intermediate pose $\mathcal{C}_\oslash$, where the axis a_{13} is parallel to g. If we move the hexapod out of $\mathcal{C}_\oslash$, by rotating the platform about the angle $\pi/4$ around the line h, which passes through the center of the platform and is orthogonal to the plane spanned by a_{13} and g, then we get into the leg singularity illustrated in Fig. 5. The value of the $GCTN$, in dependency of the rotation angle $\theta \in [0, \pi/4]$ of the rotation around h, is displayed in Fig. 6.

In the following we study a configuration close to the leg singularity and EE singularity, respectively, from the perspective of the last paragraph before Remark 5. For e.g. $\theta = 40°$, we get $GCTN \approx 0.052$, $\lambda_+ \approx 630.867$ and

$$\begin{aligned} &\mu_1 \approx 600.997, && \mu_2 \approx 7.271, && \mu_3 \approx 4.472, \\ &\mu_4 \approx 3.370, && \mu_5 \approx 3.806, && \mu_6 \approx 10.951, \end{aligned}$$

which shows that we are close to a leg singularity of the first leg. In contrast, for the configuration given by $\delta = 85°$, we get $GCTN \approx 0.034$, $\lambda_+ \approx 2267.560$ and

$$\mu_1 = \mu_3 = \mu_5 \approx 508.978, \quad \mu_2 = \mu_4 = \mu_6 \approx 246.875.$$

According to the prognosticate behaviour, we are close to an EE singularity.

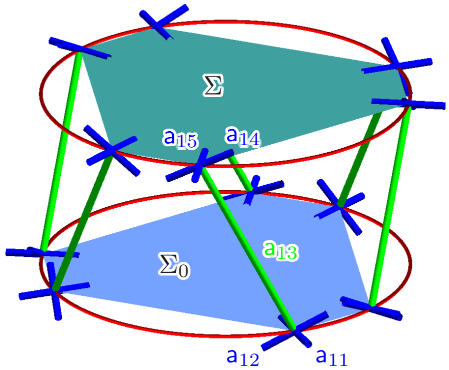

Fig. 3. The hexapod in its central configuration $\mathcal{C}_\odot$. The manipulator is far from being isotropic as $GCTN(\mathcal{C}_\odot) \approx 0.314$ holds. E.g. the corresponding octahedral manipulator ($\alpha = 0$) in this pose has a $GCTN$ of about 0.715.

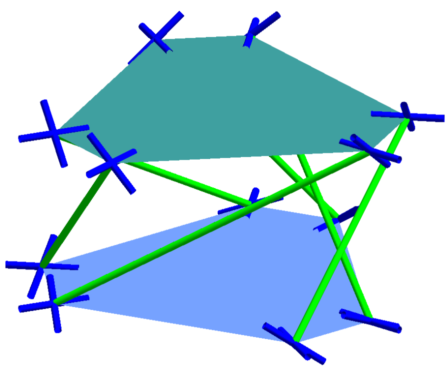

Fig. 4. A well-known EE singularity of the hexapod. Note, that the camera position for the Figs. 3, 4 and 5 is always the same with respect to Σ_0.

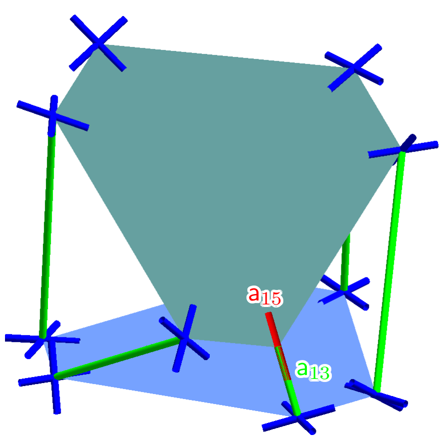

Fig. 5. Leg singularity of the hexapod: The axes $\mathfrak{a}_{13}$ (green) and $\mathfrak{a}_{15}$ (red) coincide. The axis $\mathfrak{a}_{14}$ is not displayed, as it is not uniquely determined due to a rotational self-mobility of the leg around the axis $\mathfrak{a}_{13} = \mathfrak{a}_{15}$.

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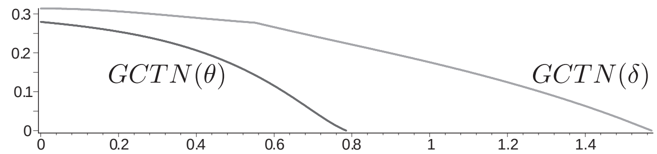

Fig. 6. The $GCTN$-graphs in dependency of the rotation angles δ and θ.

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