# Introducing the theory of bonds for Stewart Gough platforms with self-motions

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# 1. Motivation

The theory of bonds was introduced by Hegedüs, Schicho and Schröcker [1] as a new means for the analysis of overconstrained closed linkages with R-joints.

**Basic idea of bonds**

Each configuration of the overconstrained closed linkage can be identified with a point on the so-called configuration curve. Bonds are the points on this algebraic curve at some degenerate infinity, which contain a lot of information regarding the geometry of the overconstrained closed chain.

**Analogy for bonds**

To an algebraic curve $k$ in Euclidean 2-space, we associate the points of $k$ at infinity. These ideal points correspond with the bonds. If this set is a singleton, the curve is a line in direction of the ideal point. If it contains the two cyclic points, the curve is circular, and so on. ◇

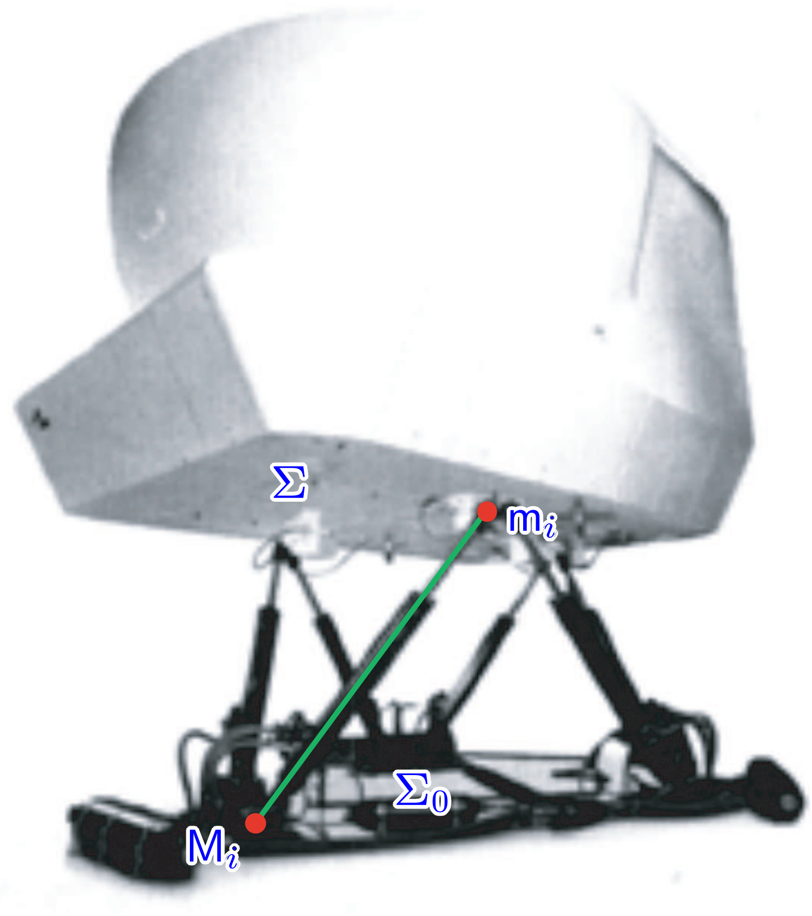
# 1. What is a self-motion of a SGP?

The geometry of a SGP is given by the six base anchor points $\mathsf{M}_i \in \Sigma_0$ and by the six platform points $\mathsf{m}_i \in \Sigma$ for $i = 1, \ldots, 6$.

A SGP is called planar, if $\mathsf{M}_1, \ldots, \mathsf{M}_6$ are coplanar and $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are coplanar.

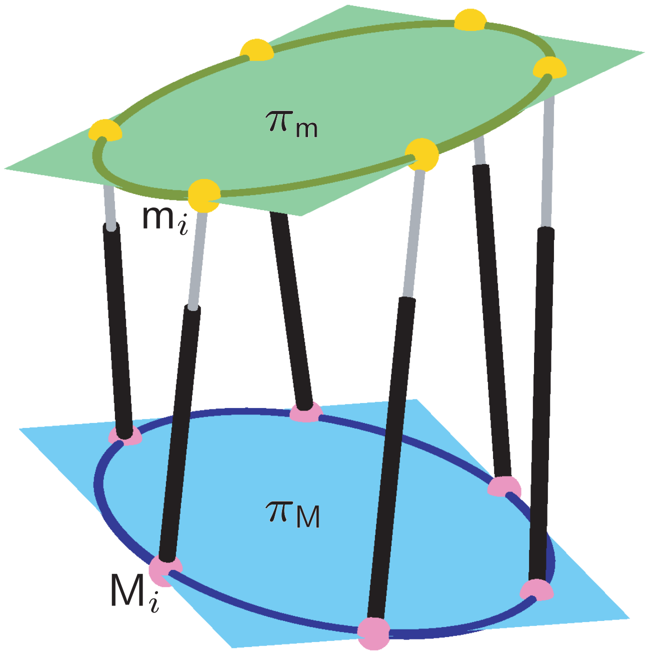
$\mathsf{M}_i$ and $\mathsf{m}_i$ are connected with a S<u>P</u>S leg.

If all <u>P</u>-joints are locked, a SGP is in general rigid. But, under particular conditions, the manipulator can perform an $d$-parametric motion ($d > 0$), which is called self-motion.

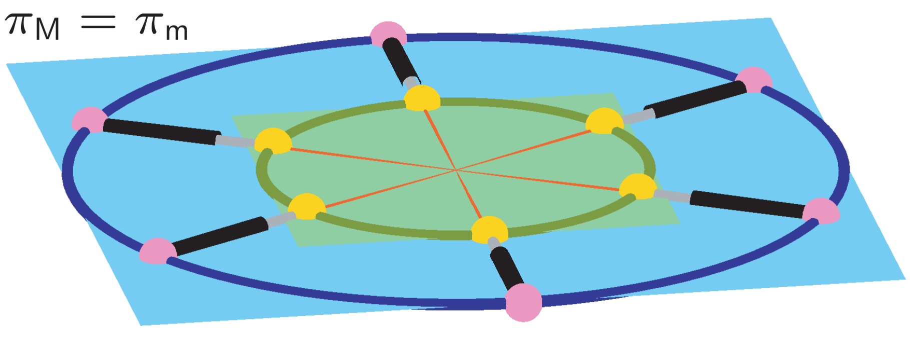


# 1. Singularity of SGPs

**Theorem 1.** Merlet [6]
A SGP is singular (infinitesimal flexible, shaky), if and only if, the carrier lines of the six SPS legs belong to a linear line complex.




If a SGP is singular in every possible configuration then it is called architecturally singular.




**Remark:** Architecturally singular SGPs possess self-motions in each pose over $\mathbb{C}$. ◇

# 1. Study parameters

We use Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ for the parametrization of SE(3). Note that $(e_0 : e_1 : e_2 : e_3)$ are the so-called Euler parameters of SO(3).

All real points of the Study parameter space $P^7$, which are located on the so-called Study quadric

$$\Psi : \ e_0 f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3 = 0,$$

correspond to an Euclidean displacement with exception of the $3$-dimensional subspace $e_0 = e_1 = e_2 = e_3 = 0$ of $\Psi$, as its points cannot fulfill the condition $N = 1$ with

$$N = e_0^2 + e_1^2 + e_2^2 + e_3^2.$$

All points of $P^7$, which cannot fulfill this normalizing condition, are located on the so-called exceptional quadric $N = 0$.

# 1. Direct kinematics of SGPs

The solution of the direct kinematics is based on Husty's quadratic homogeneous equation in Study parameters expressing that the point $\mathsf{m}_i := (a_i, b_i, c_i)$ is located on a sphere centered in $\mathsf{M}_i := (A_i, B_i, C_i)$ with radius $R_i$ (cf. [2]). This is the so-called sphere condition $\Lambda_i$ with:

$$\begin{aligned}\Lambda_i: \quad & (a_i^2 + b_i^2 + c_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)N + 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) \\ & - 2(a_iA_i + b_iB_i + c_iC_i)e_0^2 + \ldots + 4(c_iB_i - b_iC_i)e_0e_1 + \ldots \\ & + 4(a_i - A_i)(e_0f_1 - e_1f_0) + \ldots = 0.\end{aligned}$$

Now the solution of the direct kinematics over $\mathbb{C}$ can be written as the algebraic variety $V$ of the ideal $\mathcal{I}$ spanned by $\Psi, \Lambda_1, \ldots, \Lambda_6, N = 1$. In general $V$ consists of a discrete set of points with a maximum of $40$ elements.

# 2. Basic idea of bonds for SGPs

We assume that a given SGP has a $d$-dimensional self-motion. As a $d$-dimensional self-motion corresponds with a $d$-dimensional solution of the direct kinematics problem, the seven quadrics $\Psi, \Lambda_1, \ldots, \Lambda_6$ have to have a $d$-dimensional set of points in common ($=$ algebraic motion).

Now the points of this algebraic motion with $N \neq 0$ equal the kinematic image of the algebraic variety $V$. But we can also consider the points of the algebraic motion, which belong to the exceptional cone $N = 0$. These points are the so-called bonds of the $d$-dimensional self-motion.

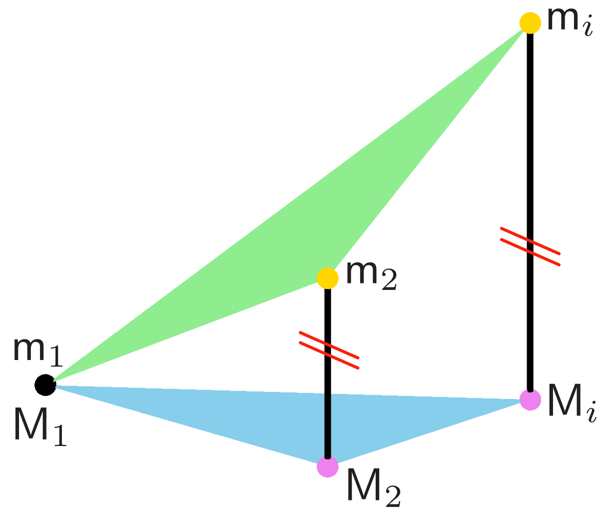
For their exact definition, one has to consider that an algebraic motion of fixed orientation ($=$ pure translational motion) is projected to a single point (with $N \neq 0$) of the Euler parameter space $P^3$ by the linear elimination of $f_0, \ldots, f_3$.
$\Longrightarrow$ *The kernel of this projection equals the group of translational motions.*

# 2a. SGPs with pure translational self-motions

All SGPs with pure translational self-motions can easily be characterized as follows:

**Theorem 2.** Nawratil [9]
A SGP possesses a pure translational self-motion, if and only if, the platform can be rotated about the center $\mathsf{m}_1 = \mathsf{M}_1$ into a pose, where the vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i = 2, \ldots, 6$ fulfill the condition $rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2}, \ldots, \overrightarrow{\mathsf{M}_6\mathsf{m}_6}) \leq 1$.

- All $1$-parametric self-motions are circular translations.
- We get a $2$-parametric translation iff the platform and the base are congruent ($\Rightarrow R_1 = \ldots = R_6$).




Therefore pure translational self-motions correspond with components of $V$.

# 2b. Definition of bonds

Based on this preparatory work, an exact definition of bonds reads as follows:

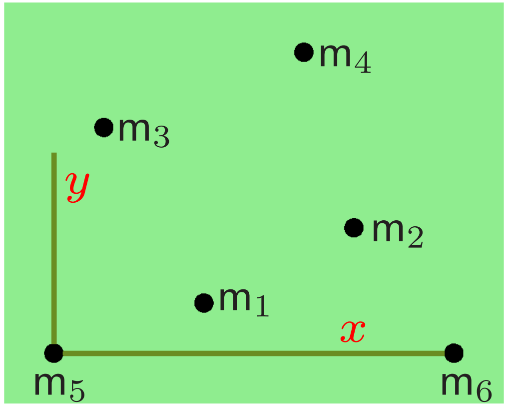
**Definition 1.** Nawratil [9]
For a parallel manipulator of SG type the set $\mathcal{B}$ of bonds is defined as:

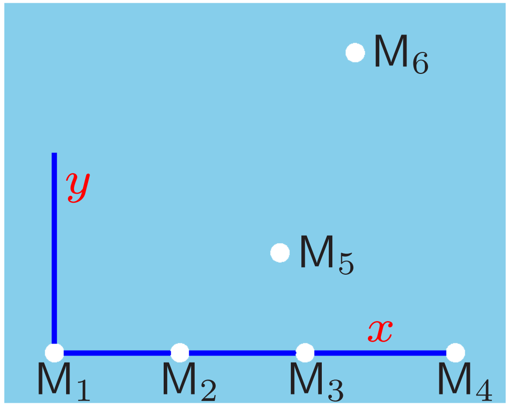
$$\mathcal{B} := ZarClo(V^\star) \;\cap\; \{(e_0 : \ldots : f_3) \in P^7 \mid \Psi = \Lambda_1 = \ldots = \Lambda_6 = N = 0\}.$$

$V^\star$ denotes the variety $V$ after the removal of all components, which correspond to pure translational motions. Moreover $ZarClo(V^\star)$ is the Zariski closure of $V^\star$; i.e. the zero locus of all algebraic equations that also vanish on $V^\star$.

Therefore the set of bonds $\mathcal{B}$ can be seen as exceptional points $(N = 0)$ of the algebraic motion, which are limits of non-translational motions.

# 3. Example: Butterfly self-motion




We get the following two 1-parametric bonds (up to conjugation of coordinates), where the ratio $u : v$ can be seen as projective parameter with $(u, v) \neq (0, 0)$:

$$\mathcal{B} = \{(uI : u : 0 : 0 : vI : v : 0 : 0), (0 : 0 : u : uI : 0 : 0 : v : vI)\}.$$

This SGP can perform a so-called butterfly self-motion if the $x$-axes of the fixed frame and moving frame coincide. The first bond corresponds to the $1$-dimensional set of rotational self-motions, where the $x$-axes have the same orientation. For the second bond they have opposite orientation.

# 3. Basic facts and results on bonds

**Theorem 3.** Nawratil [9]
The set $\mathcal{B}$ of bonds depends on the geometry of the SGP, but not on $R_1, \ldots, R_6$.

**Proof:** This follows directly from the equation of the sphere condition $\Lambda_i$. $\square$

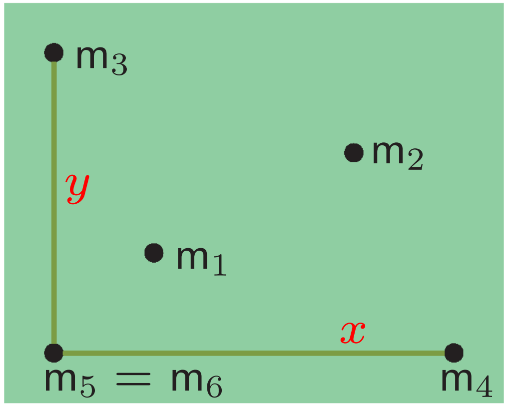
Moreover, Theorem 3 and Definition 1 already imply the following result:

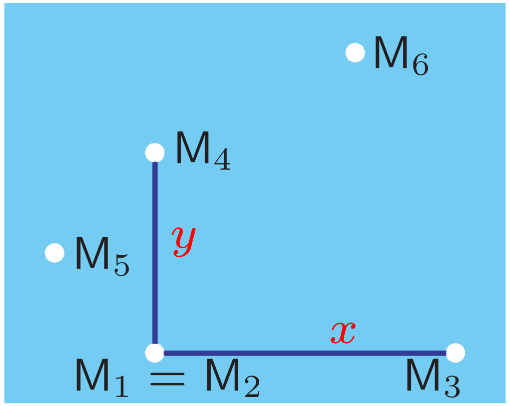
**Theorem 4.** Nawratil [9]
A SGP is free of non-translational self-motions if $\mathcal{B} = \varnothing$ holds.

Due to these properties the bond theory is suited for different tasks; e.g.:

a. Classification of SGPs with non-translational self-motions
b. Check whether a SGP is free of non-translational self-motions
c. Determine SGPs with non-translational self-motions

## 3a. Example for the classification task




We get the following four 1-parametric bonds (up to conjugation of coordinates), where the ratio $u : v$ can be seen as projective parameter with $(u, v) \neq (0, 0)$:

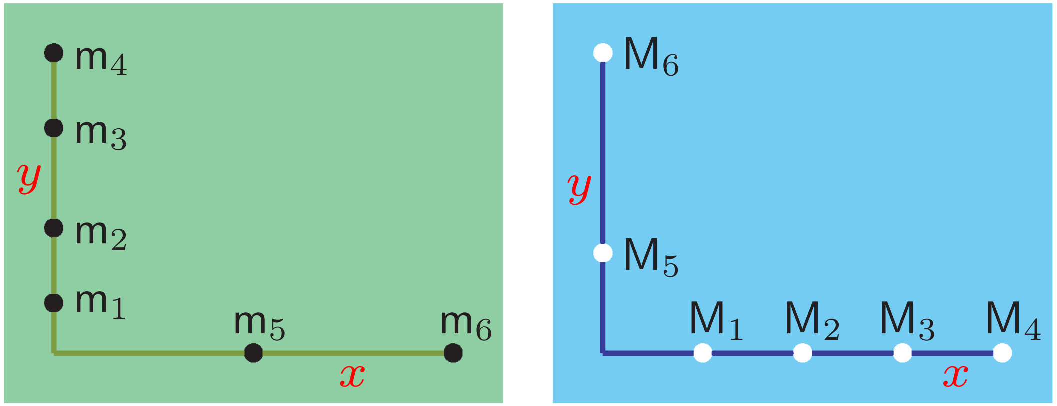
$$\mathcal{B} = \{(0 : 0 : u : uI : 0 : 0 : v : vI), (uI : u : 0 : 0 : vI : v : 0 : 0),$$
$$(0 : uI : 0 : u : 0 : vI : 0 : v), (u : 0 : uI : 0 : v : 0 : vI : 0)\}.$$

The first and second bond contain the information that the SGP possesses butterfly self-motions if the $x$-axes of the fixed and moving frame coincide. An analogous interpretation can be given for the bonds three and four with respect to the $y$-axes.

# 3a. Example for the classification task

The SGP also has a spherical self-motion if $\mathsf{M}_1 = \mathsf{M}_2$ coincides with $\mathsf{m}_5 = \mathsf{m}_6$. Therefore this spherical self-motion is also encoded within $\mathcal{B}$, which demonstrates the following: *The set of bonds is more than the sum of its single bonds.*

There exist manipulators with the same bond-set $\mathcal{B}$, which only possess the butterfly self-motions but no spherical self-motion.



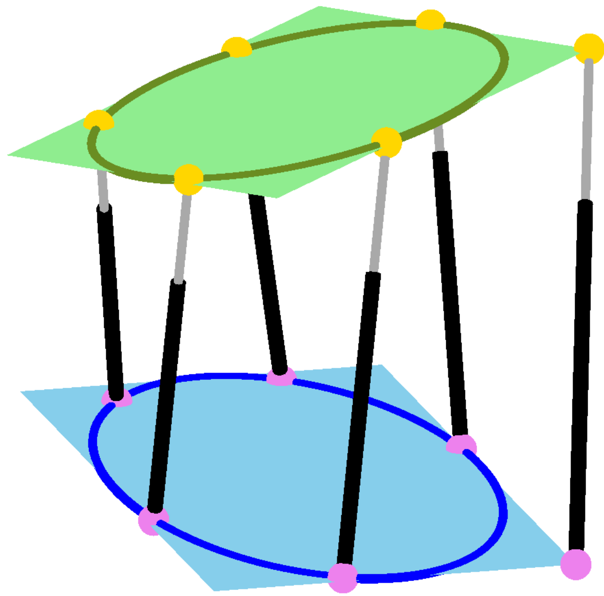

$\Longrightarrow$ For a serious classification of SGPs with non-translational self-motions, algebraic properties of $\mathcal{B}$ have to be taken into consideration, which are invariant with respect to changes of the reference frames (e.g. algebraic multiplicities of bonds).

# 3b. Examples for the checking task

Due to Theorem 4, the following examples are free of non-translational self-motions.

**Example 1:** For a SGP with a *generic* geometry (= randomly generated), we get $\mathcal{B} = \varnothing$. Moreover due to Theorem 2 a *generic* SGP is also free of pure translational self-motions. Therefore a *generic* SGP does not possess any self-motions.

**Example 2:** The planar platform and planar base of the SGP are related by a regular affinity. Due to a well known result (e.g. [7]), we can use any six platform and base anchor points related by the affinity (as long as they are not located on a conic section) for the computation of $\mathcal{B}$.
$\mathcal{B} = \varnothing$ verifies [4,7] that planar affine SGPs can only possess translatory self-motions if they are not architecturally singular.



# 3c. Basic idea of the determination task

If we want to design SGPs with non-translational self-motions, we can also make use of the necessary condition that these manipulators have to possess bonds. This criterion can for example be used for the determination of planar and spherical 3-dof R$\underline{\text{P}}$R parallel manipulators with non-translational self-motions (cf. [9]).

But in the remainder of the talk we focus on the determination of SGPs with so-called multidimensional self-motions ($d$-dimensional self-motions with $d > 1$).

Until now only the following non-architecturally singular SGP with a multidimensional self-motion is known to the author (cf. Theorem 2): The platform and the base are congruent and $R_1 = \ldots = R_6 \Longrightarrow 2$-dimensional translation.

Moreover the question is motivated by footnote 3 of [3], which reads as follows:

*Examples of 2-DOF self motions are known. If non-trivial 3-DOF self motions are possible is not known. They would correspond to solids on $S_6^2$.*

# 3c. Types of 3-dimensional self-motions

We assume that a given SGP has a $3$-dimensional self-motion $\mathcal{S}$. Therefore its corresponding algebraic motion is also $3$-dimensional and the bond-set is an algebraic variety of dimension $2$; i.e. a *bonding surface*.

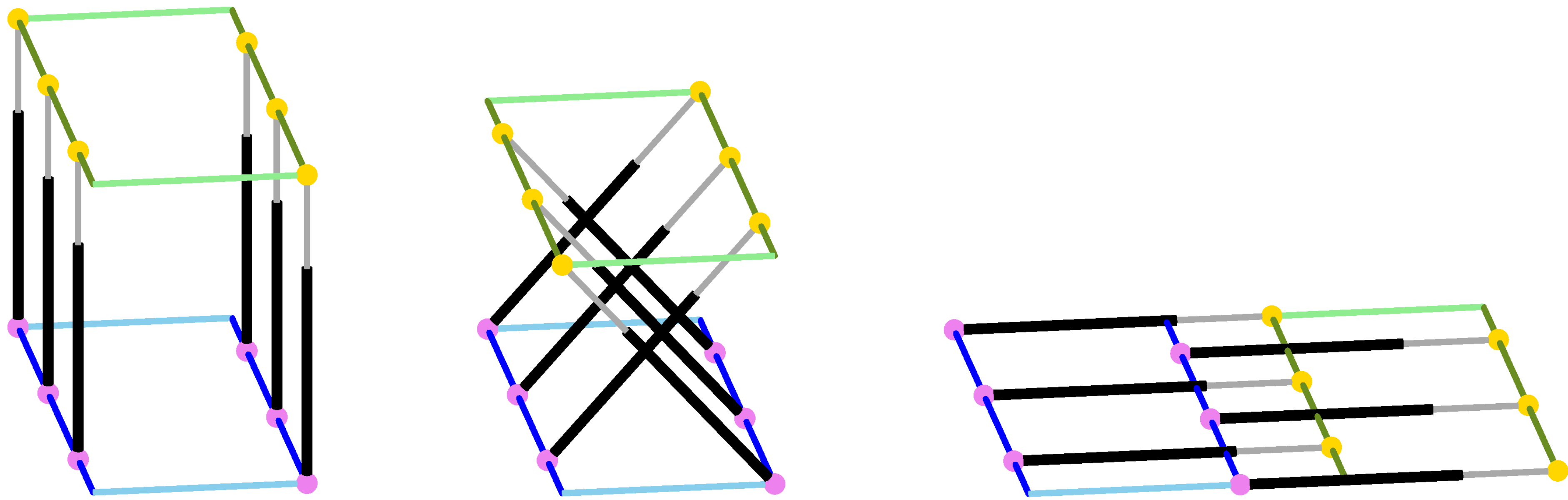
We classify $\mathcal{S}$ with respect to the dimension $\beta$ of the bonding surface after its projection into the Euler parameter space $P^3$ (by a linear elimination of $f_0, \ldots, f_3$). As we have a bonding surface in the Study parameter space $P^7$, $\beta$ can take the values:

$$\beta = 2, \qquad \beta = 1, \qquad \beta = 0, \qquad \beta = -1.$$

In order that $\beta = i$ holds for $i = -1, 0, 1$, there has to exist a $(2-i)$-dimensional translational sub-self-motion, which is contained in $\mathcal{S}$, in each pose of $\mathcal{S}$.

**Remark:** For $i = -1$ this already implies that $\mathcal{S}$ is a $3$-dimensional translation. $\diamond$

# 3c. Example for a 2-dim. self-motion with $\beta = 0$



We study a congruent SGP, where the anchor points are located on the $x$-axis and a parallel line through $(0, d, 0)$. Therefore the manipulator is architecturally singular. If all legs have equal lengths, this manipulator has two $2$-dimensional self-motions.

# 3c. Example for a 2-dim. self-motion with $\beta = 0$

**Parallelogram mode:** The already known $2$-dimensional translational self-motion. Therefore this self-motion is of type $\beta = -1$.

**Anti-parallelogram mode:** There exists a $1$-dimensional translational sub-self-motion (circular translation) in each pose of the $2$-dimensional self-motion. Its corresponding bonds are as follows, up to conjugation of coordinates:

$$\left(-\frac{2u}{d} : \frac{2uI}{d} : 0 : 0 : v : -vI : u : -uI\right), \left(\frac{2u}{d} : -\frac{2uI}{d} : 0 : 0 : v : -vI : u : uI\right).$$

By restricting us to the first four coordinate entries, we project the first and second *bonding curve* to the Euler parameter space $P^3$, which yields the points $(-1 : I : 0 : 0)$ and $(1 : -I : 0 : 0)$. This shows that the anti-parallelogram self-motion is indeed of type $\beta = 0$.

# 3c. 3-dimensional self-motions

**Theorem 5.** Nawratil [10]
Non-architecturally singular SGPs with $3$-dimensional self-motion do not exist.

**Sketch of the proof:**

Case $\beta = 2$: It turns out that the solution of the general case is equivalent to the fact that a homogeneous polynomial $P[1\,955\,651]$ of degree 16 in $e_0, e_1, e_2$ is fulfilled identically for all $e_0, e_1, e_2$. This problem can only be solved as $Q[7\,589]^2 = P[1\,955\,651]$ holds.

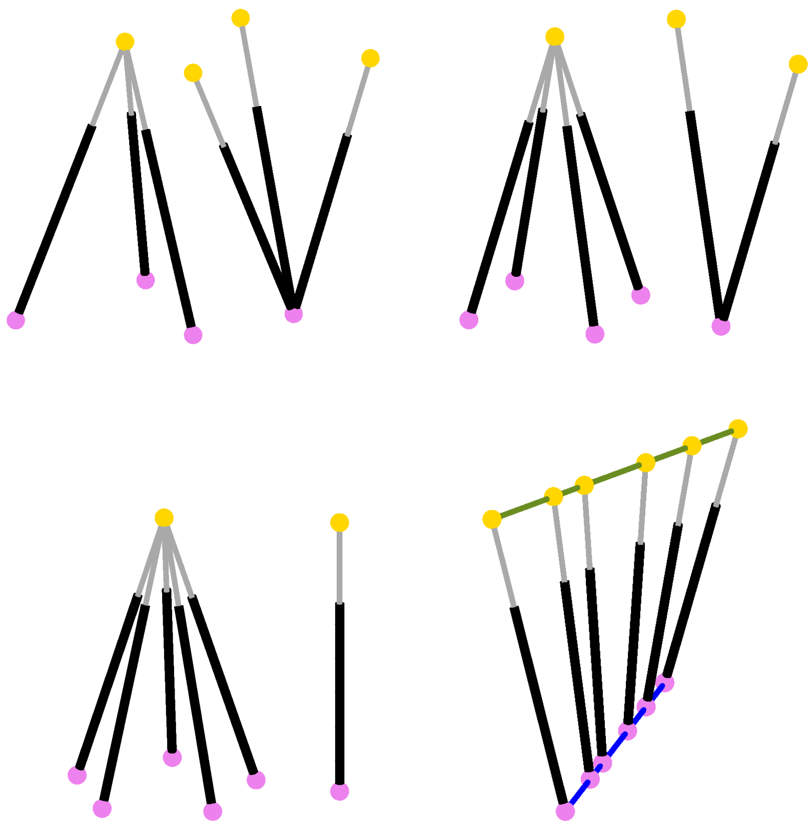
Case $\beta = 1, 0$: Due to Theorem 2 a necessary condition for the existence of these types of self-motions is that a $(1+\beta)$-dimensional set of platform orientations with $\mathsf{m}_1 = \mathsf{M}_1$ and $rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2}, \ldots, \overrightarrow{\mathsf{M}_6\mathsf{m}_6}) \leq 1$ exists.

Case $\beta = -1$: Trivial. $\square$

# 3c. 3-dimensional self-motions

**Theorem 6.** Nawratil [10]
If a SGP has a $3$-dimensional self-motion, it has to be one of the following architecturally singular designs:

1. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$ and $\mathsf{M}_4 = \mathsf{M}_5 = \mathsf{M}_6$.
2. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4$ and $\mathsf{M}_5 = \mathsf{M}_6$.
3. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4 = \mathsf{m}_5$.
4. $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are collinear, $\mathsf{M}_1, \ldots, \mathsf{M}_6$ are collinear and there exists a regular projectivity $\kappa$ with $\mathsf{M}_i \mapsto \mathsf{m}_i$ for $i = 1, \ldots, 6$.



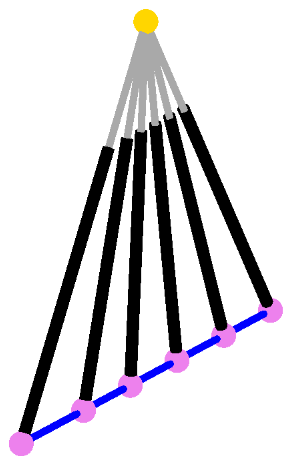
# 3c. n-dimensional self-motions with n$>$3

Based on the results for $3$-dimensional self-motions, one can prove the following:

**Theorem 7.** Nawratil [10]
Non-architecturally singular SGPs with $4$-dimensional self-motion do not exist. SGPs (architecturally singular or not) with higher-dimensional self-motions than 4 do not exist.

**Theorem 8.** Nawratil [10]
If a SGP has a $4$-dimensional self-motion, it has to be the following architecturally singular design:

- All six base anchor points are collinear and the six platform anchor points collapse into one point.



# 3c. 2-dimensional self-motions

**Architecturally singular SGPs:**

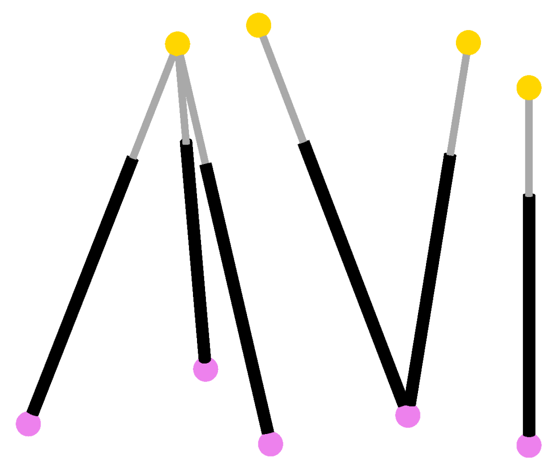
- Based on Theorem 3 of [5], it is not difficult to give a list of all SGPs with $rk(\mathbf{J}) = 4$, where $\mathbf{J}$ denotes the Jacobian matrix.
- The more challenging (still unsolved) problem is to determine all designs with $rk(\mathbf{J}) = 5$. From each of these designs one can construct non-architecturally singular SGPs with $1$-dimensional self-motions (cf. type II DM self-motion [8]).

**Non-architecturally singular SGPs:**

Beside the translational self-motion of the congruent SGP, there exists a further trivial example, which was not mentioned in the literature before, to the best knowledge of the author:

$$\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 \text{ and } \mathsf{M}_4 = \mathsf{M}_5.$$

This self-motion is spherical with center $\mathsf{m}_1 = \mathsf{M}_4$.



# 4. Conclusion and outlook

• In this paper we introduced the theory of bonds for SGPs with self-motions.

• We presented some basic facts and results on bonds and demonstrated the potential of this theory on the basis of several examples.

• Moreover we showed that for a further, deeper study of bonds, their algebraic multiplicities have to be considered as well, which is dedicated to future research.

• This concept is not limited to SGPs, but it can also be adopted for other parallel manipulators as well (e.g. spherical and planar $3$-dof R$\underline{\text{P}}$R manipulators).

• We gave a geometric characterization of all SGPs with pure translational self-motions.

• We listed all SGPs, which have $n$-dimensional self-motions with $n > 2$. The case of SGPs possessing multidimensional self-motions with $n = 2$ remains open.

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