Types of self-motions of planar Stewart Gough platforms

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$[1a]$ Stewart Gough Platform

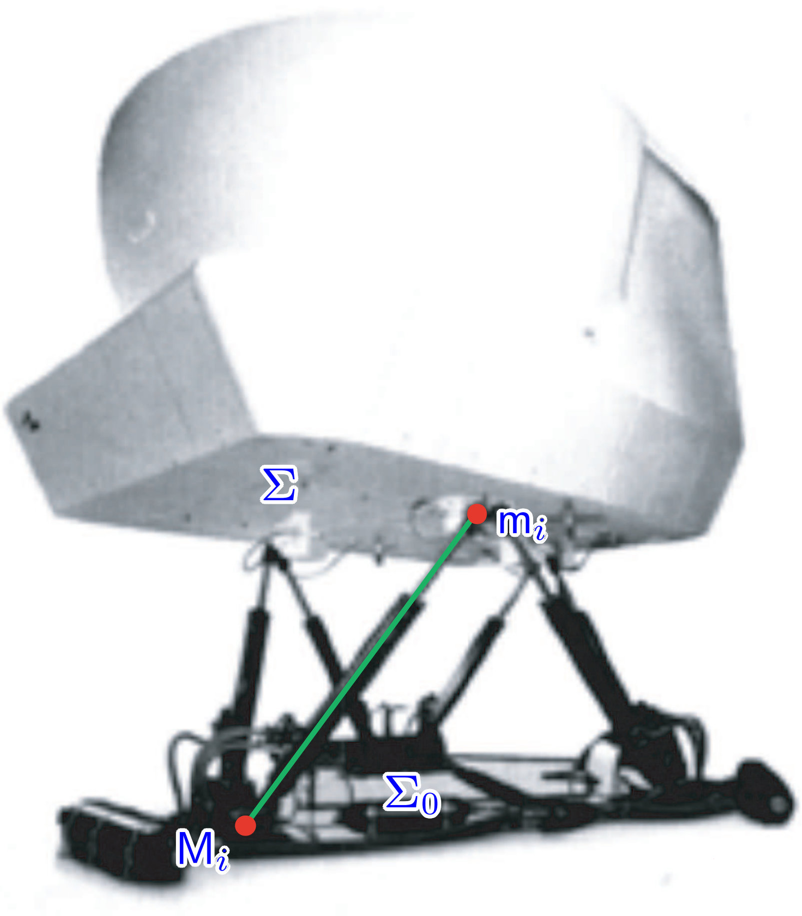
The geometry of a planar SGP is given by the six base anchor points M_i with

$\mathbf{M}_i := (A_i, B_i, 0)^T$ in the fixed space Σ_0,

and by the six platform points m_i with

$\mathbf{m}_i := (a_i, b_i, 0)^T$ in the moving space Σ.

M_i and m_i are connected with a S<u>P</u>S leg.



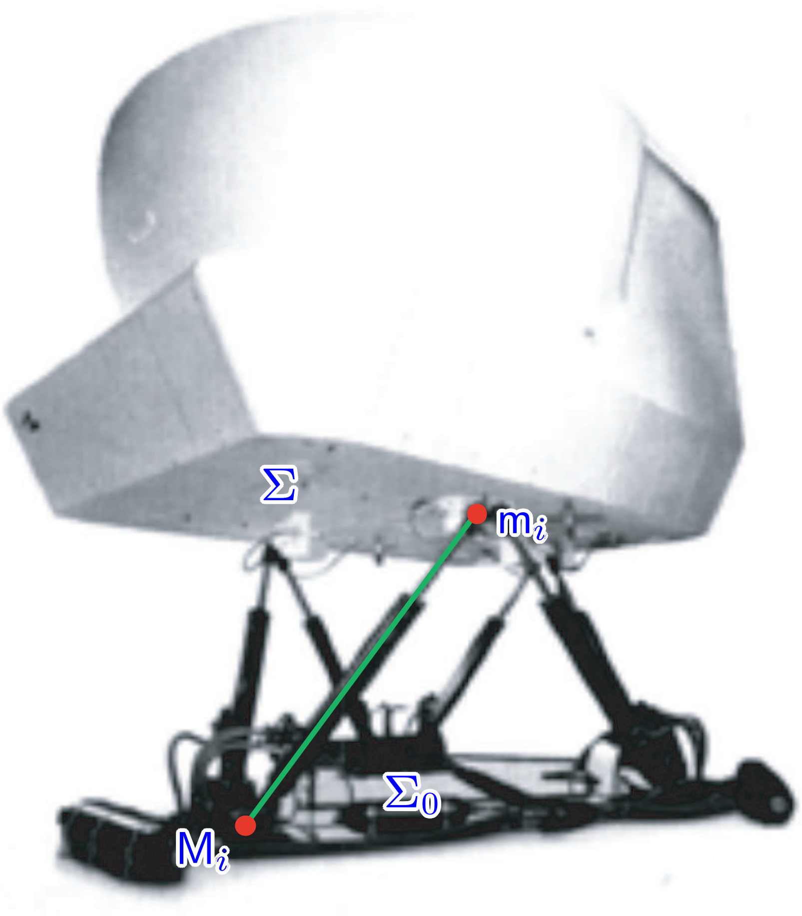
Theorem 1
A SGP is singular (infinitesimal flexible, shaky) if and only if the carrier lines of the six S<u>P</u>S legs belong to a linear line complex.

[1b] Self-motions and the Borel Bricard problem

If all P-joints are locked, a SGP is in general rigid. But, in some special cases the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion.

Note that in each pose of the self-motion, the SGP has to be singular. Moreover, all self-motions of SGPs are solutions to the famous Borel Bricard problem [3,6,7,8,9].

Borel Bricard problem (still unsolved)
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.



$[1c]$ Architecturally singular SGPs

Manipulators which are singular in every possible configuration, are called architecturally singular.

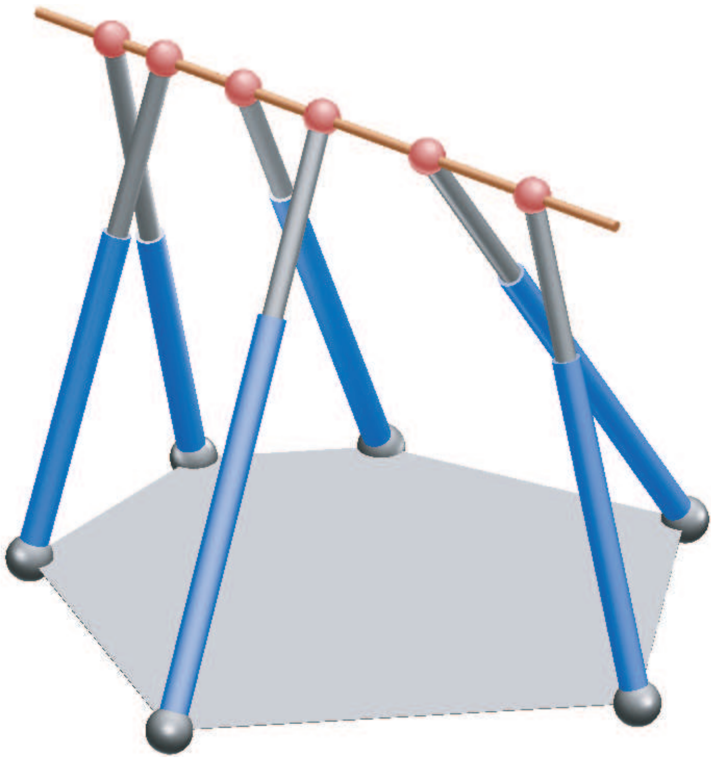
Architecturally singular SGPs are well studied:

⋆ For the planar case see [A,B,C,D],

⋆ For the non-planar case see [E,F].

It is well known, that architecturally singular SGPs possess self-motions in each pose.

Therefore we are only interested in self-motions of non-architecturally singular SGPs.



$[1d]$ Review on SGPs with self-motions

- Husty and Zsombor-Murray [G]: SGP with Schönflies self-motion
- Zsombor-Murray et al. [H]: SGP with line-symmetric self-motion (cf. Krames [I])
- Husty and Karger [J] proved that the list of Schönflies Borel Bricard motions given by Borel [3] is complete
- Karger and Husty [K]: Self-motions of the original SGP
- Karger [2,10] presented a method for designing planar SGPs with self-motions of the type $e_0 = 0$, where e_0 denotes an Euler parameter
- Nawratil [L] presented a complete list of TSSM self-motions (6-3 SGPs)

$[2a]$ Redundant planar SGPs

According to Husty [M], the "sphere constraint" that m_i is located on a sphere with center M_i can be expressed by a homogeneous quadratic equation Λ_i in the Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$.

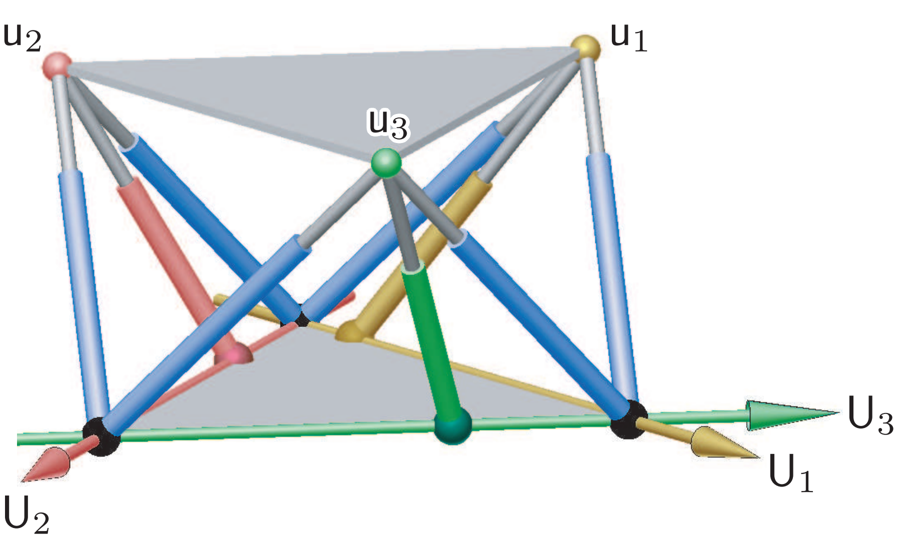
Therefore the direct kinematic problem corresponds to the solution of the system $\Lambda_1, \ldots, \Lambda_6, \Psi$ where Ψ denotes the equation of the Study quadric.

If a planar SGP is not architecturally singular, then at least a 1-parametric set of legs $\lambda_1\Lambda_1 + \ldots + \lambda_6\Lambda_6$ can be added without changing the direct kinematics [N,O].

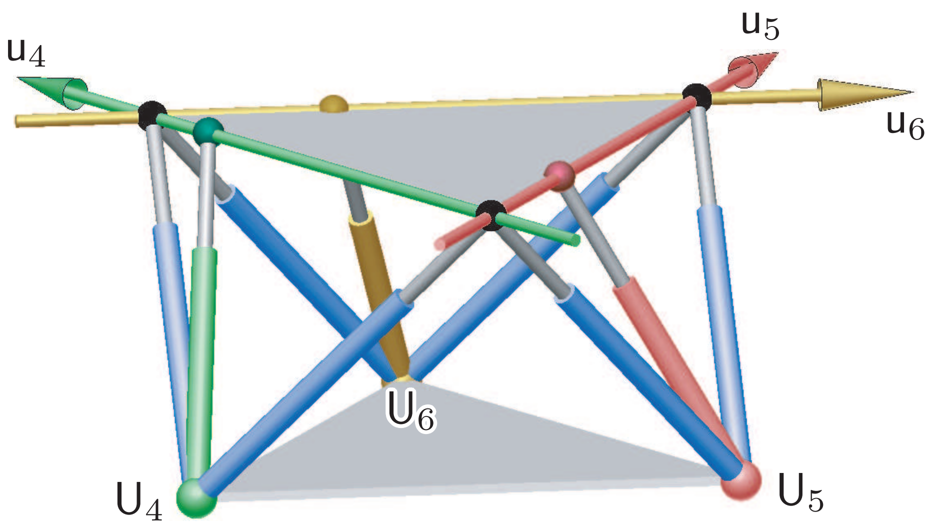
As the solvability condition of the underlying linear system of equations (Eq. (30) of [O]) is equivalent with the criterion given in Eq. (12) of [P], also the singularity surface of the SGP does not change by adding legs of this 1-parametric set.

$[2a]$ Redundant planar SGPs

Moreover, it was shown [N,O] that in general the base anchor points M_i as well as the corresponding platform anchor points m_i are located on planar cubic curves C and c, which can also split up.



Cubic C of the octahedral SGP



Cubic c of the octahedral SGP

$[2b]$ Assumptions and basic idea

Assumption 1
We assume, that there exist such cubic curves c and C in the Euclidean domain of the platform and the base, respectively.

As the correspondence between c and C has not to be a bijection, a point $\in P^3_{\mathbb{C}}$ of c resp. C is in general mapped to an non-empty set of points $\in P^3_{\mathbb{C}}$ of C resp. c. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{\,\}$.

Assumption 2
For guaranteeing a general case, we assume that each of the corresponding locations $\{u_1\}$, $\{u_2\}$, $\{u_3\}$, $\{U_4\}$, $\{U_5\}$, $\{U_6\}$ consists of a single point. Moreover, we assume that no four collinear platform points u_i or base points U_i for $i = 1, \ldots, 6$ exist.

Basic idea: Attach the special "legs" $\overline{u_i U_i}$ with $i = 1, \ldots, 6$ to SGP $m_1, \ldots, M_6$.

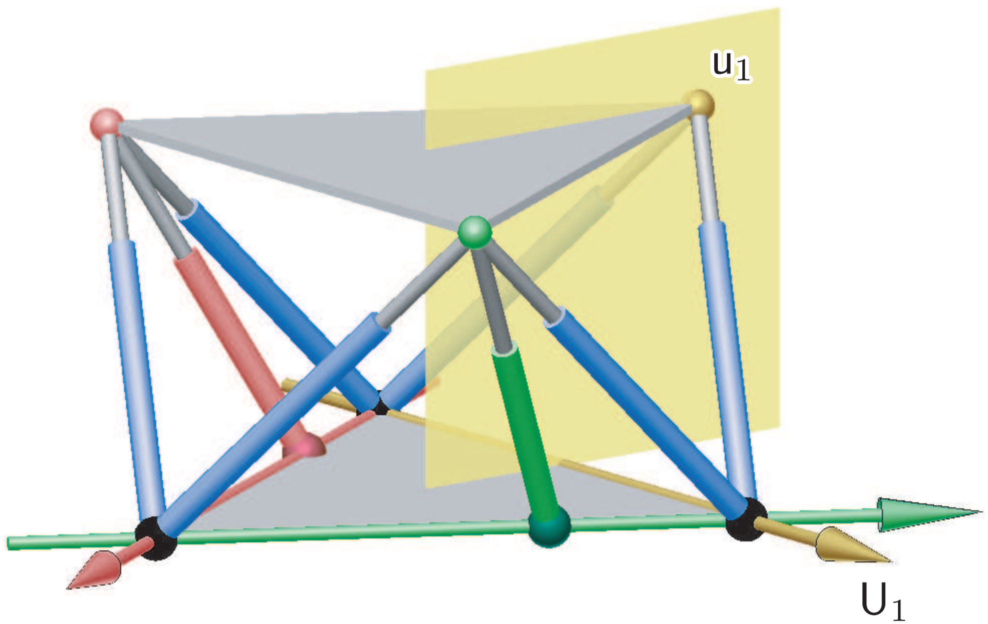
$[2c]$ Darboux constraint

The attachment of the "legs" $\overline{\mathrm{u}_i\mathrm{U}_i}$ with $i = 1, 2, 3$ corresponds with the so-called Darboux constraint, that the platform anchor point u_i moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i.

The Darboux constraint can be written as a homogeneous quadratic equation Ω_i in the Study parameters (for details see [1]).

Note that Ω_i depends only linearly on f_0, f_1, f_2, f_3.

Remark: Due to Assumption 2 not both points u_i and U_i can be ideal points. ◇



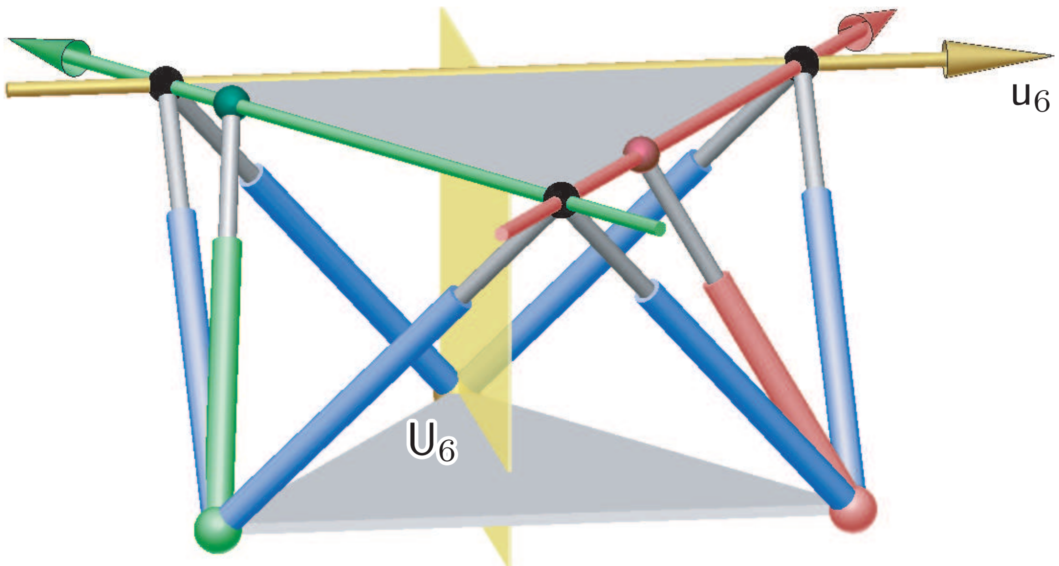
$[2c]$ Mannheim constraint

The attachment of the "leg" $\overline{\mathrm{u}_j\mathrm{U}_j}$ with $j = 4, 5, 6$ corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to u_j slides through the point U_j.

The Mannheim constraint can be written as a homogeneous quadratic equation Π_j in the Study parameters (for details see [1]).

Note that Π_j depends only linearly on f_0, f_1, f_2, f_3.

Remark: Due to Assumption 2 not both points u_j and U_j can be ideal points. ⋄



$[2d]$ Implication of the assumptions

Theorem 2
Given is a planar SGP $m_1, \ldots, M_6$ which is not architecturally singular and which fulfills Assumption 1 and 2. Then the resulting manipulator $u_1, \ldots, U_6$ is redundant and therefore architecturally singular.

Proof: As the points U_i and u_i are corresponding points of C and c we get:

$$\Omega_i = \sum_{k=1}^{6} \lambda_{i,k}\Lambda_k \quad \text{and} \quad \Pi_j = \sum_{k=1}^{6} \lambda_{j,k}\Lambda_k \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad j = 4, 5, 6.$$

As Ω_i and Π_j are only linear in $f_0, \ldots, f_3$, in contrast to Λ_k which contains the term $4(f_0^2 + f_1^2 + f_2^2 + f_3^2)$, the equations can be rewritten as:

$$\Omega_i = \sum_{k=2}^{6} \delta_{i,k}\Delta_k \quad \text{and} \quad \Pi_j = \sum_{k=2}^{6} \delta_{j,k}\Delta_k \quad \text{with} \quad \Delta_k = \Lambda_1 - \Lambda_k.$$

Therefore the set of the six polynomials $\Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5, \Pi_6$ is redundant. $\square$

$[3a]$ Types of self-motions

Definition 1
Assume $\mathcal{M}$ is a 1-parametric self-motion of a non-architecturally singular SGP $\mathsf{m}_1, \ldots, \mathsf{M}_6$. Then $\mathcal{M}$ is of type n DM (Darboux Mannheim) if the corresponding architecturally singular manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ has an n-parametric self-motion $\mathcal{U}$.

Note that $\mathcal{U}$ includes $\mathcal{M}$, because if we attach the "legs" $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i = 1, \ldots, 6$ to $\mathsf{m}_1, \ldots, \mathsf{M}_6$, we do not change the direct kinematics and singularity surface. Therefore also $\mathcal{M}$ remains unchanged. By removing the legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ the self-motion $\mathcal{M}$ can only be enlarged.

Theorem 3 (Proof is given in [1])
All 1-parametric self-motions of non-architecturally singular planar SGPs fulfilling Assumption 1 and 2 are type I or type II DM self-motions.

$[3b]$ Computation of type II DM self-motions

W.l.o.g. we can assume that the variety of a 2-parametric DM self-motion is spanned by $\Psi, \Omega_1, \Omega_2, \Omega_3$, Π_4, Π_5 (otherwise we can consider the inverse motion).

Lemma 1 (Proof is given in [1])
W.l.o.g. we can choose coordinate systems in Σ_0 and Σ with $X_2(X_2 - X_3)x_5 \neq 0$,

$$a_1 = b_1 = y_4 = A_4 = B_4 = Y_1 = h_4 = g_5 = 0, \quad X_1 = Y_2 = Y_3 = x_4 = y_5 = 1,$$

where $(0 : X_i : Y_i : 0)$ and $(0 : x_i : y_i : 0)$ are the projective coordinates of the ideal points U_i and u_i, respectively.

We solve $\Psi, \Omega_1, \Omega_2, \Pi_4$ for $f_0, \ldots, f_3$ and plug the obtained expressions in the remaining two equations which yield $\Omega_3^\star[40]$ (degree 2) and $\Pi_5^\star[96]$ (degree 4).

Finally, we compute the resultant of $\Omega_3^\star$ and $\Pi_5^\star$ with respect to one of the Euler parameters. For e_0 this yields $\Gamma[117\,652]$ (degree 8).

$[3b]$ Computation of type II DM self-motions

In the following, we list the coefficients of $e_1^i e_2^j e_3^k$ of Γ, which are denoted by Γ_{ijk}:

$$\Gamma_{080} = F_1[8]F_2[18]^2, \qquad \Gamma_{800} = (b_2 - b_3)^2(L_1 - g_4)^2F_3[8],$$

$$\Gamma_{170} = F_2[18]F_4[283], \qquad \Gamma_{710} = (b_2 - b_3)(L_1 - g_4)F_5[170],$$

$$\Gamma_{620}[2054], \quad \Gamma_{602}[1646], \quad \Gamma_{260}[6126], \quad \Gamma_{062}[4916],$$

$$\Gamma_{026}[5950], \quad \Gamma_{116}[3066], \quad \Gamma_{530}[4538], \quad \Gamma_{512}[4512],$$

$$\Gamma_{152}[6514], \quad \Gamma_{440}[7134], \quad \Gamma_{422}[6314], \quad \Gamma_{242}[7622],$$

$$\Gamma_{044}[6356], \quad \Gamma_{314}[6934], \quad \Gamma_{224}[7096], \quad \Gamma_{134}[6656],$$

$$\Gamma_{206}[5950], \quad \Gamma_{350}[7166], \quad \Gamma_{404}[5766], \quad \Gamma_{332}[6982].$$

Based on these 24 equations $\Gamma_{ijk} = 0$ (in 14 unknowns), we were already able to compute first results for type II DM self-motions in [5], which raise the hope of giving a complete classification of these self-motions in the future.

$[3b]$ SGPs with type II DM self-motions

Assuming we have computed a 2-parametric DM self-motion, the question remains open how to construct a SGP with a 1-parametric self-motion from it.

Clearly, we can attach an arbitrary finite leg $\overline{\mathrm{m}_6\mathrm{M}_6}$ to the manipulator $\mathrm{u}_1, \ldots, \mathrm{U}_5$. The resulting planar manipulator $\mathrm{u}_1, \ldots, \mathrm{U}_5, \mathrm{m}_6, \mathrm{M}_6$ is not architecturally singular as $(\mathrm{m}_6, \mathrm{M}_6) \neq (\mathrm{u}_6, \mathrm{U}_6)$ holds.

Analogous considerations as in [N,O] yield that we can attach at least a 1-parametric set $\mathcal{L}$ of legs to $\mathrm{u}_1, \ldots, \mathrm{U}_5, \mathrm{m}_6, \mathrm{M}_6$, without changing the direct kinematics.

Replace "legs" $\overline{\mathrm{u}_i\mathrm{U}_i}$ bei finite legs $\overline{\mathrm{m}_i\mathrm{M}_i}$ $(i = 1, \ldots, 5)$ of $\mathcal{L}$ such that the resulting SGP $\mathrm{m}_1, \ldots, \mathrm{M}_6$ is not architecturally singular.

$[4]$ Known examples of type II DM self-motions

The self-motions $\mathcal{K}$ computed by Karger [2,10] with $e_0 = 0$ are of type II DM.

Karger [10] wrote that the general condition for the geometry of the SGP yielding a self-motion of $\mathcal{K}$ is a very complicated algebraic condition (approx. 1000 terms).

Moreover, he noted that it would be interesting to find further special cases beside the original SGP [K] and the homological configuration [6,7], for which the condition has a geometric interpretation.

Based on our approach we can give easily a nice geometric interpretation for a subset of $\mathcal{K}$ as follows: If we set $e_0 = 0$ the equations $\Omega_3^\star$ and $\Pi_5^\star$ have to vanish identically. Doing so, we only cover a subset $\mathcal{S}$ of $\mathcal{K}$ as for the general case U_1 must not be located on the x-axis of the fixed frame.

[4] Known examples of type II DM self-motions

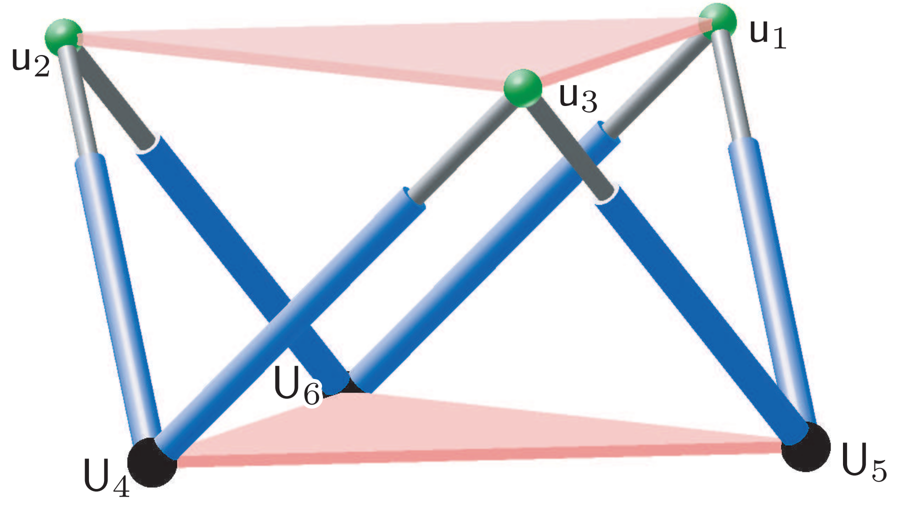
Theorem 4 (Proof is given in [1])
The self-motions $\mathcal{S}$ fulfilling Assumption 1 and 2 are line-symmetric motions and can be parametrized with respect to the homogeneous parameter $e_1 : e_2$.

Moreover, the self-motions $\mathcal{S}$ fulfilling Assumption 1 and 2 are octahedral.

Definition 2
A DM self-motion is called octahedral if following triples of points are collinear:

$(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_6)$, $(\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_5)$, $(\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4)$,

$(\mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_3)$, $(\mathsf{U}_5, \mathsf{U}_6, \mathsf{U}_1)$, $(\mathsf{U}_4, \mathsf{U}_6, \mathsf{U}_2)$.



$[4]$ Known examples of type II DM self-motions

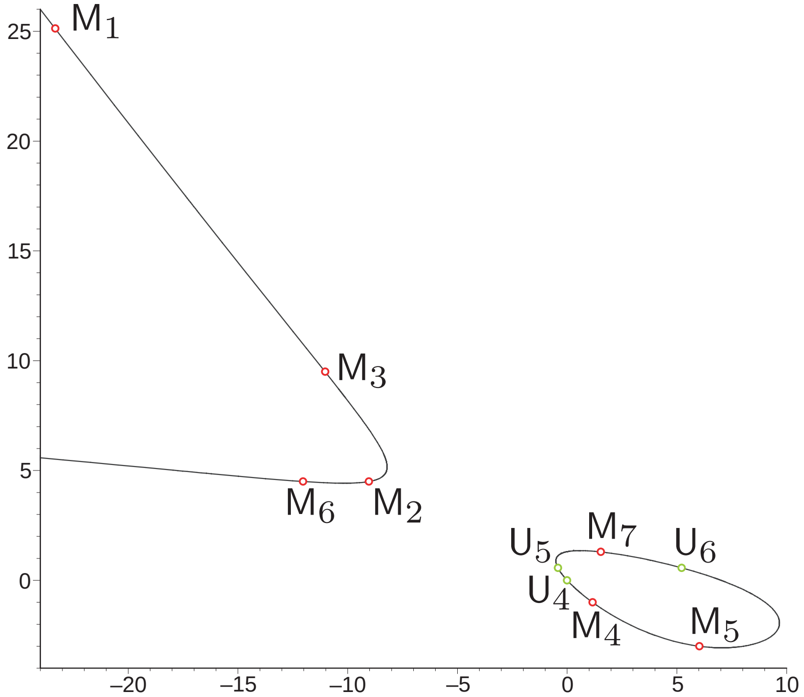
Theorem 5 (Proof is given in [1])
Assume that a self-motion of $\mathcal{S}$ is given which fulfills Assumption 1 and 2. If all anchor points of the corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ are real then it is always possible to attach a leg (e.g. $\overline{\mathsf{u}_2\mathsf{U}_4}$) to $\mathsf{u}_1, \ldots, \mathsf{U}_6$ such that we get a self-motion of a type 1 Bricard octahedron.

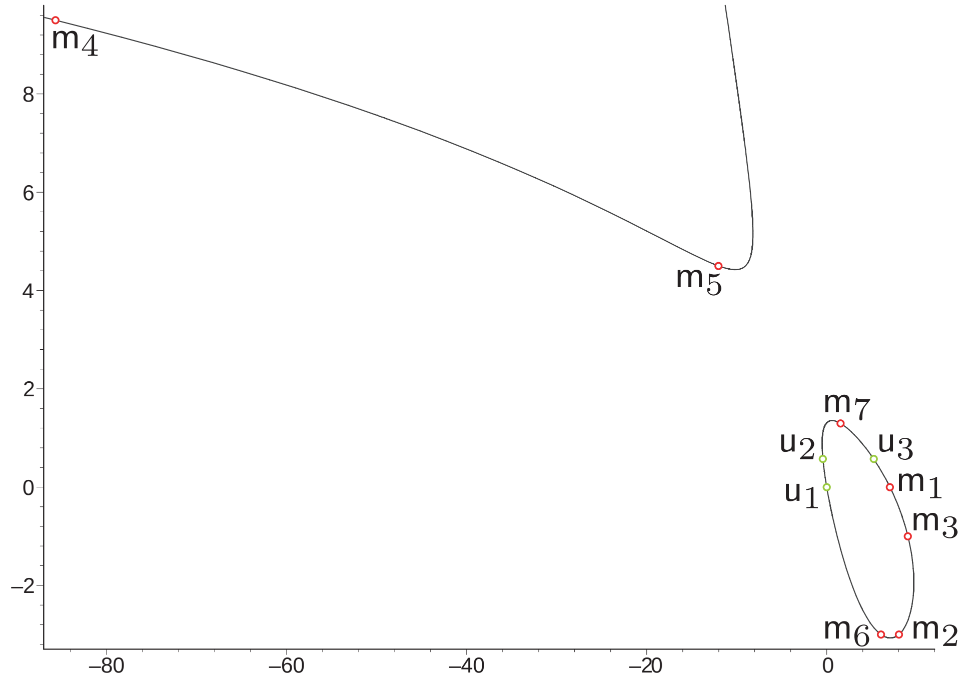
Corollary 1
All Bricard octahedra of type 1 have a type II DM self-motion.

Therefore we can construct easily non-architecturally singular SGP with a type II DM self-motion from any Bricard octahedron of type 1.

Remark: As all self-motions of type I DM and II DM, known to the speaker, are octahedral, the question arises if this property is a necessary condition for a general planar SGP (cf. Assumption 1 and 2) in order to have a 1-parametric self-motion? ◇

$[4]$ Example





Remark: $\mathrm{m}_1, \ldots, \mathrm{M}_6$ is a non-architecturally singular SGP with a 1-parametric self-motion. $\mathrm{m}_2, \ldots, \mathrm{M}_7$ is an architecturally singular SGP with a 1-parametric self-motion, where $e_0 = 0$ characterizes only one branch of the self-motion. For more details see [1]. ◇

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For [1-10] see the abstract. The remaining references [A-P] are as follows:

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