A new approach to the classification of architecturally singular parallel manipulators

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Abstract This is a new approach to the classification of architecturally singular parallel manipulators. We prove that a complete classification is possible in a geometric way if we distinguish the cases whether the linear line complex spanned by the carrier lines of the legs is always singular or not. The proof is based on the result that 5-legged planar parallel manipulators of Stewart Gough type which belong in every possible configuration to a singular linear line complex must possess 4 collinear anchor points. Moreover we list all types of 5-legged planar parallel manipulators with this property.

1 Introduction

It is well known (see e.g. Merlet [5]) that parallel manipulators of Stewart Gough type are singular if and only if the carrier lines of the prismatic legs belong to a linear line complex. Manipulators which are singular at every possible configuration are called architecturally singular (cf. Ma and Angeles [4]).

Karger presented in [1, Theorem 1] the four sufficient and necessary conditions for architecturally singular planar parallel manipulators with no 4 anchor points aligned. Moreover Karger proved in [2, Theorem 1 and 2] that architecturally singular non-planar manipulators must have 4 collinear anchor points. Finally in [2, Theorem 3], all types of architecturally singular manipulators, planar or non-planar, with 4 collinear anchor points are listed. Considered in retrospect, one can say that Karger divided the set $\mathscr{A}$ of architecturally singular manipulators into two classes with respect to the criterion of possessing 4 collinear anchor points or not.

Another attempt for the determination of $\mathscr{A}$ was done by Röschel and Mick [8]. They divided this set into planar and non-planar manipulators. But they were only able to give a geometric characterization for the planar case, which reads as follows: Planar Stewart Gough Platforms are architecturally singular iff (M_i, m_i), $i = 1, \ldots, 6$, are four-fold conjugate pairs of points with respect to a 3-dimensional linear manifold of correlations or one of the two sets $\{M_i\}$ and $\{m_i\}$ of anchor points is aligned.

In our approach we subdivide $\mathscr{A}$ with respect to the criterion whether the linear line complex spanned by the carrier lines of the legs is always singular or not. The reclassification done in section 4 is based on the theorem (cf. section 3) that 5-legged planar parallel manipulators of Stewart Gough type which belong in every possible configuration to a singular linear line complex must have 4 collinear anchor points.

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2 Fundamentals

A 5-legged planar parallel manipulator of Stewart Gough type consists of two sets $\mathscr{B} := \{M_1,\ldots,M_5\}$ and $\mathscr{P} := \{m_1,\ldots,m_5\}$ of coplanar anchor points in the Euclidean 3-space E^3 where the carrier planes of these sets are called base Σ_0 and platform Σ, respectively. W.l.o.g. we can choose coordinate systems in Σ_0 and Σ such that the points M_i and m_i have coordinates $\mathbf{M}_i = (A_i, B_i, 0)^T \in \mathbb{R}^3$ and $\mathbf{m}_i = (a_i, b_i, 0)^T \in \mathbb{R}^3$, $i = 1,\ldots,5$, with $a_1 = b_1 = b_2 = A_1 = B_1 = B_2 = 0$.

By using Euler Parameters (e_0, e_1, e_2, e_3) for the parametrization of the spherical motion group the coordinates $\mathbf{m}_i'$ of the platform points with respect to the fixed system Σ_0 can be written as $\mathbf{m}_i' = K^{-1}\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t}$ with

$$\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_1e_3+e_0e_2) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix}, \qquad (1)$$

the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and $K := e_0^2 + e_1^2 + e_2^2 + e_3^2$. Then Plücker coordinates of the lines $[M_i, m_i]$ are given by $\underline{\mathbf{l}}_i := (\mathbf{l}_i, \widehat{\mathbf{l}}_i) \in \mathbb{R}^6$ with $\mathbf{l}_i := \mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - K\mathbf{M}_i$ and $\widehat{\mathbf{l}}_i := \mathbf{M}_i \times \mathbf{l}_i$ for $i = 1,\ldots,5$.

A linear line complex $\mathscr{C} := (\mathbf{c}, \widehat{\mathbf{c}})$ is a three-dimensional linear manifold of lines with Plücker coordinates $(\mathbf{l}, \widehat{\mathbf{l}})$ which satisfy the linear equation $\mathbf{c}\cdot\widehat{\mathbf{l}} + \widehat{\mathbf{c}}\cdot\mathbf{l} = 0$, where $(\mathbf{c}, \widehat{\mathbf{c}}) \neq (\mathbf{0}, \mathbf{0})$ are the homogeneous coordinates of $\mathscr{C}$.

If the coordinates of $\mathscr{C}$ meet the Plücker condition $\mathbf{c}\cdot\widehat{\mathbf{c}} = 0$ it is called singular (otherwise regular). For $(\mathbf{c}, \widehat{\mathbf{c}}) \in \mathbb{R}^6$, apart from a common complex factor, we call a singular linear line complex *real*; for $(\mathbf{c}, \widehat{\mathbf{c}}) \in \mathbb{C}^6$ *complex*. For the rest of the article we summarize real and complex singular linear line complexes under the notation of singular linear line complexes.

If the Plücker coordinates $\underline{\mathbf{l}}_i \in \mathbb{R}^6$ of the 5 lines are given, a linear line complex $\mathscr{C}$ is uniquely determined as the solution of the linear system

$$\mathbf{c}\cdot\widehat{\mathbf{l}}_i + \widehat{\mathbf{c}}\cdot\mathbf{l}_i = 0 \quad \text{with} \quad \widehat{\mathbf{c}} = (c_1, c_2, c_3), \mathbf{c} = (c_4, c_5, c_6) \quad \text{for} \quad i = 1,\ldots,5 \qquad (2)$$

provided the Plücker coordinates are linearly independent, i.e. $rk(\underline{\mathbf{l}}_1 \ldots \underline{\mathbf{l}}_5) = 5$. In this case we always get a real singular linear line complex $\mathscr{C}$ if $\mathbf{c}\cdot\widehat{\mathbf{c}} = 0$ holds. As the homogeneous coordinates of this real singular linear line complex meet the Plücker condition the 6-tuple $(\mathbf{c}, \widehat{\mathbf{c}})$ also corresponds to a line in E^3; the so called axis. This axis may be a proper Euclidean line or it may be an ideal line. Therefore a singular linear line complex consists of all lines $(\mathbf{l}, \widehat{\mathbf{l}})$ which in the projective extension of E^3 intersect the axis $(\mathbf{c}, \widehat{\mathbf{c}})$ because the condition $\mathbf{c}\cdot\widehat{\mathbf{l}} + \widehat{\mathbf{c}}\cdot\mathbf{l} = 0$ is nothing else than the intersection condition of Sommerville.

If $rk(\underline{\mathbf{l}}_1 \ldots \underline{\mathbf{l}}_5) < 5$ then there always exists a singular linear line complex $(\mathbf{c}, \widehat{\mathbf{c}})$. We discuss this case in more detail for the obtained solution at the end of section 3.

2.1 Preparatory work and notation

The best way to compute the linear line complex $\mathscr{C}$ spanned by $\underline{\mathbf{l}}_1,\ldots,\underline{\mathbf{l}}_5$ is as follows: If we denote a generic line of $\mathscr{C}$ by $\underline{\mathbf{g}} := (g_4, g_5, g_6, g_1, g_2, g_3)$ then due to the linear dependence of $\underline{\mathbf{g}}, \underline{\mathbf{l}}_1,\ldots,\underline{\mathbf{l}}_5$ the expression $det(\underline{\mathbf{g}}, \underline{\mathbf{l}}_1,\ldots,\underline{\mathbf{l}}_5)$ must vanish. Now the coefficient of g_i in this equation equals c_i of the complex $(\mathbf{c}, \widehat{\mathbf{c}})$ in (2).

For the computation of $Q:\ c_1c_4 + c_2c_5 + c_3c_6 = 0$ (with MAPLE) we use the abbreviation r_{ij} for the entries of the matrix (1). Using this notation the equation Q has 1 043 682 terms. We denote the coefficients of $t_1^i t_2^j t_3^k$ by Q_{ijk} where $i + j \leq 4$ and $i + j + k \leq 5$ hold.

For the proof of the theorems given in section 3 we need the following 14 coefficients:

$Q_{005}[8634]$, $Q_{311}[2796]$, $Q_{400}[774]$, $Q_{310}[3900]$, $Q_{030}[14664]$,
$Q_{401}[582]$, $Q_{131}[5154]$, $Q_{040}[3174]$, $Q_{130}[7968]$, $Q_{003}[70717]$,
$Q_{041}[2004]$, $Q_{221}[5364]$, $Q_{301}[7800]$, $Q_{300}[4821]$,

where the number in the square brackets gives the number of terms. The Euler Parameters are substituted into these expressions at those points of the proofs where they are needed because then some geometric constraints must hold which simplify these expressions considerably. The resulting expressions will split up into several factors F_i where we denote the coefficients of $e_0^i e_1^j e_2^k e_3^l$ of F_i by F_i^{ijkl}. In most cases one of these factors is the homogenizing factor K which we cross out.

Moreover the collinearity condition of the points x,y,z is denoted by $coll(x,y,z)=0$.

3 Theorems on 5-legged planar parallel manipulators

Theorem 1. *If the legs of a 5-legged planar parallel manipulator belong in every possible configuration to a singular linear line complex $\mathscr{C}$ then 3 anchor points must be collinear.*

Proof. Assuming no 3 points are collinear ($\Rightarrow a_2b_3b_4b_5A_2B_3B_4B_5 \neq 0$), the proof is done by contradiction. Computing Q_{400} and Q_{401} yields $r_{31}a_2^2C[24]F_1[168]$ and $a_2C[24]F_2[126]$.

1. $F_2=F_3=0$: Computing the resultant of F_1^{2200} and F_2^{2200} with respect to b_3 yields $A_2B_3B_4B_5b_4b_5coll(M_3,M_4,M_5)(b_4-b_5)$. For $b_4=b_5$ the condition $F_1^{2200}=0$ yields the contradiction.
2. $C=0$: In this case we calculate the resultant of C^{1100} and C^{1010} with respect to B_3 which yields $b_3B_4B_5coll(m_3,m_4,m_5)(b_4B_5-B_4b_5)$. This implies $b_4=b_5B_4/B_5$. Now C^{1100} and C^{1010} can only vanish without contradiction (w.c.) for $b_3=b_5B_3/B_5$. Then Q_{040} factors into $b_5G_1[60]G_2[28]/B_5^4$.
   a. $G_2=0$: Now $G_2^{3300}=A_2B_3B_4b_5^3coll(M_3,M_4,M_5)$ yields the contradiction.
   b. $G_1=0$: We compute Q_{311} which splits up into $r_{13}r_{31}b_5H[48]/B_5^3$. Now the resultant of G_1^{1100} and H^{1100} with respect to a_3 yields
   $$A_2B_3B_4B_5coll(M_3,M_4,M_5)[B_5(A_2a_4-a_2A_4)+B_4(a_2A_5-A_2a_5)].$$
   We solve the last expression for a_4. Now G_1^{1100} and H^{1100} can only vanish w.c. for $a_3=[B_3(a_5A_2-a_2A_5)+a_2A_3B_5]/(A_2B_5)$. Then Q_{300} factors into
   $$a_2^2b_5B_3B_4coll(M_3,M_4,M_5)[e_1e_3(b_5+B_5)+e_0e_2(b_5-B_5)]L_1[12]L_2[214]/A_2^3B_5^3.$$
   Now $L_1^{2110}+L_1^{1201}=2A_2b_5$ as well as $L_2^{2200}=A_2^2B_3B_4b_5^2coll(M_3,M_4,M_5)$ yield the contradiction. □

Theorem 2. *If the legs of a 5-legged planar parallel manipulator belong in every possible configuration to a singular linear line complex $\mathscr{C}$ then 4 anchor points must be collinear.*

Proof. Assuming no 4 anchor points are collinear, the proof is done by contradiction:

Part A: We start with the case that m_1,m_2,m_3 as well as M_1,M_2,M_3 are collinear and pairwise distinct, i.e. $b_3=B_3=0$. Therefore we can assume $A_2A_3a_2a_3(a_2-a_3)(A_2-A_3)B_4B_5b_4b_5\neq 0$. Moreover we can assume that there do not exist any 3 collinear base points M_i,M_j,M_k and corresponding collinear platform points m_i,m_j,m_k where two points coincide, for $i,j,k\in\{1,\dots,5\}$. We consider the following coefficients:

$$Q_{310}=r_{31}^2r_{13}a_2a_3B_4B_5(A_2-A_3)F_1[8]F_2[24], \qquad Q_{040}=r_{31}F_3[32]F_4[140], \quad (3)$$
$$Q_{311}=r_{31}r_{13}B_4B_5(a_2A_3-A_2a_3)F_1[8]F_2[24], \qquad Q_{041}=r_{31}F_3[32]F_5[40]. \quad (4)$$

As $F_1[8]=0$ implies $m_4=m_5$ we have to discuss the following 4 cases:

1. $m_4=m_5, F_3=0$: For $a_4=a_5$ and $b_4=b_5$ we get $F_3^{1100}=b_5^2(A_4-A_5)(a_3A_2-a_2A_3)$ and $F_3^{1010}=b_5(A_4-A_5)[A_2(a_3a_2-a_5a_3)+A_3(a_5a_2-a_3a_2)]$. For $A_4=A_5$ the coefficient $Q_{131}=r_{31}r_{23}b_5(B_4-B_5)G_1[12]G_2[20]$ yields an easy contradiction. The remaining factors of F_3^{1100} and F_3^{1010} cannot vanish without contradiction.
2. $m_4=m_5$, $F_4=F_5=0$: The resultant of F_4^{2200} and F_5^{1010} with respect to A_2 yields $a_2a_3A_3(A_4B_5-B_4A_5)(a_2-a_3)coll(M_3,M_4,M_5)$. This expression can only vanish w.c. for $A_4=B_4A_5/B_5$. Now $F_4^{2200}=A_2A_3(a_2-a_3)(B_4-B_5)$ yields the contradiction.
3. $F_2=F_4=F_5=0$: Moreover we can assume $m_4\neq m_5$. The resultant of F_2^{1100} and F_4^{2200} with respect to a_2 implies $B_4=B_5$. Now the resultant of F_2^{1010} and F_4^{2200} with respect to a_2 yields $A_2B_5^2a_3^2(A_4-A_5)(A_2-A_3)(b_4-b_5)$.

   a. $A_4=A_5$: Now the coefficients $F_4^{2020}=a_2a_3A_5B_5(a_4-a_5)(A_2-A_3)$ and $F_4^{2110}=a_2a_3A_5B_5(b_4-b_5)(A_2-A_3)$ imply $A_5=0$. Then $F_5^{1100}=A_2A_3B_5(b_4-b_5)(a_2-a_3)$ and $F_5^{1010}=A_2A_3B_5(a_4-a_5)(a_2-a_3)$ yield the contradiction.
   b. $b_4=b_5$, $A_4\neq A_5$: Now the resultant of F_4^{2200} and F_4^{2110} with respect to a_2 yields $a_3^2b_5A_2B_5^2(A_4-A_5)^2(A_2-A_3)$, a contradiction.

4. $F_2=F_3=0$: Again we can assume $m_4\neq m_5$. Due to $F_2^{1100}=b_4b_5(B_4-B_5)(a_2A_3-a_3A_2)$ and $F_3^{1100}=b_4b_5(A_4-A_5)(a_2A_3-a_3A_2)$ we have to distinguish two cases:

   a. $M_4=M_5$: The resultant of F_2^{1010} and F_3^{1010} with respect to a_2 implies $a_4=b_4a_5/b_5$. Then $F_2^{1010}=0$ yields the contradiction.
   b. $a_2=a_3A_2/A_3, M_4\neq M_5$: Now $F_2^{1010}=0$ implies $b_4=b_5B_4/B_5$ and from $F_3^{1010}=0$ we get $a_4=[B_4(a_5A_3-a_3A_5)-B_5a_3A_4]/(A_3B_5)$. Then Q_{030} yields $r_{31}(A_2-A_3)^2(A_4-A_5)H_1[8]H_2[12]H_3[12]$. As all factors H_i $i=1,2,3$ yield an easy contradiction we set $A_4=A_5$. Then Q_{003} splits up into $r_{12}(B_4-B_5)^2L_1[10]L_2[12]L_3[20]$. Again all factors L_i $i=1,2,3$ yield an easy contradiction.

Part B: We discuss the case $m_1=m_2$ and M_1,M_2,M_3 collinear, i.e. $a_2=b_3=B_3=0$. Moreover we can assume $a_3b_4b_5A_2B_4B_5(a_4b_5-a_5b_4)\neq 0$. Now Q_{040} splits up into

$$a_3^2A_2^2r_{31}(a_5r_{31}+b_5r_{32})(a_4r_{31}+b_4r_{32})coll(M_3,M_4,M_5)N[12]. \tag{5}$$

1. $N=0$: From N^{1100} we get $A_4=A_5$. Then N^{1010} implies $A_3=A_4=A_5$, i.e. the collinearity of M_3,M_4,M_5. Now Q_{311} factors into $r_{13}r_{31}a_3^2A_2^2B_4B_5P_1[8]P_2[8]$. As $P_1=0$ implies $m_4=m_5$, a contradiction, we consider P_2^{1100} which yields $B_4=B_5$. From $P_2^{1100}=B_4(a_4b_5-a_5b_4)$ we get the contradiction.
2. M_3,M_4,M_5 collinear: Moreover we can exclude the case $A_3=A_4=A_5$.

   a. $B_4\neq B_5$: Under this assumption we can compute A_3 from the collinearity condition. Then Q_{041} factors into $r_{31}a_3^2A_2^2B_4B_5(A_4-A_5)P_1[8]P_2[8]/(B_4-B_5)^2$. As the $P_i=0$ for $i=1,2$ imply contradictions we set $A_4=A_5$ which yields $A_3=A_4=A_5$.
   b. $B_4=B_5$: Now the collinearity condition immediately implies $A_4=A_5$. We compute again Q_{041} which yields $a_3^2A_2^2B_5(A_3-A_5)(a_4b_5-a_5b_4)P_1[8]r_{31}^2$. As $P_1=0$ yields a contradiction we get again the case $A_3=A_4=A_5$.

W.l.o.g. we can assume for the remaining two parts that there do not exist any 3 legs those base anchor points *and* platform anchor points are collinear.

Part C: We consider the case where three platform points m_1,m_2,m_3 are collinear ($\Rightarrow$ $b_3=0$, $A_2B_3\neq 0$) and pairwise distinct. Moreover we assume that there do not exist two coinciding platform or base points. Therefore we can assume $a_2a_3(a_2-a_3)b_4b_5A_2B_3\neq 0$. We start by considering $Q_{400}=r_{31}a_2^2B_3S_1[12]S_2[112]$ and $Q_{401}=a_2B_3S_1[12]S_3[90]$.

1. $S_1=0$: S_1^{1100} implies $B_4=B_5$. Then S_1^{1010} simplifies to $B_4coll(m_3,m_4,m_5)$. From the collinearity condition we compute a_4 and insert it into Q_{310} which yields $r_{13}r_{31}(b_5r_{32}+(a_3-a_5)r_{31})T_1[28]T_2[24]/b_5$. The coefficients T_i^{1100} $i=1,2$ imply easy contradictions.

2. $S_2 = S_3 = 0$: The coefficients S_i^{2200} $i = 2,3$ immediately imply the contradiction.

Part D: Now we consider the case $m_1 = m_2$ and M_1, M_2, M_3 not collinear. We can assume $A_2B_3a_3b_4b_5 \neq 0$. As in part B we consider Q_{040} given in (5).

1. Again $N = 0$ can only vanish for $A_3 = A_4 = A_5$. Then the coefficient Q_{311} splits up into $r_{31}a_3A_2^2U_1[12]U_2[56]$. Now the coefficients of both factor yield easy contradictions.
2. M_3, M_4, M_5 collinear: Moreover we can exclude the case $A_3 = A_4 = A_5$. As we can assume w.l.o.g. that $A_4 \neq A_5$ we compute B_3 form the collinearity condition and plug it into Q_{041} which yield again easy contradictions. □

This theorem is sufficient for the reclassification done in the next section. The following theorem is given for the sake of completeness.

Theorem 3. *If the legs of a 5-legged planar parallel manipulator belong in every possible configuration to a singular linear line complex $\mathscr{C}$ then it is one of the following cases (after permutation of indices and exchanging $\mathscr{B}$ and $\mathscr{P}$):*

1. *$m_1, \dots, m_5$ are collinear,*
2. *$m_1 = m_2 = m_3$,*
3. *$m_1 = m_2, m_3, m_4$ collinear and $M_3 = M_4$,*
4. *$m_1 = m_2$, $m_3 = m_4$, M_1, M_2, M_5 collinear and M_3, M_4, M_5 collinear,*
5. *$m_1, \dots, m_4$ and $M_1, \dots, M_4$ collinear and pairwise distinct with $CR(m_1, \dots, m_4) = CR(M_1, \dots, M_4)$ where CR denotes the cross ratio.*

Proof. For the proof we assume that no 5 points are collinear and that no 3 points coincide, which correspond to the trivial cases 1 and 2, respectively.

Part A: In this part we assume that the 4 collinear points m_1, m_2, m_3, m_4, i.e. $b_3 = b_4 = 0$, are pairwise distinct. Moreover we assume that there do not exist any four collinear base or platform points where two points coincide. Therefore $b_5a_2a_3a_4(a_2 - a_3)(a_2 - a_4)(a_3 - a_4) \neq 0$ must hold. Computation of Q_{401} yields $r_{31}a_2(a_2 - a_3)b_5B_3B_4F[36]$.

1. We assume $B_3B_4 \neq 0$ and compute the resultant of F^{1100} and F^{1010} with respect to A_3 which yields $a_2b_5B_3B_4B_5(a_3 - a_4)[a_2(A_5B_4 - A_4B_5) - a_4A_2(B_4 - B_5)]$. As for $B_5 = 0$ the conditions $F^{1100} = 0$ and $F^{1010} = 0$ yield a contradiction we can assume $B_5 \neq 0$. Now we can express A_4 from the last factor of the resultant without loss of generality. Then F^{1100} implies $A_3 = (a_3A_2(B_5 - B_3) + a_2B_3A_5)/(a_2B_5)$. Now Q_{400} factors into $r_{31}^3a_2B_3B_4b_5(a_3 - a_4)A_2G[44]$. As for $A_2 = 0$ we get $M_1 = M_2, M_3, M_4, M_5$ collinear we compute the resultant of G^{1100} and G^{1010} with respect to B_3 which implies $B_4 = B_5$. From G^{1100} we get the contradiction.
2. $B_3 = 0$: Now Q_{040} splits up into $r_{31}^3b_5R[6]H[44]$ with

$$R := A_2A_3(a_2a_4 - a_3a_4) + A_2A_4(a_3a_4 - a_2a_3) + A_3A_4(a_2a_3 - a_2a_4), \tag{6}$$

   expressing the cross ratio relation $CR(m_1, m_2, m_3, m_4) = CR(M_1, M_2, M_3, M_4)$.

   a. $H = 0$, $R \neq 0$: We compute the resultant of H^{1100} and H^{1010} with respect to A_2 which yields $a_2a_3a_4^2b_5A_3A_5B_4(a_2 - a_3)coll(M_3, M_4, M_5)$.
      i. $B_4 = 0$: From H^{1100} we get b_5B_5R, a contradiction.
      ii. $A_3 = 0, B_4 \neq 0$: Now H^{1100} implies $A_5 = A_4B_5(a_2 - a_4)/(a_4B_4)$. Then $H^{110} = a_2a_3A_2A_4B_5(a_2 - a_4)$ cannot vanish without contradiction.
      iii. $A_5 = 0, A_3B_4 \neq 0$: Now H^{1100} implies $B_5 = a_4A_2A_3B_4(a_2 - a_3)/R$. Then Q_{041} yields an easy contradiction.
      iv. $A_3A_5B_4 \neq 0$: W.l.o.g. we can compute A_5 from $coll(M_3, M_4, M_5) = 0$. Then H^{1100} implies $B_4 = B_5(A_3a_4 - a_3A_4)/(a_4A_3)$. H^{1010} yields the contradiction.

   b. $R = 0$: This cases splits up into the following subcases:
      i. $a_4(A_4A_3 - A_2A_3) + a_3(A_2A_4 - A_3A_4) \neq 0$: Under this assumption we can compute a_2 from $R = 0$. Then Q_{310} can only vanish w.c. for

- $B_4 = 0$: This yields item 5 of Theorem 3.
- $B_5 = 0$, $B_4 \neq 0$: Now Q_{005} yields $r_{13}r_{31}a_3^2a_4A_2^2B_4^2b_5A_3(A_2-A_3)^2(a_3-a_4)L[20]$. Then the resultant of L^{1100} and L^{1010} with respect to A_4 yields $a_3a_4b_5A_3A_5(a_3-a_4)(A_3-A_5)$, a contradiction.

ii. $a_4(A_4A_3-A_2A_3)+a_3(A_2A_4-A_3A_4)=0$, $a_3A_4-a_4A_3\neq 0$: Now we can express A_2. Then R can only vanish w.c. for
- $A_4 = 0$: For this case Q_{130} yields an easy contradiction.
- $A_3 = A_4 \neq 0$: Now Q_{130} implies $A_3 = A_4 = A_5$. Q_{005} yields the contradiction.

iii. $a_4(A_4A_3-A_2A_3)+a_3(A_2A_4-A_3A_4)=0$, $A_3 = a_3A_4/a_4$: Then the first equation implies $A_4 = 0$. Again Q_{130} yields an easy contradiction.

Part B: In this part we have 4 collinear points where two coincide. W.o.l.g. we set $a_2 = b_3 = b_4 = 0$. We can assume $A_2a_3a_4b_5 \neq 0$. Then the computation of Q_{040} yields $r_{31}^3A_2^2a_3^2a_4^2b_5(A_3-A_4)(a_5r_{31}+b_5r_{32})coll(M_3,M_4,M_5)$.

1. $A_3 = A_4$: Q_{301} splits up into $r_{31}r_{23}A_2^2a_3^2a_4^2b_5(B_3-B_4)^2(A_4-A_5)(a_5r_{31}+b_5r_{32})$. As $M_3 = M_4$ yields item 3 of Theorem 3 we can assume $M_3 \neq M_4$ for the remaining case study. If $A_3 = A_4 = A_5$ holds Q_{221} factors into $r_{23}r_{31}^2a_3a_4b_5A_2^2(B_3-B_4)N[20]$. Then the resultant of N^{1100} and N^{1010} with respect to B_5 yields $a_3a_4b_5B_3B_4(B_3-B_4)(a_3-a_4)$.

   a. $B_3 = 0$: Now $N^{1100} = a_3b_5B_4B_5$. For $B_5 = 0$ we get a special case of item 3.
   b. $a_3 = a_4$: Then N^{1100} yields $a_3b_5B_5(B_3-B_4)$. This yields again $B_5 = 0$ and we get a special case of item 4 of Theorem 3.

2. M_3, M_4, M_5 collinear, $A_3 \neq A_4$: We express B_5 from the collinearity condition. Now Q_{131} equals $r_{31}^2A_2^2a_3a_4b_5(r_{13}(A_3-A_4)+2(B_3-B_4)r_{23})P[24]$. Then the resultant of P^{1100} and P^{1010} with respect to B_4 yield $a_3a_4b_5B_3B_4(a_3-a_4)(A_3-A_5)$.

   a. $B_3 = 0$: In this case we get $P^{1100} = a_3b_5B_4(A_5-A_3)$. This implies $A_3 = A_5$, which is again a special case of item 3 of Theorem 3.
   b. $A_3 = A_5$, $B_3 \neq 0$: Now P^{1100} cannot vanish without contradiction.
   c. $a_3 = a_4$, $B_3(A_3-A_5)\neq 0$: We get $P^{1100} = a_4b_5(-B_3A_5+B_3A_4+B_4A_5-B_4A_3)$. If the last factor vanishes we get item 4 of Theorem 3. □

In the following we give a geometric interpretation of the 5 cases listed in Theorem 3: In the cases 1-4 the five carrier lines $[M_i, m_i]$ always belong to a real singular linear line complex.

ad 1) The axis equals $[m_1,\dots,m_5]$.
ad 2) The axis is given by the intersection line of the planes $[m_1, M_4, m_4]$ and $[m_1, M_5, m_5]$.
ad 3) The axis equals $[m_1 = m_2, s]$ where s denotes the intersection point, which may be an ideal point, of the line $[M_5, m_5]$ and the plane $[m_3, m_4, M_3 = M_4]$.
ad 4) The axis is determined by the intersection line of the planes $[M_1, M_2, m_1 = m_2]$ and $[M_3, M_4, m_3 = m_4]$ which passes through M_5.
ad 5) The lines $[M_1, m_1], \dots, [M_4, m_4]$ belong to a regulus $\mathscr{R}$. We must distinguish 3 cases:

   a. If $[M_5, m_5]$ intersects $\mathscr{R}$ in real points s_i then the 5 lines belong to 2 real singular linear line complexes $\mathscr{C}_i$, $i = 1, 2$. The axis of $\mathscr{C}_i$ corresponds to the line of the complementary regulus $\mathscr{R}'$ of $\mathscr{R}$ which contains s_i.
   b. If $[M_5, m_5]$ touches $\mathscr{R}$ in the point s then the 5 lines belong to a real singular linear line complex $\mathscr{C}$. The axis equals the generator of $\mathscr{R}'$ which contains s.
   c. If $[M_5, m_5]$ intersects $\mathscr{R}$ in conjugate complex points s and $\overline{s}$ then the 5 lines belong to conjugate complex singular linear line complexes $\mathscr{C}$ and $\overline{\mathscr{C}}$. Therefore all 5 lines intersect two conjugate complex lines (cf. [7]).

Remark 1. An interesting question arises: Can the free parameters of the base and platform points of the 5^{th} case be chosen such that the five lines always belong to a real singular linear line complex, if neither all base nor all platform points are collinear? This problem remains open and is dedicated to future research.

4 Reclassifying architecturally singular parallel manipulators

For a classification of $\mathscr{A}$ it is sufficient to classify all architecturally singular parallel manipulator with $rk(\underline{\mathbf{l}}_1,\ldots,\underline{\mathbf{l}}_6)=5$ because manipulators with $rk(\underline{\mathbf{l}}_1,\ldots,\underline{\mathbf{l}}_6)<5$ can only be special cases of those. In the following we want to subdivide $\mathscr{A}$ with respect to the criterion if the legs belong in every possible configuration to a *real* singular linear line complex (subset $\mathscr{A}_S$) or not (subset $\mathscr{A}_R$). Now the classification can be done as follows:

Corollary 1. *An architecturally singular parallel manipulator with $rk(\underline{\mathbf{l}}_1,\ldots,\underline{\mathbf{l}}_6)=5$ belongs to the subset $\mathscr{A}_S$ if it fulfills one of the following conditions (after permutation of indices and exchanging M_i with m_i for $i=1,\ldots,6$):*

(a) $m_1,\ldots,m_6$ are collinear,
(b) $m_1=m_2=m_3$ and $M_4=M_5=M_6$,
(c) $m_1=m_2=m_3$, $m_1,\ldots,m_5$ are collinear and $M_4=M_5$,
(d) $m_1=m_2=m_3=m_4$,
(e) $m_1=m_2$, $m_3=m_4$, $M_5=M_6$, M_1,M_2,M_5 and M_3,M_4,M_5 are collinear.

Otherwise it belongs to the subset $\mathscr{A}_R$. Then the point pairs (M_i,m_i), $i=1,\ldots,6$ are exactly 11-fold conjugate pairs of points with respect to a 10-dimensional linear manifold of correlations, i.e. $rk(\mathbf{u}_1,\ldots\mathbf{u}_6)=5$ with

$$\mathbf{u}_i := (1,a_i,b_i,c_i,A_i,B_i,C_i,a_iA_i,a_iB_i,a_iC_i,b_iA_i,b_iB_i,b_iC_i,c_iA_i,c_iB_i,c_iC_i)^T,$$

where (a_i,b_i,c_i) and (A_i,B_i,C_i) are the coordinates of the platform resp. base points.

Proof. Due to Theorem 2 and the result of Karger [2, Theorem 1 and 2] there do not exist architecturally singular parallel manipulators with no 4 points collinear which belong in each configuration to a singular linear line complex.

On the other hand all types of architecturally singular parallel manipulators with 4 points collinear were listed by Karger [2, Theorem 3]. For the items 1-10 of this list the conditions given for the classification are also sufficient conditions for an manipulator to be architecturally singular. For the entries 11 and 12 (the degenerated planar cases) such sufficient conditions where given by the author in [6]. It can easily be checked by computations that these conditions yield $rk(\underline{\mathbf{l}}_1,\ldots,\underline{\mathbf{l}}_6)=5$ in all 12 cases.

Moreover it is not difficult to determine the line (axis) intersecting all legs of the manipulators given in (a)-(e), which correspond to the entries 1, 2, 4, 5, 9 of Karger's list. For more details see [2, Subsection 5.1].

Now we prove the second part of this corollary. For the planar cases (i.e. architecturally singular parallel manipulators with no 4 points on a line plus the entries 3, 11 and 12 of Karger's list) $rk(\mathbf{u}_1,\ldots\mathbf{u}_6)=5$ follows from the geometric characterization given by Röschel and Mick [8]. For the remaining non-planar cases (i.e. entries 6, 7, 8, 10 of Karger's list) $rk(\mathbf{u}_1,\ldots\mathbf{u}_6)=5$ can be verified by computation. Now we must show that for the cases 3, 6, 7, 8, 10, 11, 12 the carrier lines of the legs do not belong to a real singular linear line complex in every possible configuration. The proof can easily be done by contradiction as follows:

If the manipulator belongs in every possible configuration to a singular linear line complex then the 5-legged manipulator[1] which results from removing the i^{th} leg must also have this property. Hence, we compute the linear line complex $(\mathbf{c},\widehat{\mathbf{c}})\in\mathbb{R}^6$ and the equation $Q_i:\mathbf{c}\cdot\widehat{\mathbf{c}}=0$ as described in subsection 2.1. Now for each of the above mentioned cases the equations Q_i for $i=1,\ldots,6$ must be fulfilled identically.

By means of computation it can be verified that this is not the case. □

Remark 2. We hope that the subdivision of $\mathscr{A}$ into $\mathscr{A}_S$ and $\mathscr{A}_R$ can be helpful for finding a purely geometric way for the determination of $\mathscr{A}$.

[1] Note that the platform and the base of the resulting 5-legged parallel manipulator must not be planar.

5 Future work

1. Determining all 5-legged non-planar parallel manipulators which belong in every possible configuration to a singular linear line complex. We conjecture that this problem only has the solutions 1-3 of Theorem 3.
2. It would also be interesting to determine the whole set $\mathscr{S}$ of parallel manipulators possessing the following property: In each singular configuration of the manipulator the carrier lines of the legs belong to a (real) singular linear line complex. This set $\mathscr{S}$ is not empty because the set $\mathscr{A}_S$ is a subset of $\mathscr{S}$. But $\mathscr{S}$ also contains parallel manipulators which are not architecturally singular, e.g. the following one:
   - $m_1 = m_2 = m_3$, the base is planar and M_4, M_5, M_6 are collinear.

   Due to the examples known to the author, we conjecture that all parallel manipulators belonging to the set $\mathscr{S}$ have 4 collinear anchor points.
3. Solving the problems formulated in Remark 1 and 2.

6 Conclusion

We presented a new approach to the classification of the set $\mathscr{A}$ of architecturally singular parallel manipulators. We proved that a complete classification is possible in a geometric way (cf. Corollary 1) if we subdivide $\mathscr{A}$ with respect to the criterion if the legs belong in every possible configuration to a *real* singular linear line complex (subset $\mathscr{A}_S$) or not (subset $\mathscr{A}_R$).

The proof was based on the fact that 5-legged planar parallel manipulators of Stewart Gough type which belong in every possible configuration to a singular linear line complex must possess 4 collinear anchor points (cf. Theorem 1 and 2). Moreover we listed all types of 5-legged planar parallel manipulators with this property (cf. Theorem 3).

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