Parallel manipulators in terms of dual Cayley-Klein parameters

G. Nawratil

Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Austria, e-mail: nawratil@geometrie.tuwien.ac.at

Abstract. Cayley-Klein parameters are an alternative to Euler parameters for describing the spherical motion group. Based on Study's and Kotelnikov's "Principle of Transference" one can use dual Cayley-Klein parameters for the motion study of oriented lines in Euclidean 3-space. In this paper we focus on the transformation of points in terms of dual Cayley-Klein parameters and show that these parameters imply a very compact symbolic expression of the sphere condition, which is the central equation for computational algebraic kinematics of parallel manipulators of Stewart-Gough type. Moreover it is shown that the compactness of this formulation is passed on to the symbolic expression of the singularity loci. We also adopt our results to the analogue in planar kinematics and point out the difference to the well-known approach of isotropic coordinates.

Key words: Dual Cayley-Klein parameters, Sphere condition, Circle condition, Singularity loci

1 Introduction

It is well-known that planar displacements of the Euclidean plane can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}, \tag{1}$$

where $(x_0, y_0)^T$ (resp. $(x, y)^T$) are the coordinates of a point P with respect to the fixed frame (resp. moving frame), φ is the angle of rotation and $(m, n)^T$ the translation vector. By interpreting the Euclidean plane as Gaussian plane, Eq. (1) can be rewritten as:

$$x + yi \mapsto x_0 + y_0 i = e^{i\varphi}(x + yi) + (m + ni), \tag{2}$$

where i denotes the complex unit. In addition one can set:

$$p_0 := x_0 + y_0 i, \quad \overline{p}_0 := x_0 - y_0 i \tag{3}$$

and replace the original coordinates x_0 and y_0 by:

$$x_0 = (p_0 + \overline{p}_0)/2, \quad y_0 = (p_0 - \overline{p}_0)/(2i). \tag{4}$$

The obtained pair $(p_0, \overline{p}_0)$ are the so-called isotropic coordinates of P with respect to the fixed system. Analogously one gets the isotropic coordinates $(p, \overline{p}) := (x + yi, x - yi)$ of P with respect to the moving system and the isotropic coordinates $(\tau, \overline{\tau}) := (m + ni, m - ni)$ of the translation. Thus Eq. (2) equals $p \mapsto p_0 = e^{i\varphi} p + \tau$. In order to make this formulation algebraic, we replace $e^{i\varphi}$ by the complex number κ, which has to fulfill the normalizing condition $\kappa\overline{\kappa} = 1$. Hence we get the compact notation:

$$p \mapsto p_0 = \kappa p + \tau \quad \text{with} \quad \kappa\overline{\kappa} = 1. \tag{5}$$

A historical overview on planar kinematics based on isotropic coordinates is given in the work [21] by Wampler, in which these coordinates are used to determine the degree and circularity of curves traced by planar linkages. Further references and historical remarks can be found in the book of Wunderlich [23] where these coordinates are called minimal coordinates. Beside [18] most of the recent work using isotropic coordinates was done by Wampler (cf. [22] and all self-references therein).

1.1 Motivation and outline of the paper

Based on the algebraic formulation Eq. (5) we can derive the basic equation for the study of planar parallel manipulators with RPR legs (Fig. 1 left); namely the condition that a point P of the moving system is located on a circle with radius R centered at the point B with fixed coordinates $(u_0, v_0)^T$. This so-called circle condition reads as follows:

$$(\kappa p + \tau - b_0)(\overline{\kappa}\,\overline{p} + \overline{\tau} - \overline{b}_0) - R^2 = 0, \tag{6}$$

where $(b_0, \overline{b}_0) := (u_0 + v_0 i, u_0 - v_0 i)$ denote the isotropic coordinates of B with respect to the fixed system. Expanding this equation shows that it has 10 terms and that it is inhomogeneous quadratic in the motion parameters $\kappa, \overline{\kappa}, \tau, \overline{\tau}$.

Nevertheless the symbolic expression of Eq. (6) is very compact, a lot of recent publications (e.g. [2, 7, 9, 11, 19]) use the circle condition formulated in terms of Blaschke-Grünwald (BG) parameters, which has 26 terms. A motive for doing this is that one ends up with a homogenous quadratic equation in the BG parameters, thus methods of projective algebraic geometry can be applied. This gives reason to ask for a formulation, which has both benefits (compactness and homogeneity). We present such a formulation as a special case of a more general approach taken for spatial kinematics. In detail the paper is structured as follows:

We close Section 1 by giving a very brief review on the quaternionic formulation of displacements in Euclidean spaces of dimension 2 and 3. In Section 2 we discuss the transformation of points with respect to dual Cayley-Klein (CK) parameters and use them in Section 3 for presenting the most compact symbolic expression of the sphere condition and the singularity loci of Stewart-Gough (SG) manipulators (Fig. 1 right), which is known to the author. Moreover, the obtained results can easily be adopted for planar kinematics, thus we also get a solution to our motivating question; namely a homogenous circle condition with only 10 terms.

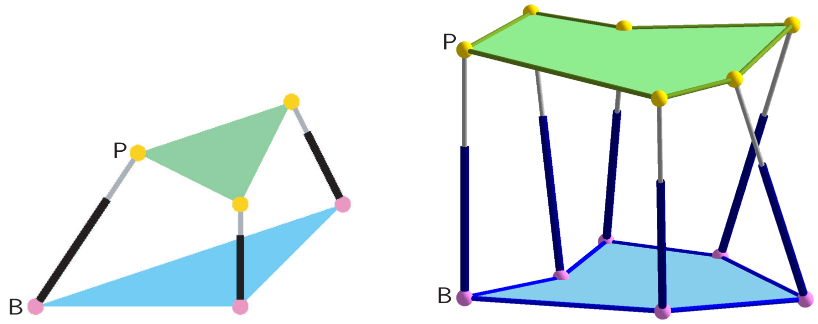

Fig. 1 (Left) 3-dof RPR planar parallel manipulator: The platform is connected via three RPR-legs to the base. (Right) SG manipulator: The platform is connected via six SPS-legs to the base. For the planar as well as the spatial mechanism the anchor points of the legs are denoted by P and B, respectively, and in both cases only the prismatic joints are active.

1.2 Quaternionic formulation of displacements

$\mathfrak{Q} := q_0 + q_1\mathfrak{i} + q_2\mathfrak{j} + q_3\mathfrak{k}$ with $q_0,\ldots,q_3 \in \mathbb{R}$ is an element of the skew field of quaternions $\mathbb{H}$, where $\mathfrak{i},\mathfrak{j},\mathfrak{k}$ are the so-called quaternion units. The conjugated quaternion to $\mathfrak{Q}$ is given by $\widetilde{\mathfrak{Q}} := q_0 - q_1\mathfrak{i} - q_2\mathfrak{j} - q_3\mathfrak{k}$. Moreover, $\mathfrak{Q}$ is called unit-quaternion for $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

Displacements in spatial kinematics can be formulated in terms of dual quaternions $\mathbb{H} + \varepsilon\mathbb{H}$, where ε is the dual unit with the property $\varepsilon^2 = 0$. An element $\mathfrak{E} + \varepsilon\mathfrak{T}$ of $\mathbb{H} + \varepsilon\mathbb{H}$ with $\mathfrak{E} := e_0 + e_1\mathfrak{i} + e_2\mathfrak{j} + e_3\mathfrak{k}$ and $\mathfrak{T} := t_0 + t_1\mathfrak{i} + t_2\mathfrak{j} + t_3\mathfrak{k}$ is called dual unit-quaternion if $\mathfrak{E}$ is an unit-quaternion and following condition holds:

$$e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3 = 0. \tag{7}$$

It is well-known (e.g. [10, Section 3.3.2.2]) that displacements of points in the Euclidean 3-space can be expressed by dual unit-quaternions $\mathfrak{E} + \varepsilon\mathfrak{T}$ as follows:

$$\mathfrak{P} \mapsto \mathfrak{P}_0 = \mathfrak{E} \circ \mathfrak{P} \circ \widetilde{\mathfrak{E}} + (\mathfrak{T} \circ \widetilde{\mathfrak{E}} - \mathfrak{E} \circ \widetilde{\mathfrak{T}}), \tag{8}$$

where $\circ$ denotes the quaternion multiplication and $\mathfrak{P} := x\mathfrak{i} + y\mathfrak{j} + z\mathfrak{k}$ (resp. $\mathfrak{P}_0 := x_0\mathfrak{i} + y_0\mathfrak{j} + z_0\mathfrak{k}$) is the embedding of a point P with Cartesian coordinates $\mathbf{p} = (x,y,z)^T$ (resp. $\mathbf{p}_0 = (x_0,y_0,z_0)^T$) with respect to the moving (resp. fixed) frame into $\mathbb{H}$.

As both dual unit-quaternions $\pm(\mathfrak{E} + \varepsilon\mathfrak{T})$ correspond to the same Euclidean motion, one considers the homogeneous 8-tuple $(e_0,\ldots,e_3,t_0,\ldots,t_3)\mathbb{R}$, which are the well-known Study parameters [20] of the Euclidean motion group SE(3). Note that $(e_0,\ldots,e_3)\mathbb{R}$ are the so-called Euler parameters of the spherical motion group.

Restricting the Study parameters to planar Euclidean displacements within the plane $x_3 = 0$ implies $e_1 = e_2 = t_0 = t_3 = 0$ (cf. [10, Remark 3.38]) , thus one ends up with the homogenous quadruple $(e_0,e_3,t_1,t_2)\mathbb{R}$, which are the already mentioned BG parameters [1, 6].

2 Dual Cayley-Klein parameters

According to the recently published work [17], which also contains a historical overview and a detailed list of references on CK parameters, the formulation of spherical displacements of points based on Euler parameters $(e_0,\ldots,e_3)\mathbb{R}$; i.e.

$$\mathfrak{P} \mapsto \mathfrak{E}\circ\mathfrak{P}\circ\widetilde{\mathfrak{E}} \quad \text{with} \quad e_0^2+e_1^2+e_2^2+e_3^2=1 \tag{9}$$

can be rewritten in terms of CK parameters $\alpha,\beta\in\mathbb{C}$ as follows:

$$\mathbf{P} \mapsto \mathbf{EPE}^* \quad \text{with} \quad \alpha\overline{\alpha}+\beta\overline{\beta}=1, \tag{10}$$

where

$$\mathbf{P} := \begin{pmatrix} z & \overline{p} \\ p & -z \end{pmatrix}, \quad \mathbf{E} := \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} \quad \text{and} \quad \mathbf{E}^* := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}. \tag{11}$$

Note that the upper index * denotes the transposed conjugate of a matrix. Moreover, $\mathbf{P}$ is the embedding of the point P with Cartesian coordinates $\mathbf{p}=(x,y,z)^T$ into the set of complex 2×2 matrices, which can be seen as a spatial generalization of isotropic coordinates according to [17]. The introduction of CK parameters can be completed by giving their relations to the Euler parameters, which read as follows:

$$\alpha := e_0+e_3 i \quad \text{and} \quad \beta := e_2+e_1 i. \tag{12}$$

Remark 1. Note that their exists the alternative formulation $\mathbf{P}\mapsto\mathbf{E}^*\mathbf{PE}$, where the matrix $\mathbf{P}$ of Eq. (11) and the formula for β of Eq. (12) are replaced by:

$$\mathbf{P} := \begin{pmatrix} zi & pi \\ -\overline{p}i & -zi \end{pmatrix} \quad \text{and} \quad \beta := -e_2+e_1 i. \tag{13}$$

We prefer the other convention, as the connection with the isotropic coordinates in case of planar kinematics is straightforward. Moreover, one can compute $\|\mathbf{p}\|^2$ simply as $-\det\mathbf{P}$. $\diamond$

Due to the "Principle of Transference", which dates back to Kotelnikov [12] and Study [20], this formulation of a spherical displacement of points can also be applied to the spatial displacements of oriented lines by dualizing the complete framework; i.e. complex numbers are substituted by dual complex numbers. Up to the author's knowledge the resulting dual CK parameters have only be used for this purpose [3, 17], but never for the description of displacements of points in Euclidean 3-space. For doing this, we use the relation to quaternions and a more detailed formulation of Eq. (8), which reads as follows (cf. [10, page 498]):

$$1+\varepsilon\mathfrak{P} \mapsto 1+\varepsilon\mathfrak{P}_0 = (\mathfrak{E}+\varepsilon\mathfrak{T})\circ(1+\varepsilon\mathfrak{P})\circ(\widetilde{\mathfrak{E}}-\varepsilon\widetilde{\mathfrak{T}}). \tag{14}$$

A straightforward translation into terms of complex 2×2 matrices yields:

$$(\mathbf{I}i+\varepsilon\mathbf{P}) \mapsto (\mathbf{I}i+\varepsilon\mathbf{P}_0) = (\mathbf{E}+\varepsilon\mathbf{T})(\mathbf{I}i+\varepsilon\mathbf{P})(\mathbf{E}^*-\varepsilon\mathbf{T}^*), \tag{15}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix and

$$\mathbf{P}_0 := \begin{pmatrix} z_0 & \overline{p}_0 \\ p_0 & -z_0 \end{pmatrix}, \quad \mathbf{T} := \begin{pmatrix} \overline{\gamma} & -\delta \\ \overline{\delta} & \gamma \end{pmatrix} \quad \text{with} \quad \gamma := t_0 + t_3 i, \quad \delta := t_2 + t_1 i. \tag{16}$$

Expanding and simplifying Eq. (15) implies:

$$(\mathbf{I}i + \varepsilon\mathbf{P}) \mapsto (\mathbf{I}i + \varepsilon\mathbf{P}_0) = \mathbf{I}i + \varepsilon(\mathbf{EPE}^* + i\mathbf{TE}^* - i\mathbf{ET}^*). \tag{17}$$

In order that our later obtained symbolic expressions (e.g. Eq. (26)) are free of the complex unit i we make the following redefinition:

$$\mathbf{S} := i\mathbf{T} = \begin{pmatrix} \lambda & \overline{\mu} \\ \mu & -\overline{\lambda} \end{pmatrix} \quad \text{and} \quad \mathbf{S}^* := -i\mathbf{T}^* = \begin{pmatrix} \overline{\lambda} & \overline{\mu} \\ \mu & -\lambda \end{pmatrix} \tag{18}$$

with

$$\lambda := t_3 + t_0 i \quad \text{and} \quad \mu := t_1 + t_2 i, \tag{19}$$

thus we finally get the desired representation, which is summarized next.

Theorem 1. *Any spatial displacement of points* P *can be written as:*

$$\mathbf{P} \mapsto \mathbf{P}_0 = \mathbf{EPE}^* + \mathbf{SE}^* + \mathbf{ES}^*, \tag{20}$$

where the four involved parameters $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ *fulfill the normalizing condition* $\Phi = 1$ *with*

$$\Phi := \alpha\overline{\alpha} + \beta\overline{\beta} \tag{21}$$

and the analogue of the Study condition (7), which is given by $\Psi = 0$ *with*

$$\Psi := (\alpha\lambda - \overline{\alpha}\overline{\lambda}) + (\beta\mu - \overline{\beta}\overline{\mu}). \tag{22}$$

Moreover, the mapping of Eq. (20) is a spatial displacement of points for each quadruple $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ *fulfilling* $\Phi = 1$ *and* $\Psi = 0$.

For the planar case we get $\beta = 0$ and $\lambda = 0$ due to $e_1 = e_2 = 0$ and $t_0 = t_3 = 0$, respectively (cf. end of Section 1.2). Therefore the following corollary holds:

Corollary 1. *Any planar displacement of points* P *can be written as:*

$$p \mapsto p_0 = \alpha(\alpha p + 2\mu) \quad \textit{with} \quad \alpha, \mu \in \mathbb{C} \quad \textit{and} \quad \alpha\overline{\alpha} = 1. \tag{23}$$

Moreover, the mapping of Eq. (23) is a planar displacement of points for each bituple $\alpha, \mu \in \mathbb{C}$ *fulfilling* $\alpha\overline{\alpha} = 1$.

Remark 2. Based on Corollary 1 we can point out the relation

$$\kappa = \alpha^2 \quad \text{and} \quad \tau = 2\alpha\mu \tag{24}$$

between the parameters $\kappa, \tau \in \mathbb{C}$ of Eq. (5) and the parameters $\alpha, \mu \in \mathbb{C}$ of Eq. (23), which is a non-linear one. ◇

3 Application to parallel manipulators

For symbolic computations in robotics, we consider $\overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu}$ as independent variables; i.e. they are uncoupled from $\alpha, \beta, \lambda, \mu$. Under this assumption Study's kinematic mapping (e.g. [16, Section 2]) can be reformulated as follows:

Corollary 2. *There is a bijection between SE(3) and 8-tuples of complex numbers* $(\alpha, \beta, \lambda, \mu, \overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu})\mathbb{R}$ *fulfilling* $\Psi = 0$ *with* $(\alpha, \beta, \overline{\alpha}, \overline{\beta}) \neq (0,0,0,0)$ *and the condition that the quadruple* $(\overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu})$ *is the conjugate quadruple of* $(\alpha, \beta, \lambda, \mu)$.

Based on this result the sphere condition, that the platform point P is located on a sphere with radius R centered in the base point B with fixed coordinates $\mathbf{b}_0 = (u_0, v_0, w_0)^T$, can be computed as (cf. end of Remark 1):

$$\Phi^2 R^2 + \det(\mathbf{P}_0 - \mathbf{B}_0) = 0 \quad \text{with} \quad \mathbf{B}_0 := \begin{pmatrix} w_0 & \overline{b}_0 \\ b_0 & -w_0 \end{pmatrix}, \tag{25}$$

where the coefficient Φ^2 of R^2 homogenizes the equation. Doing the corresponding tricky summation of Husty [8] (see also [15]) by adding Ψ^2 to the left hand-side, shows that Φ factors out. The remaining quadratic factor Σ reads as follows:

$$\begin{aligned} &\alpha^2 p\overline{b}_0 - \beta^2 p b_0 + \overline{\alpha}^2 \overline{p} b_0 - \overline{\beta}^2 \overline{p}\overline{b}_0 + (\alpha\overline{\alpha} + \beta\overline{\beta})(R^2 - z^2 - w_0^2 - b_0\overline{b}_0 - p\overline{p}) \\ &+ 2(\alpha\overline{\alpha} - \beta\overline{\beta}) z w_0 - 2\alpha\beta p w_0 + 2\alpha\overline{\beta} z \overline{b}_0 + 2\overline{\alpha}\beta z b_0 - 2\overline{\alpha}\overline{\beta}\overline{p} w_0 \\ &- 2(\beta\mu + \overline{\beta}\overline{\mu})(w_0 + z) + 2(\alpha\lambda + \overline{\alpha}\overline{\lambda})(w_0 - z) + 2(\overline{\alpha}\overline{\mu} + \beta\lambda) b_0 \\ &+ 2(\alpha\mu + \overline{\beta}\,\overline{\lambda})\overline{b}_0 + 2(\beta\overline{\lambda} - \alpha\overline{\mu}) p + 2(\overline{\beta}\lambda - \overline{\alpha}\mu)\overline{p} - 4(\lambda\overline{\lambda} + \mu\overline{\mu}). \end{aligned} \tag{26}$$

Therefore the sphere condition $\Sigma = 0$ has only 38 terms in contrast to its formulation based on Study parameters, which has 80 terms (cf. [8]). An example for pointing out the beneficial effects of this reduction of terms is the symbolic elimination process in the direct kinematics of SG platforms.

Example 1. As each leg imply a sphere condition we get six sphere equations $\Sigma_i = 0$ with $i = 1, \ldots, 6$. It is well-known [8] that the differences of two sphere conditions are only linear in the translational parameters. Therefore the system of five equations $\Psi = \Sigma_5 - \Sigma_1 = \Sigma_4 - \Sigma_1 = \Sigma_3 - \Sigma_1 = \Sigma_2 - \Sigma_1 = 0$ linear in $\lambda, \mu, \overline{\lambda}, \overline{\mu}$ can only have a non-trivial solution if the determinant of the 5×5 coefficient matrix vanishes. This determinant splits up into Φ and a factor with 53 280 terms, which is homogenous of degree 4 in $\alpha, \beta, \overline{\alpha}, \overline{\beta}$. In contrast, the corresponding quartic expression based on Study parameter has 258 720 terms (cf. [5, Section 3.2]). ⋄

By setting $z = w_0 = \beta = \overline{\beta} = \lambda = \overline{\lambda} = 0$ we get from Eq. (26) the circle condition, which can be written similarly to Eq. (6) as:

$$(\alpha p + 2\mu - \overline{\alpha} b_0)(\overline{\alpha}\,\overline{p} + 2\overline{\mu} - \alpha\overline{b}_0) - \alpha\overline{\alpha} R^2 = 0. \tag{27}$$

This equation has both benefits; i.e. the compactness of the isotropic formulation and the homogeneity of the approach based on BG parameters (cf. Section 1.1).

In the following we show that the compactness of the proposed formulation passes on to the symbolic expression of the singularity loci of SG platforms. Therefore we compute the Plücker coordinates of the line spanned by the base anchor point and the corresponding platform anchor point. The direction vector $\mathbf{l} = (l_1, l_2, l_3)^T$ is given by $(\mathbf{p}_0 - \Phi \mathbf{b}_0)$, where the coefficient Φ is again used for homogenization, and the moment vector $\mathbf{m} := (m_1, m_2, m_3)^T$ reads as $\mathbf{b}_0 \times \mathbf{l}$. Thus each entry of the 6-tuple $(\mathbf{l}, \mathbf{m})\mathbb{R}$ fulfilling the Plücker condition $\langle \mathbf{l}, \mathbf{m} \rangle = 0$ is homogenous of degree 2 in the dual CK parameters $\alpha, \beta, \lambda, \mu, \overline{\alpha}, \overline{\beta}, \overline{\lambda}, \overline{\mu}$. By defining:

$$\begin{aligned}
l := l_1 + l_2 i &= \alpha^2 p - \overline{\beta}^2 \overline{p} - \alpha\overline{\alpha} b_0 - \beta\overline{\beta} b_0 + 2\alpha\overline{\beta} z + 2\overline{\lambda}\,\overline{\beta} + 2\mu\alpha, \\
m := m_2 + m_1 i &= (\overline{\alpha}^2 \overline{p} - \beta^2 p) w_0 + (\overline{\beta}\overline{\alpha}\overline{p} + \beta\alpha p)\overline{b}_0 + (\beta\overline{\beta} - \alpha\overline{\alpha}) z \overline{b}_0 \\
&\quad + 2(\overline{\alpha}\beta z + \overline{\alpha}\,\overline{\mu} + \beta\lambda) w_0 + (\beta\mu + \overline{\beta}\overline{\mu} - \alpha\lambda - \overline{\alpha}\overline{\lambda})\overline{b}_0, \qquad (28) \\
n := 2m_3 i &= 2(\alpha\mu + \overline{\beta}\,\overline{\lambda} + \alpha\overline{\beta} z_0)\overline{b}_0 - 2(\overline{\alpha}\,\overline{\mu} + \beta\lambda + \overline{\alpha}\beta z_0) b_0 \\
&\quad + (\alpha^2 \overline{b}_0 + \beta^2 b_0) p - (\overline{\alpha}^2 b_0 + \overline{\beta}^2 \overline{b}_0)\overline{p},
\end{aligned}$$

we can replace $(\mathbf{l}, \mathbf{m})\mathbb{R}$ by the more compact 6-tuple $\mathbf{f} := (l, \bar{l}, l_3, m, \overline{m}, n)\mathbb{R}$ with

$$l_3 = \alpha\overline{\alpha}(z - w_0) - \beta\overline{\beta}(z + w_0) - \alpha\beta b_0 - \overline{\alpha}\overline{\beta}\overline{b}_0 + \alpha\lambda + \overline{\alpha}\overline{\lambda} - \beta\mu - \overline{\beta}\overline{\mu} \qquad (29)$$

fulfilling $lm - \bar{l}\overline{m} + l_3 n = 0$. As each leg of the SG platform implies such a 6-tuple, we get $\mathbf{f}_1, \ldots, \mathbf{f}_6$. As a consequence the manipulator is in a singular pose (cf. [14]), if and only if:

$$\det(\mathbf{F}) = 0 \quad \text{with} \quad \mathbf{F} := (\mathbf{f}_1, \ldots, \mathbf{f}_6). \qquad (30)$$

The expression $\det(\mathbf{F})$ splits up into Φ^2 and a homogenous octic factor F in the dual CK parameters. Moreover, F has 542 496 terms if the platform and base anchor points are chosen as follows with respect to the moving and fixed frame: the first anchor point is located in the origin, the second one on the x-axis and the third one in the xy-plane. In contrast, F reformulated in Study parameters has 1 748 184 terms.

Note that $F = 0$ can be seen as an alternative singularity locus expression to [4, 13]. Finally, the singularity loci of the planar analogue (Fig. 1 left) can be computed as the determinant of a 3×3 matrix (cf. [9]), as $m = \overline{m} = l_3 = 0$ hold.

4 Conclusion

We discussed the transformation of points in terms of dual CK parameters (Section 2) and showed that these parameters imply a very compact symbolic expression of the sphere condition and the singularity loci of SG platforms (Section 3). These parameters cannot only be restricted to planar motions, but they can also be extended for kinematics in Euclidean 4-space according to [16]. The proposed representation is especially of interest for the determination of SG platforms with self-motions (e.g. [5]), but maybe it is also beneficial for the symbolic study of other mechanisms.

Acknowledgements The author is supported by Grant No. P 24927-N25 of the Austrian Science Fund FWF within the project ”Stewart Gough platforms with self-motions”.

References

1. Blaschke, W.: Euklidische Kinematik und nichteuklidische Geometrie. Zeitschrift für Mathematik und Physik **60** 61–91 and 203–204 (1911)
2. Brunnthaler, K., Pfurner, M., Husty, M.L.: Synthesis of planar four-bar mechanisms. CSME Transactions **30**(2) 297–313 (2006)
3. Denavit, J.: Displacement Analysis of Mechanisms based on (2×2) Matrices of Dual Numbers. VDI-Berichte **29** 81–88 (1958)
4. DiGregorio, R.: Singularity-locus expression of a class of parallel mechanisms. Robotica **20** 323–328 (2002)
5. Gallet, M., Nawratil, G., Schicho, J.: Liaison Linkages. Journal of Symbolic Computation **79**(1) 65–98 (2017)
6. Grünwald, J.: Ein Abbildungsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen Geometrie verknüpft. Sitz.-Ber. der math.-nat. Klasse der kaiserlichen Akademie der Wissenschaften Wien **120** 677–741 (1911)
7. Hayes, M.J.D., Husty, M.L.: On the kinematic constraint surfaces of general three-legged planar robots. Mechanism and Machine Theory **38**(5) 379–394 (2003)
8. Husty, M.: An algorithm for solving the direct kinematics of general Stewart-Gough platforms. Mechanism and Machine Theory **31**(4) 365–380 (1996)
9. Husty, M., Gosselin, C.: On the Singularity Surface of Planar 3-RPR Parallel Mechanisms. Mechanics Based Design of Structures and Machines **36** 411–425 (2008)
10. Husty, M., Karger, A., Sachs, H., Steinhilper, W.: Kinematik und Robotik. Springer (1997)
11. Husty, M., Mielczarek, S., Hiller, M.: Constructing an overconstrained planar 4RPR manipulator with maximal forward kinematics solution set. CD-ROM Proceedings of the 10th International Workshop on Robotics in Alpe-Adria-Danube Region RAAD (2001)
12. Kotelnikov, A.P.: Screw Calculus and Some Applications to Geometry and Mechanics. Annals of the Imperial University of Kazan (1895)
13. Li, H., Gosselin, C.M., Richard, M.J., Mayer St-Onge, B.: Analytic Form of the Six-Dimensional Singularity Locus of the General Gough-Stewart Platform. Journal of Mechanical Design **128** 279–287 (2006)
14. Merlet, J.-P.: Singular Configurations of Parallel Manipulators and Grassmann Geometry. The International Journal of Robotics Research **8**(5) 45–56 (1989)
15. Nawratil, G.: Kinematic Mapping of SE(4) and the Hypersphere Condition. Advances in Robot Kinematics (J. Lenarcic, O. Khatib eds.), pages 11–19, Springer (2014)
16. Nawratil, G.: Fundamentals of quaternionic kinematics in Euclidean 4-space. Advances in Applied Clifford Algebras **26**(2) 693–717 (2016)
17. Pennestri, E., Valentini, P.P., Figliolini, G., Angeles, J.: Dual Cayley-Klein parameters and Möbius transform: Theory and applications. Mechanism and Machine Theory **106** 50–67 (2016)
18. Plecnik, M.M, McCarthy, J.M.: Numerical Synthesis of Six-Bar Linkages for Mechanical Computation. Journal of Mechanisms and Robotics **6** 031012 (2014)
19. Schröcker, H.-P., Husty, M., McCarthy, J.M.: Kinematic mapping based assembly mode evaluation of planar four-bar mechanisms. Journal of Mechanical Design **129**(9) 924–929 (2007)
20. Study, E.: Geometrie der Dynamen. Teubner (1903)
21. Wampler, C.W.: Isotropic coordinates, circularity, and Bezout numbers: Planar kinematics from a new perspective. Proceedings of ASME Design Engineering Technical Conference and Computers in Engineering Conference, 96-DETC/MECH-1210 (1996)
22. Wampler, C.W.: The geometry of singular foci of planar linkages. Mechanism and Machine Theory **39** 1139–1153 (2004)
23. Wunderlich, W.: Ebene Kinematik. Hochschultaschenbücher-Verlag (1970)