A new approach to the classification of architecturally singular parallel manipulators

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Table of Contents

[1] Singular configurations of SGPs

The geometry of a Stewart Gough Platform is given by the six base anchor points

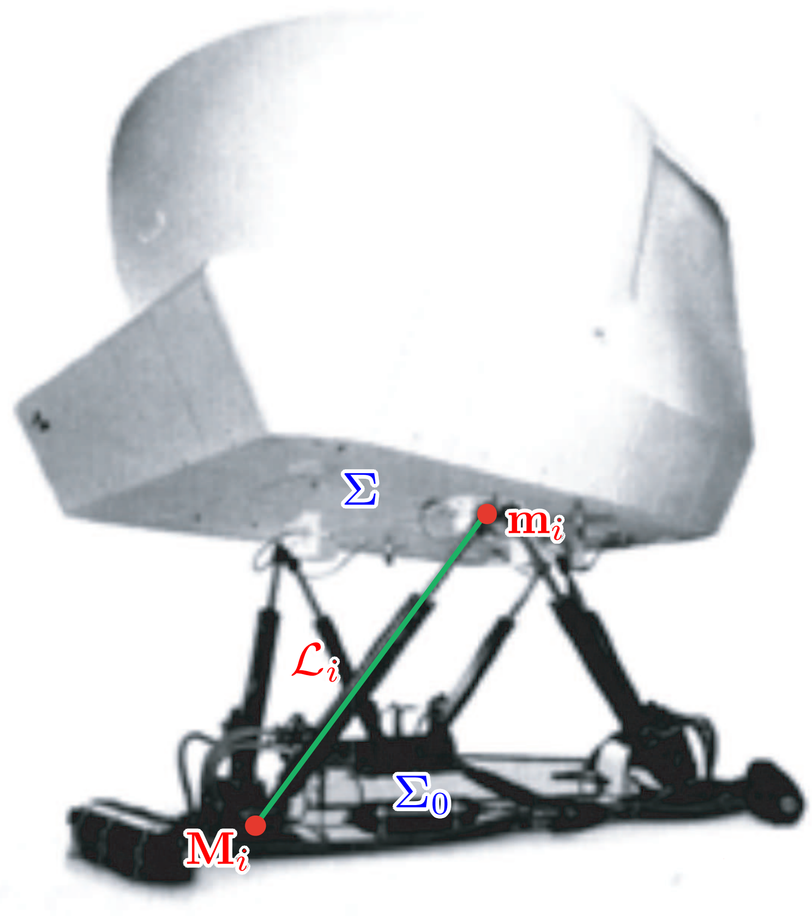
$\mathbf{M}_i := (A_i, B_i, C_i)^T$ in the fixed space Σ_0

and by the six platform anchor points

$\mathbf{m}_i := (a_i, b_i, c_i)^T$ in the moving space Σ.

Theorem MERLET [1992]
A SGP is singular iff the carrier lines $\mathcal{L}_i$ of the six legs belong to a linear line complex.

Manipulators which are singular at every possible configuration are called architecturally singular (cf. MA AND ANGELES [1992]).



[1] Review: Architecturally singular SGPs

Results of Karger

Karger divided the set $\mathcal{A}$ of architecturally singular manipulators into two classes with respect to the criterion of possessing 4 collinear anchor points or not:

- KARGER [2003] presented in Theorem 1 the four sufficient and necessary conditions for architecturally singular planar SGPS with no 4 anchor points aligned.

- KARGER [2008] proved in Theorem 1 and 2 that architecturally singular non-planar manipulators must have 4 collinear anchor points.

- KARGER [2008] listed in Theorem 3 all types of architecturally singular manipulators, planar or non-planar, with 4 collinear anchor points.

[1] Review: Architecturally singular SGPs

Results of Röschel and Mick

They divided the set $\mathcal{A}$ into planar and non-planar manipulators and gave the following geometric characterization for the planar case:

Theorem Röschel and Mick [1998]
Planar SGPs are architecturally singular iff (M_i, m_i), $i = 1, \ldots, 6$, are four-fold conjugate pairs of points with respect to a 3-dimensional linear manifold of correlations or one of the two sets $\{M_i\}$ and $\{m_i\}$ of anchor points is aligned.

In our approach we subdivide $\mathcal{A}$ with respect to the criterion whether the linear line complex spanned by the carrier lines of the legs is always singular or not.

[2] Plücker coordinates

By using Euler Parameters for the parametrization of SO_3 the coordinates $\mathbf{m}_i'$ of the platform points m_i with respect to Σ_0 can be written as $\mathbf{m}_i' = K^{-1}\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t}$

$$\text{with} \quad \mathbf{R} := (r_{ij}) = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\ 2(e_1e_2 - e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 + e_0e_1) \\ 2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix},$$

the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and $K := e_0^2 + e_1^2 + e_2^2 + e_3^2$.

Plücker coordinates of $\mathcal{L}_i$ are given by $\underline{\mathbf{l}}_i = (\mathbf{l}_i, \widehat{\mathbf{l}}_i) = (\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - K\mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$

Remark: $\underline{\mathbf{l}}_i$ meet the Plücker condition $l_i \cdot \widehat{\mathbf{l}}_i = 0$

[2] Linear line complexes

A linear line complex $\mathcal{C}$ with homogeneous coordinates $(\mathbf{c}, \widehat{\mathbf{c}})$ is a 3-dimensional linear manifold of lines $(\mathbf{l}, \widehat{\mathbf{l}})$ which satisfy the linear equation $\mathbf{c} \cdot \widehat{\mathbf{l}} + \widehat{\mathbf{c}} \cdot \mathbf{l} = 0$.

If $\mathcal{C}$ meets the Plücker condition $\mathbf{c} \cdot \widehat{\mathbf{c}} = 0$ it is called singular (otherwise regular). For $(\mathbf{c}, \widehat{\mathbf{c}}) \in \mathbb{R}^6$ we call a singular linear line complex *real*; for $(\mathbf{c}, \widehat{\mathbf{c}}) \in \mathbb{C}^6$ *complex*.

The 6-tuple $(\mathbf{c}, \widehat{\mathbf{c}})$ of a real singular linear line complex corresponds to a line in E^3; the so called axis, which may be a proper Euclidean line or may be an ideal line.

Therefore a real singular linear line complex consists of all lines $(\mathbf{l}, \widehat{\mathbf{l}})$ which in the projective extension of E^3 intersect the axis $(\mathbf{c}, \widehat{\mathbf{c}})$.

[2] Preparatory work and notation

For the computation of the linear line complex $\mathcal{C} := (\mathbf{c}, \widehat{\mathbf{c}})$ spanned by the 5 lines $\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_5$ we use the abbreviation r_{ij} for the entries of the matrix $\mathbf{R}$.

Using this notation the equation $Q: \ \mathbf{c} \cdot \widehat{\mathbf{c}} = 0$ has $1\,043\,682$ terms.

We denote the coefficients of $t_1^i t_2^j t_3^k$ by Q_{ijk} and compute the following ones:

$$
\begin{array}{lllll}
Q_{005}[8634], & Q_{311}[2796], & Q_{400}[774], & Q_{310}[3900], & Q_{030}[14664], \\
Q_{401}[582], & Q_{131}[5154], & Q_{040}[3174], & Q_{130}[7968], & Q_{003}[70717], \\
Q_{041}[2004], & Q_{221}[5364], & Q_{301}[7800], & Q_{300}[4821], &
\end{array}
$$

where the number in the square brackets gives the number of terms.

[3] Theorems on on 5-legged planar SGPs

Theorem 1.
If the legs of a 5-legged planar SGP belong in every possible configuration to a singular linear line complex $\mathcal{C}$ then 3 anchor points must be collinear.

This theorem is improved by

Theorem 2.
If the legs of a 5-legged planar SGP belong in every possible configuration to a singular linear line complex $\mathcal{C}$ then 4 anchor points must be collinear.

For the proofs of these theorems we refer to the paper.

Remark: Theorem 2 is sufficient for the reclassification done later on.

[3] Theorems on on 5-legged planar SGPs

Theorem 3.
If the legs of a 5-legged planar SGP belong in every possible configuration to a singular linear line complex $\mathcal{C}$ then it is one of the following cases:

1. $m_1, \ldots, m_5$ are collinear,
2. $m_1 = m_2 = m_3$,
3. $m_1 = m_2, m_3, m_4$ collinear and $M_3 = M_4$,
4. $m_1 = m_2$, $m_3 = m_4$, M_1, M_2, M_5 collinear and M_3, M_4, M_5 collinear,
5. $m_1, \ldots, m_4$ and $M_1, \ldots, M_4$ collinear and pairwise distinct with $CR(m_1, \ldots, m_4) = CR(M_1, \ldots, M_4)$ where CR denotes the cross ratio.

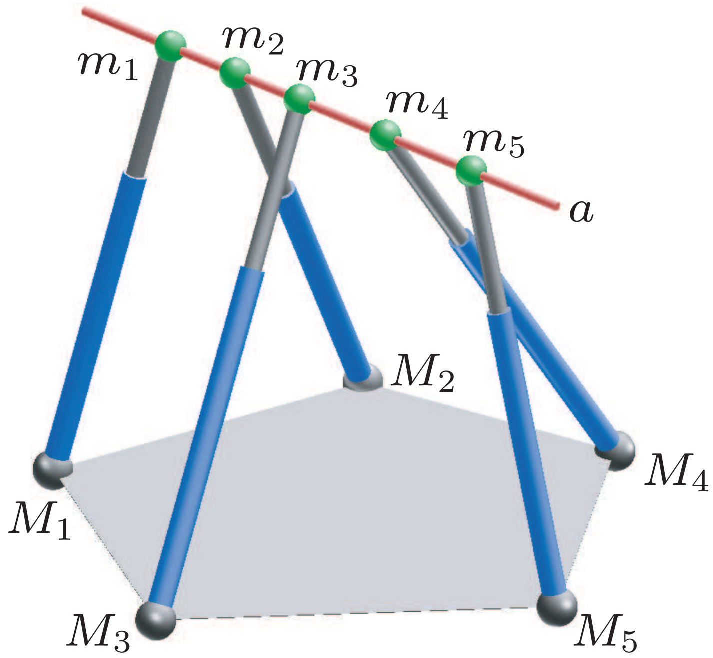
For the proof of this theorem we refer to the paper.

[3] Geometric interpretation of the 5 cases

Cases 1-4: Lines $[M_i, m_i]$ $i = 1, \ldots, 5$ belong to a real singular linear line complex.

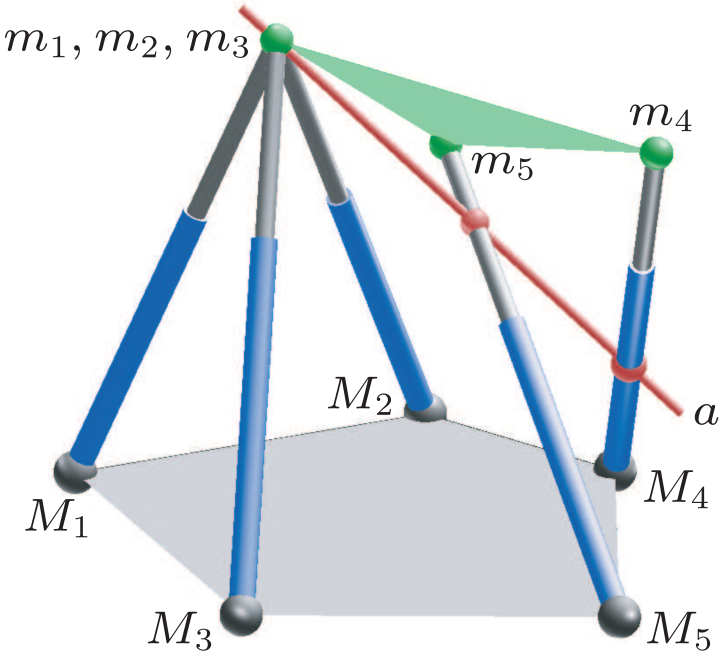
1. $m_1, \ldots, m_5$ are collinear

$a = [m_1, \ldots, m_5]$



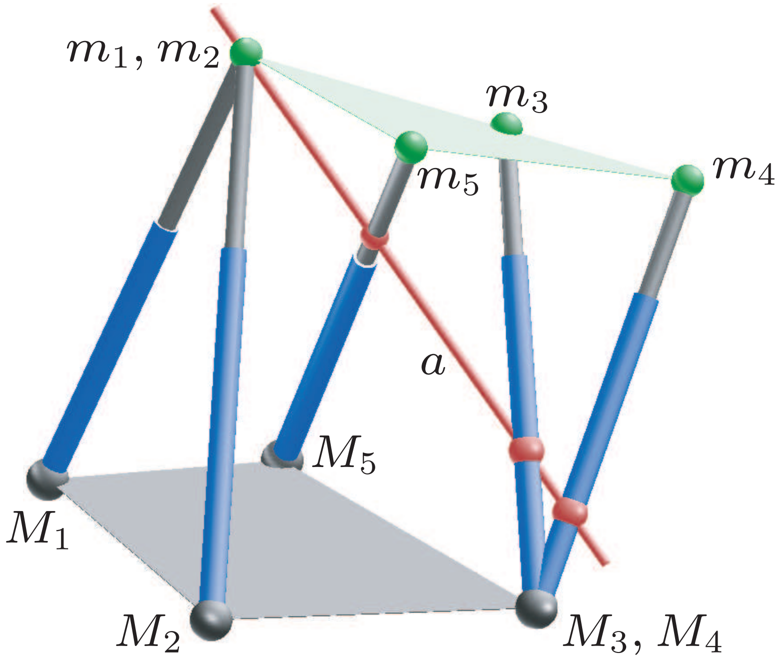
2. $m_1 = m_2 = m_3$

$a = ([m_1, M_4, m_4], [m_1, M_5, m_5])$

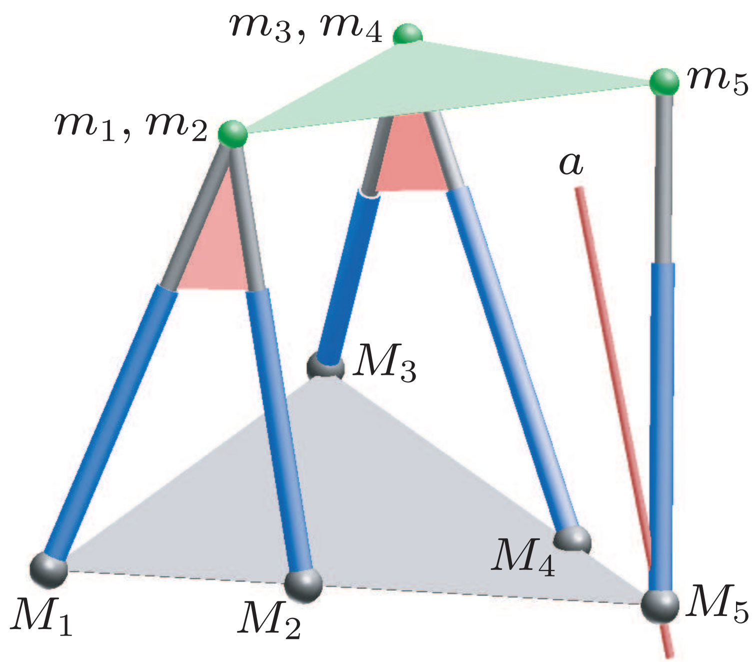


[3] Geometric interpretation of the 5 cases

3. $m_1 = m_2, m_3, m_4$ coll. $\wedge$ $M_3 = M_4$
$a = [m_{1,2}, s]$ where s is given by
$s := ([M_5, m_5], [m_3, m_4, M_{3,4}])$.



4. $m_1 = m_2,\ m_3 = m_4,\ M_1, M_2, M_5$ collinear and M_3, M_4, M_5 collinear.
$a = ([M_1, M_2, m_{1,2}], [M_3, M_4, m_{3,4}])$



[3] Geometric interpretation of the 5 cases

5. $[M_1, m_1], \ldots, [M_4, m_4]$ belong to a regulus $\mathcal{R}$

- **a.** $[M_5, m_5]$ intersects $\mathcal{R}$ in two real points s_1, s_2
- **b.** $[M_5, m_5]$ touches $\mathcal{R}$ in the point s
- **c.** $[M_5, m_5]$ intersects $\mathcal{R}$ in conjugate complex points s and $\overline{s}$.

Interesting question arises:

Can the free parameters of the base and platform points of the 5^{th} case be chosen such that the five lines always belong to a real singular linear line complex, if neither all base nor all platform points are collinear?

Remark 1. Solving this problem is part of recent research.
We conjecture that such parameters do not exist.

[4] Reclassifying architecturally singular SGPs

Corollary 1.
An architecturally singular parallel manipulator with $rk(\underline{\mathbf{l}}_1, \ldots, \underline{\mathbf{l}}_6) = 5$ belongs to the subset $\mathcal{A}_S$ if it fulfills one of the following conditions:

(a) $m_1, \ldots, m_6$ are collinear,

(b) $m_1 = m_2 = m_3$ and $M_4 = M_5 = M_6$,

(c) $m_1 = m_2 = m_3$, $m_1, \ldots, m_5$ are collinear and $M_4 = M_5$,

(d) $m_1 = m_2 = m_3 = m_4$,

(e) $m_1 = m_2$, $m_3 = m_4$, $M_5 = M_6$, M_1, M_2, M_5 and M_3, M_4, M_5 are collinear.

Otherwise it belongs to the subset $\mathcal{A}_R$. Then the point pairs (M_i, m_i), $i = 1, \ldots, 6$ are exactly 11-fold conjugate pairs of points with respect to a 10-dimensional linear manifold of correlations.

[4] Reclassifying architecturally singular SGPs

Proof.
Theorem 2 and Karger [2008] Theorem 2 $\Longrightarrow$
$\nexists$ architecturally singular SGPs with no 4 points collinear which belong in each configuration to a singular linear line complex.

All types of architecturally singular SGPs with 4 points collinear were listed by Karger [2008] Theorem 3. For these manipulators the corollary was proven separately by computations resp. the Theorem of Röschel and Mick [1998].
For details we refer to the paper. $\square$

Remark 2. We hope that the subdivision of $\mathcal{A}$ into $\mathcal{A}_S$ and $\mathcal{A}_R$ can be helpful for finding a purely geometric way for the determination of $\mathcal{A}$.

[5] Future work

- Finding a purely geometric way for the determination of $\mathcal{A}$.

- Determining all 5-legged non-planar SGPs which belong in every possible configuration to a singular linear line complex.
  We conjecture that this problem only has the solutions 1-3 of Theorem 3.

- Can the free parameters of the base and platform points of the 5^{th} case, i.e.

  $\star$ $m_1, \ldots, m_4$ and $M_1, \ldots, M_4$ collinear $\wedge$ $CR(m_1, \ldots, m_4) = CR(M_1, \ldots, M_4)$

  be chosen such that the 5 lines always belong to a real singular linear line complex?

[5] Future work

- Determining the whole set $\mathcal{S}$ of SGPs possessing the following property: In each singular configuration the carrier lines of the legs belong to a (real) singular linear line complex.

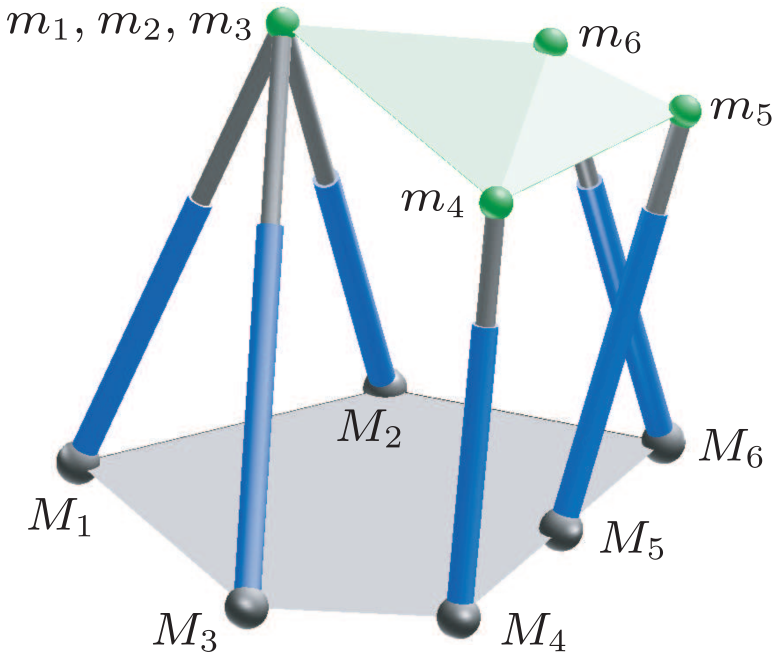
  $\mathcal{S}$ contains $\mathcal{A}_S$ and the following one:

  ⋆ $m_1 = m_2 = m_3$,

  ⋆ base is planar and

  ⋆ M_4, M_5, M_6 are collinear.

  We conjecture that all SGPs belonging to $\mathcal{S}$ have 4 collinear anchor points.



[6] Conclusion

- We presented a new approach to the classification of the set $\mathcal{A}$ of architecturally singular SGPs, which was subdivided with respect to the criterion if the legs belong in every possible configuration to a singular linear line complex or not (cf. Corollary 1).

- The proof was based on the fact that 5-legged planar parallel manipulators which belong in every possible configuration to a singular linear line complex must possess 4 collinear anchor points (cf. Theorem 1 and 2).

- Moreover we listed all types of 5-legged planar parallel manipulators with this property (cf. Theorem 3).

[7] References

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