Journal for Geometry and Graphics
Volume VOL (YEAR), No. NO, 1–8.

On the degenerated cases of architecturally singular planar parallel manipulators

Georg Nawratil
Institute of Discrete Mathematics and Geometry, Vienna University of Technology
Wiedner Hauptstrasse 8-10/104, Vienna, A-1040, Austria
nawratil@geometrie.tuwien.ac.at

Abstract. In Theorem 3 of A. Karger, Architecturally singular non-planar parallel manipulators, Mechanism and Machine Theory, Vol. 43, No.3, pp. 335-346, 2008 all types of architecturally singular parallel manipulators with four collinear anchor points are listed. As the manipulator is assumed to be architecturally singular in the cited theorem the given conditions for each entry of this list are sufficient for classification. However, we prove that in contrast to the items 1 to 10, the degenerated planar cases (items 11 and 12) do not contain conditions which are sufficient for an architecturally singular manipulator design. We propose sufficient conditions for these two cases in order to decide whether a given manipulator (with 4 points collinear) is architecturally singular or not by checking only the list instead of calculating the determinant of the Jacobian in all poses. Moreover we give a geometric interpretation of the degenerated planar cases.

Key Words: architecturally singular parallel manipulators, Stewart Gough Platform

1. Introduction

It is well known (see, e.g., Merlet [1]) that parallel manipulators of Stewart Gough type are singular if and only if the carrier lines of the prismatic legs belong to a linear line complex. Manipulators which are singular at every possible configuration are called architecturally singular (cf. Ma and Angeles [2]).

Karger presented in [3, Theorem 1] the four sufficient and necessary conditions for architecturally singular planar parallel manipulators with no four points on a line. Moreover Karger proved in [4, Theorem 1 and 2] that architecturally singular non-planar parallel manipulators must have four points on a line. Finally in [4, Theorem 3], all types of architecturally singular manipulators, planar or non-planar, with four collinear anchor points are listed. In the following we give those extracts of this theorem which are of central importance for the understanding of this article. The notation is the same as in [4].

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Theorem 3 of [4] Let a Stewart Gough Platform with four collinear points in the platform be architecture singular. Then it is one of the listed 12 cases. For details of 1 to 10 we refer to the original paper [4].
11. $m_1 = m_2$, $M_5 = M_6$, m_1, m_3, m_4 and M_3, M_4, M_5 are collinear, platform and base are planar and

$$P_0 = (a_5b_6 - a_6b_5)[B_3B_4(a_4 - a_3) + B_5(B_4a_3 - B_3a_4)] + a_3a_4B_5(B_3 - B_4)(b_6 - b_5) = 0. \quad (1)$$

12. $m_1, \dots, m_4$ are collinear, $M_5 = M_6$, base and platform are planar, and the two equations (linear in A_5, B_5) remain:

$$P_1 := jA_5 + kB_5 + l = 0^1 \quad \text{and} \quad P_2 := j'A_5 + k'B_5 + l' = 0^2 \quad \text{with} \quad (2)$$

$$\begin{aligned}
j &:= a_2B_3B_4(b_6 - b_5)(a_4 - a_3)) \\
k &:= A_4a_2B_3[b_6(a_3 - a_5) + b_5(a_6 - a_3)] - a_4A_2B_3[b_6(a_3 - a_5) + b_5(a_6 - a_3)] \\
&\quad + B_4(A_2a_3 - A_3a_2)[b_6(a_4 - a_5) + b_5(a_6 - a_4)] \\
l &:= A_2B_4B_3(b_5a_6 - b_6a_5)(a_4 - a_3) \\
j' &:= - A_4a_2B_3[b_6(a_4 - a_5) + b_5(a_6 - a_4)] + B_4A_3a_2[b_6(a_3 - a_5) + b_5(a_6 - a_3)] \\
&\quad + A_2(B_3a_4 - B_4a_3)[b_6(a_2 - a_5) + b_5(a_6 - a_2)] \\
k' &:= (b_5 - b_6)[A_4A_3a_2(a_3 - a_4) + A_4A_2a_3(a_4 - a_2) + A_2A_3a_4(a_2 - a_3)] \\
l' &:= A_2(b_6a_5 - b_5a_6)[B_4A_3(a_2 - a_3) - B_3A_4(a_2 - a_4)].
\end{aligned}$$

If the determinant D of the system (2) is non-zero, the values of A_5, B_5 are uniquely determined. However, explicit formulas for the solution are not given for their complexity. The determinant of this system consists of two factors $D = F_1F_2$. The first one is equal to

$$F_1 := B_4(a_2A_3 - a_3A_2) + B_3(a_4A_2 - a_2A_4). \quad (3)$$

If F_1 vanishes, we obtain a special case of the general solution. If the other factor is zero, we obtain two solutions:
(a) m_1, m_5, m_6 are collinear, M_2, M_3, M_4 are collinear and

$$A_5 = B_5\frac{B_4A_3a_2(a_3 - a_4) + B_4A_2a_3(a_4 - a_2) + B_3A_2a_3(a_2 - a_4)}{a_2B_3B_4(a_3 - a_4)}, \quad (4)$$

(b) m_2, m_5, m_6 are collinear, M_1, M_3, M_4 are collinear and

$$A_5 = A_2 + B_5\frac{B_4A_3a_2(a_3 - a_4) + B_4A_2a_3(a_4 - a_2) + B_3A_2a_3(a_2 - a_4)}{a_2B_3B_4(a_3 - a_4)}. \quad (5)$$

Remark 1: In equation (4) a typing error of [4, Equ. (19)] is corrected. ⋄

2. Sufficient conditions for the degenerated cases

Firstly, we demonstrate that the conditions of case 12 are not sufficient for a manipulator to be architecturally singular. In subsection 2.2 we present such a sufficient set of conditions for this degenerated case. As case 11 can be regarded as a subcase of 12 it can easily be shown that the conditions of case 11 are also not sufficient for an architecturally singular design (subsection 2.3). Finally, a deeper insight is gained by considering the problem of formulating sufficient conditions for the degenerated cases from a more geometric point of view.

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[1] In the original text of [4] this equation is denoted by g_{33}.
[2] In the original text of [4] this equation is denoted by h_{39}.

2.1. The conditions of case 12 are not sufficient

It is true that for non-vanishing determinant A_5 and B_5 are uniquely determined and yield an architecturally singular manipulator. But if the determinant D vanishes then there exist three cases where all conditions of case 12 are fulfilled but the manipulator is not architecturally singular. These cases are as follows:

Case I: Assuming $a_2B_3(b_5 - b_6) \neq 0$:

$$A_4 := [B_4(a_2A_3 - a_3A_2) + A_2B_3a_4] \,/\, (a_2B_3) \quad \text{and} \tag{6}$$

$$A_5 := [B_5(A_3a_2 - A_2a_3)(b_5 - b_6) + A_2B_3(a_6b_5 - b_6a_5)] \,/\, [a_2B_3(b_5 - b_6)]\,. \tag{7}$$

Case II: Assuming $B_3b_5 \neq 0$:

$$a_2 := 0, \quad a_4 := a_3B_4/B_3 \quad \text{and} \quad a_6 := [a_3B_5(b_5 - b_6) + B_3a_5b_6] \,/\, (B_3b_5)\,. \tag{8}$$

Case III:

$$B_3 := 0, \quad B_4 := 0 \quad \text{and} \quad b_5 := b_6. \tag{9}$$

The verification can be done by inserting the given expressions into P_1 and P_2 of Equ. (2) and into the singularity condition J (cf. [4, Equ. (4)]).

Remark 2: If we interchange platform and base, case III yields a planar Stewart Gough Platform with cylindrical singularity surface. This means that the resulting manipulator possesses for each orientation of the platform a cylindrical singularity surface with rulings parallel to a given fixed direction in the space of translations. Such manipulators were studied by the author in [5, 6, 7]. Moreover, it can easily be verified by calculating the determinant J of the Jacobian that manipulators of the other two cases (I and II) cannot have such a singularity surface without being architecturally singular. ◇

These three cases can be computed by performing the following case study:

- One starts by solving the first factor $F_1 = 0$ of the determinant D say for A_4 as it was done in case I. To do so, we must assume $a_2B_3 \neq 0$.
- Then we factorize the polynomials $P_1 = 0$, $P_2 = 0$ as well as the singularity condition $J = 0$ (cf. [4, Equ. (4)]). We skip the greatest common divisor of the three polynomials P_1, P_2, J from P_1 and P_2, and call the resulting polynomials again P_1 and P_2, respectively. This can be done because we are only interested in manipulators with $P_1 = 0, P_2 = 0$ and $J \neq 0$.
- Then we compute the greatest common divisor of P_1 and P_2. If $gcd(P_1, P_2) := C_1 \ldots C_n$ splits up into n factors C_i we get n solutions resulting from $C_i = 0$. For example in case I there is only one common factor ($n = 1$) which we solve for A_5.
- We proceed by skipping the greatest common divisor $gcd(P_1, P_2)$ of P_1 and P_2. For the remaining factors G_i, H_j of $P_1 := G_1 \ldots G_g$ and $P_2 := H_1 \ldots H_h$ we discuss all possible combinatorial cases. In each of these gh cases we solve the system $G_i = 0, H_j = 0$ and check if J vanish or not. In the latter case we get also an entry of the above list.

In this way all possible cases were studied. The resulting manipulator designs causing $P_1 = 0, P_2 = 0$ and $J \neq 0$ can then be summarized to the above three given cases. This list is complete under the following assumptions:

i) If $M_1 = M_2, M_3, M_4$ are collinear we can assume $B_3 = B_4 = 0$ (w.l.o.g.).

ii) If M_1, M_2, M_3, M_4 are not collinear we can assume $B_3 \neq 0$ (w.l.o.g.).

iii) Moreover, we can assume $b_5 \neq 0$ (w.l.o.g.) because if $b_5 = b_6 = 0$ holds all platform anchor points are collinear and we get an architecturally singular manipulator.

But F_1 is only one factor of the determinant D. For reasons given later (see subsection 2.4) the same procedure as described above for F_2 (instead of F_1) can only yield one of these three solutions.

2.2. Sufficient conditions for case 12

These three cases demonstrate that the given conditions of case 12 are not sufficient. In the following we add two conditions and prove that the resulting set of equations is sufficient. We extract these conditions from J as follows:
We denote the coefficients of $t_1^i t_2^j t_3^k$ in J by Q_{ijk} as in [4]. From Q_{002} we can factor out the homogenizing factor $K := x_0^2 + x_1^2 + x_2^2 + x_3^2$ and denote the resulting factor again Q_{002}. It should be noted that x_0, x_1, x_2, x_3 are the Euler Parameters and that t_1, t_2, t_3 parametrize the group of translations. Finally, we denote the coefficient of $x_0^i x_1^j x_2^k x_3^l$ of Q_{002} by P_{ijkl} and compute the following two equations P_3 and P_4:

$$\begin{aligned} P_3 &:= P_{5010} + P_{1050} = a_2(ua_6 + vb_6 + w) = 0 \quad \text{and} \\ P_4 &:= P_{4011} + P_{1140} = u'a_6 + v'b_6 + w' = 0 \quad \text{with} \end{aligned} \tag{10}$$

$$\begin{aligned} u :=& b_5[B_3B_4(A_2 - A_6)(a_3 - a_4) + B_3B_5a_4(A_2 - A_4) + B_4B_5a_3(A_2 - A_3) \\ v :=& a_3a_4B_6(A_2B_3 - B_4A_2 - A_4B_3 + B_4A_3) + \\ & a_3a_5B_4(B_6A_2 - B_6A_3 + A_6B_3 - A_2B_3) + \\ & a_4a_5B_3(A_4B_6 - B_6A_2 - B_4A_6 + B_4A_2) \\ w :=& a_3a_4b_5B_6(B_4A_2 - A_2B_3 + A_4B_3 - B_4A_3) \\ u' :=& b_5[a_2a_3A_4(B_3A_5 + A_2B_5 - A_3B_5 - A_2B_3) + \\ & a_2a_4A_3(B_5A_4 - A_2B_5 - B_4A_5 + A_2B_4) + \\ & a_3a_4A_2(B_3A_4 - B_5A_4 - B_3A_5 - B_4A_3 + A_3B_5 + B_4A_5)] \\ v' :=& a_2a_3a_4A_5(B_3A_4 - A_2B_3 - B_4A_3 + A_2B_4) + \\ & a_2a_3a_5A_4(A_2B_3 - B_3A_5 - B_5A_2 + A_3B_5) + \\ & a_2a_4a_5A_3(B_4A_5 - A_2B_4 + B_5A_2 - B_5A_4) + \\ & a_3a_4a_5A_2(B_5A_4 - B_3A_4 + B_3A_5 + B_4A_3 - A_3B_5 - B_4A_5) \\ w' :=& a_2a_3a_4b_5A_5(A_2B_3 - B_3A_4 + B_4A_3 - A_2B_4). \end{aligned}$$

As all conditions $P_i = 0$ with $i = 1, \dots, 4$ are linear combinations of some coefficients of J it is clear that if J vanishes also the conditions $P_1 = P_2 = P_3 = P_4 = 0$ are satisfied. Therefore these conditions are necessary. In the following we prove the sufficiency of these conditions.

Proof: We have to prove that $J = 0$ is satisfied if the conditions m_1, m_2, m_3, m_4 collinear, $M_5 = M_6$ as well as $P_1 = P_2 = P_3 = P_4 = 0$ hold. This must only be shown for the above given three cases, because for any other case the conditions m_1, m_2, m_3, m_4 collinear, $M_5 = M_6$ and $P_1 = P_2 = 0$ are already sufficient due to the performed case study. Hence we compute for each case P_3, P_4 and J and factorize the resulting expressions. We end up with the following results:

ad Case I: P_3 vanishes under $A_2 = 0$ or if the following equation is fulfilled:

$$\begin{aligned}
&a_2\,(b_5-b_6)\,[a_3a_4B_5\,(b_5-b_6)\,(B_3-B_4)+a_3a_5B_4b_6\,(B_3-B_5)+\\
&a_3a_6B_4b_5\,(B_5-B_3)+a_4a_5B_3b_6\,(B_5-B_4)+a_4a_6B_3b_5\,(B_4-B_5)]+\\
&B_5\,(b_5-b_6)\,[a_3^2B_4\,(a_4b_5-a_6b_5+b_6a_5-b_6a_4)+\\
&a_4^2B_3\,(a_6b_5-b_5a_3-b_6a_5+b_6a_3)]+B_3B_4(a_5b_6-a_6b_5)^2(a_3-a_4)=0.
\end{aligned} \tag{11}$$

P_4 equals $P_3(A_3a_2 - a_3A_2)$ and $gcd(J, P_3) = P_3$ which finishes this case.

ad Case II: Now P_3 vanishes and $gcd(P_4, J) = P_4$ with

$$P_4 = A_2{a_3}^2B_4B_5\,(b_6-b_5)\,(B_4A_3-A_3B_5-A_4B_3+A_5B_3+A_4B_5-B_4A_5)\,. \tag{12}$$

ad Case III: Again P_3 vanishes and $gcd(P_4, J) = P_4$ with

$$P_4 = B_5b_6\,(a_6-a_5)\,(a_4A_2A_3(a_3-a_2)+a_3A_2A_4(a_4-a_2)+a_2A_3A_4(a_4-a_3))\,. \tag{13}$$

This finishes the proof of the sufficiency. □

2.3. Sufficient conditions for case 11

Case 11 can be regarded as a subcase of 12 because we get case 11 by adding the conditions $m_1 = m_2$ and M_3, M_4, M_5 collinear to case 12. Therefore the result of the last section can be used to prove that the conditions of case 11 are also not sufficient and that such a sufficient set of conditions is given by: $m_1 = m_2, m_3, m_4$ collinear, $M_3, M_4, M_5 = M_6$ collinear, $P_1 = 0$ and $P_2 = 0$.

Proof: After setting $a_2 = 0$, P_3 vanishes and we get $P_1 = A_2P_0$ with P_0 of (1), and

$$P_4 = a_3A_2a_4\,(b_6a_5-b_5a_6)\,(B_5A_4-B_4A_5+B_3A_5-B_5A_3-B_3A_4+A_3B_4)\,. \tag{14}$$

Due to the last factor P_4 vanishes if M_3, M_4, M_5 are collinear.

If we assume $B_4 \neq B_5$ we can compute A_3 from this condition. Then we substitute the obtained expressions into P_2 (P_0 does not depend on A_3). Computing $gcd(P_2, J) = P_1$ finishes the first part of the proof.

For $B_4 = B_5$ the collinearity condition splits up into $(B_3-B_5)(A_4-A_5)$. For $B_3 = B_4 = B_5$ the factor P_0 vanishes (and therefore P_1) and $gcd(P_2, J) = P_2$. This case denoted by IV shows that the condition $P_0 = 0$ is not sufficient. For $A_4 = A_5$ we get

$$P_1 = A_2B_5a_3\,(b_5a_6-a_5b_6+b_6a_4-b_5a_4)\,(B_3-B_5) \tag{15}$$

which can also be written as $gcd(P_1, P_2, J)(B_3 - B_5)$. Therefore P_1 can only vanish for the case $B_3 = B_4 = B_5$ which we have already discussed (subcase of case IV). □

Remark 3: The verification that the conditions of the degenerated cases given by Karger [4] are sufficient for the classification of architecturally singular manipulators can be done by constructing architecturally singular manipulators from the designs given in cases I,II, III and IV and checking if the resulting manipulators are in the list.

No unknown case appears and therefore the original conditions are sufficient for classification and the list given by Karger is complete. ⋄

We sum up the results of the above section by formulating the following Theorem:

Theorem 1 A parallel manipulator of Stewart Gough type with four collinear points in the platform is architecturally singular if and only if it is one of the following: Items 1 to 10 of the list given in [4, Theorem 3] or in the following degenerated cases:

11. $m_1 = m_2$, $M_5 = M_6$, m_1, m_3, m_4 and M_3, M_4, M_5 are collinear, platform and base are planar, and the two equations $P_1 = 0$ and $P_2 = 0$ given in (2) remain.

12. $m_1, \ldots, m_4$ are collinear, $M_5 = M_6$ base and platform are planar, and four equations $P_i = 0$ with $i = 1, \ldots, 4$ remain, they are given in (2) and (10), respectively. For the subclassification of this case we refer to [4, Theorem 3].

Remark 4: As already mentioned in subsection 2.3 case 11 can be regarded as a subcase of 12. This becomes more clear by considering the geometric interpretation given in section 3.1. Therefore a more reasonable listing would be to classify case 11 as a special case of 12. But in order to avoid confusion we stick to the proposed classification of Karger. ◇

2.4. Geometric point of view

Röschel and Mick presented in [8, Theorem 4.2] the following geometric characterization of architecturally singular manipulators with planar base and platform:

Theorem 4.2 of [8] Planar Stewart Gough Platforms are architecturally singular iff the pairs (M_i, m_i), $i = 1, \ldots, 6$, are four-fold conjugate pairs of points with respect to a 3-dimensional linear manifold of correlations or one of the two sets $\{M_i\}$ and $\{m_i\}$ is aligned.

Analytically the first part of this theorem can be expressed by the following condition (cf. [8, Remark 1 of section 4]):

$$rk \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & a_2 & 0 & A_2 & a_2A_2 & 0 & 0 & 0 & 0 \\ 1 & a_3 & b_3 & A_3 & a_3A_3 & b_3A_3 & B_3 & a_3B_3 & b_3B_3 \\ 1 & a_4 & b_4 & A_4 & a_4A_4 & b_4A_4 & B_4 & a_4B_4 & b_4B_4 \\ 1 & a_5 & b_5 & A_5 & a_5A_5 & b_5A_5 & B_5 & a_5B_5 & b_5B_5 \\ 1 & a_6 & b_6 & A_6 & a_6A_6 & b_6A_6 & B_6 & a_6B_6 & b_6B_6 \end{pmatrix} < 6. \tag{16}$$

Let $\mathbf{c}_i$ denote the i^{th} column of this matrix. For case 12 in Karger's list we set $b_3 = b_4 = 0$, $A_5 = A_6$ and $B_5 = B_6$. Then $rk(\mathbf{c}_3, \mathbf{c}_6, \mathbf{c}_9) = 1$ and $rk(\mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_7, \mathbf{c}_8) = 5$.

The conditions $P_1 = 0$ and $P_2 = 0$ of (2) can also be computed as $det(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_7, \mathbf{c}_8) = 0$ and $det(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_7) = 0$, respectively. These two conditions are not only necessary but also sufficient if the columns $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_7$ are linearly independent. This is the case (provided not all platform or base anchor points are collinear) if

$$det \begin{pmatrix} a_2 & A_2 & 0 \\ a_3 & A_3 & B_3 \\ a_4 & A_4 & B_4 \end{pmatrix} \neq 0. \tag{17}$$

It should be noted that this determinant is exactly F_1 of (3). This is the reason why the second factor F_2 of the determinant D does not yield further solutions in the case study of subsection 2.1.

If the columns $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_7$ are linearly dependent then we need the additional conditions $P_3 = 0$ and $P_4 = 0$ given in (10), which can also be computed as $det(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_5, \mathbf{c}_7, \mathbf{c}_8) = 0$ and $det(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_8) = 0$, respectively. The sufficiency of these conditions was already shown in subsection 2.2.

3. Geometric interpretation of the degenerated planar cases

The geometric interpretation of the cases 1 to 10 of Karger's list was already given in [4] but for the degenerated cases the problem remained open. It turns out that the geometric interpretation of these cases is not as complicated as might be expected:

3.1. Geometric interpretation of case 12

General Solution

In this case we have a projective correspondence between the points m_1, m_2, m_3, m_4, s and the conic section passing through $M_1, M_2, M_3, M_4, M_5 = M_6$ where $s = (a_s, 0)$ denotes the intersection point of the lines $[m_5, m_6]$ and $[m_1, \ldots, m_4]$.

Proof: We use formula (17) of [4] characterizing case 10 and replace a_5 by a_s. The resulting equations (denoted by $U_1 = 0$ and $U_2 = 0$) are linear in A_5 and B_5. If the determinant of the system $P_1 = 0$ and $P_2 = 0$ is non-zero, then the systems $U_1 = U_2 = 0$ and $P_1 = P_2 = 0$ yield the same solution for A_5 and B_5. For the case that s is at infinity the result can be obtained by introducing homogeneous coordinates. □

As in case 10 the lines spanned by pairs (M_i, m_i) of corresponding points generate a ruled surface and all its generators can be added without restricting the self-motion.

Remark 5: The special case of the general solution has the same geometric interpretation.

Special Cases

ad (a) In the special case (a) given by the corrected formula (4) the geometric interpretation is as follows: The algebraic condition corresponds to the following relation:

$$CR(m_1, m_2, m_3, m_4) = CR(S, M_2, M_3, M_4), \tag{18}$$

where CR denotes the cross ratio and S the intersection point of the lines $[M_1, M_5 = M_6]$ and $[M_2, M_3, M_4]$. The proof can easily be done by computation.

By the given cross ratio relation a projective correspondence between the lines $[m_1, \ldots, m_4]$ and $[M_2, \ldots, M_4]$ is determined. Again corresponding points generate a ruled surface and all its generators can be added without restricting the self-motion.

ad (b) For the special case (b) the algebraic condition can be interpreted as:

$$CR(m_1, m_2, m_3, m_4) = CR(S, M_1, M_3, M_4), \tag{19}$$

where S denotes the intersection point of the lines $[M_2, M_5 = M_6]$ and $[M_1, M_3, M_4]$. Therefore the special case (a) and (b) are identical from the geometrical point of view.

3.2. Geometric interpretation of case 11

The algebraic condition of this case is equivalent to

$$CR(m_1 = m_2, s, m_3, m_4) = CR(S, M_5 = M_6, M_3, M_4), \tag{20}$$

where s is the intersection point of the lines $[m_5, m_6]$ and $[m_1, \ldots, m_4]$ and S the common point of the lines $[M_1, M_2]$ and $[M_3, \ldots, M_6]$. More precisely the analytical condition $P_0 = 0$ given by Karger means

$$CR(m_1 = m_2, s, m_3, m_4) = CR(S^p, M_5^p = M_6^p, M_3^p, M_4^p), \tag{21}$$

where p denotes the orthogonal projection onto the y-axis. The second equation $P_2 = 0$ is needed for the case $M_3^p = M_4^p = M_5^p = M_6^p$, i.e. $B_3 = B_4 = B_5 = B_6$. The proof is straight forward by computation.

As the general solution of case 12 is related to case 10, this case is related to the special case of 10. The cross ratio relation determines again a projective correspondence between the lines $[m_1, \ldots, m_4]$ and $[M_3, \ldots, M_6]$. Again corresponding points generate a ruled surface and all its generators can be added without restricting the self-motion.

Moreover it should be noted that in all cases the lines $[m_i, M_i]$ determined by $M_5 = M_6 = M_i$ and m_5, m_6, m_i collinear can also be added because the point $M_5 = M_6$ can only move on a circle with axis $[m_5, m_6]$.

4. Conclusion

We showed that the conditions of case 11 and 12 of the list of architecturally singular manipulators with four points collinear presented in [4] are sufficient for classification but not sufficient for a parallel manipulator of Stewart Gough type to be architecturally singular. Moreover we gave a sufficient set of equations for these cases. This was not only done for the sake of completeness but in order to decide whether a given manipulator (with 4 points collinear) is architecturally singular or not by checking only the list instead of calculating the determinant of the Jacobian in all poses. Moreover, the geometric interpretation of the degenerated planar cases was given which was missing up to recent.

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Received