BOND THEORY FOR PENTAPODS AND HEXAPODS

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Abstract. This paper deals with the old and classical problem of determining necessary conditions for the overconstrained mobility of some mechanical device. Methods from algebraic geometry have already been used to investigate these mobility conditions, and we propose some tools specifically developed for the so–called n–pods. In particular, we focus on the conditions which are imposed on an n–pod by the existence of points of particular kind, which lie on the boundary of the set of self–motions of the n–pod when we consider a specific compactification of the group of direct isometries of $\mathbb{R}^3$. Furthermore we set up a photogrammetry–like technique which allows to establish necessary conditions for movability of 5–pods.

Introduction

The objects we focus on in this paper are mechanical manipulators called *n–pods*. As described in [2], the geometry of this kind of devices is defined by the coordinates of the n base anchor points $p_i = (a_i, b_i, c_i) \in \mathbb{R}^3$ and of the n platform anchor points $P_i = (A_i, B_i, C_i) \in \mathbb{R}^3$ in one of their possible configurations. All pairs of points (p_i, P_i) are connected by a rigid body, called *leg*, so that for all possible configurations the distance $d_i = \|p_i - P_i\|$ is preserved.

Notation. We think of an n–pod L as a triple

$$L = \Big((p_1, \ldots, p_n), (P_1, \ldots, P_n), (d_1, \ldots, d_n) \Big)$$

where p_i, P_i and d_i are defined as above.

We are interested in describing the *self–motions* of a given n–pod L, namely which direct isometries σ of $\mathbb{R}^3$ satisfy the condition

$$\|\sigma(p_i) - P_i\| \; = \; \|p_i - P_i\| \; = \; d_i \qquad \text{for all } i \in \{1, \ldots, n\} \tag{1}$$

In particular we want to understand what is the *dimension* of the set of these isometries, namely the *mobility* of L, and what conditions we have to impose on the base and platform points to reach a prescribed mobility.

We first study some geometric properties of the group of direct isometries of $\mathbb{R}^3$, and in particular we notice that it is possible to embed it in projective space in a way specifically tuned for dealing with n–pods (Section 1). At this point, we prove that the inspection of boundary points of the set of self–motions of an n–pod L gives information on base and platform points (Section 2). Eventually, we develop a photogrammetry theory based on identifications (under Möbius transformations) of 2D projections of 3D configurations of points, and we use it to establish a result on pentapods with mobility 2 (Section 3).

Kinematical main results. From the kinematical point of view this paper contains the following two main results:

Result 1. *If the mobility of a pentapod is* 2 *or higher, then one of the following conditions holds:*

(a) *The platform and the base are similar.*
(b) *The platform and the base are planar and affine equivalent.*
(c) *There exists* $m \leq 5$ *such that* $p_1, \ldots, p_m$ *are collinear and* $P_{m+1}, \ldots, P_5$ *are equal, up to permutation of indices and interchange of platform and base.*

Result 2. *If an* n*–pod is mobile, then one of the following conditions holds:*

(i) *There exists at least one pair of orthogonal projections* π_l *and* π_r *such that the projections of the platform points* $p_1, \ldots, p_n$ *by* π_l *and of the base points* $P_1, \ldots, P_n$ *by* π_r *are Möbius equivalent.*
(ii) *There exists* $m \leq n$ *such that* $p_1, \ldots, p_m$ *are collinear and* $P_{m+1}, \ldots, P_n$ *are collinear, up to permutation of indices.*

1. A new compactification of SE_3

The elements of the group SE_3 of direct isometries of $\mathbb{R}^3$ correspond bijectively to points in an open subset of the Study quadric in $\mathbb{P}^7_{\mathbb{R}}$ (where we take as coordinates the so–called Study parameters of the isometries), defined as the set $\{e_0f_0+e_1f_1+e_2f_2+e_3f_3=0\}$. This embedding of SE_3 turns out to be very useful in the study of mobility properties of objects coming from kinematics. However, in our situation a different compactification will lead us to a better comprehension of the phenomena which can appear. We introduce another projective embedding of (the complexification of) SE_3 considering the following polynomials:

$$\begin{cases} n_0 = f_0^2 + f_1^2 + f_2^2 + f_3^2, & \\ c_{ij} = e_i e_j & \text{for } 0 \leq i \leq j \leq 3, \\ b_{ij} = e_i f_j - e_j f_i & \text{for } 0 \leq i < j \leq 3. \end{cases} \tag{2}$$

They give rise to a rational map

$$\Phi : \underbrace{\mathbb{P}^7_{\mathbb{C}}}_{\text{coordinates } (e_0:\ldots:e_3:f_0:\ldots:f_3)} \dashrightarrow \underbrace{\mathbb{P}^{16}_{\mathbb{C}}}_{\text{coordinates } (n_0:\{b_{ij}\}:\{c_{ij}\})}$$

Definition 1.1. We define X to be the Zariski closure of the image of SE_3 (embedded in $\mathbb{P}^7_{\mathbb{C}}$) via Φ.

Lemma 1.2. *The variety* X *is a compactification of* SE_3 *and Equation (1) becomes linear in the new coordinates. We denote by* B *the* boundary *of* X*, namely the set* $X \setminus \mathrm{SE}_3$*.*

Using the newly created compactification we can define the concepts of *configuration set*, *mobility* (also known as *internal mobility* or *internal degrees of freedom*) and *bond* of an n–pod.

Definition 1.3. The *configuration set* K_L of an n–pod L is defined as:

$$K_L = \{\sigma \in \mathrm{SE}_3 \text{ satisfying Equation (1) for all } (p_i, P_i)\} \subseteq X$$

The *mobility* of an n–pod L is defined as the dimension of K_L.

Definition 1.4. The set of *bonds* B_L of an n–pod L is defined as the intersection of the Zariski closure of K_L with the boundary B of X.

Remark 1.5. Belonging to the boundary, bonds do not represent direct isometries of $\mathbb{R}^3$, but we will see in Section 2 that we can give a precise geometric meaning to their presence.

The following result is the key tool to prove all theorems in Section 2:

Proposition 1.6. *The natural left and right actions of* SE_3 *on itself extend to linear actions on* X.

2. Geometric interpretation of Bonds

According to the local behavior of the variety X, bonds can be classified into four groups:

- **inversion bonds:** at least one c_{ij}–coordinate is not zero, and $n_0 \neq 0$ (X is smooth at these points);
- **butterfly bonds:** at least one c_{ij}–coordinate is not zero, and $n_0 = 0$ (these are double points of X);
- **similarity bonds:** all c_{ij}–coordinates are zero and the following matrix has rank 2

$$M = \begin{pmatrix} b_{01} & b_{02} & b_{03} & b_{12} & b_{13} & b_{23} \\ b_{23} & -b_{13} & b_{12} & b_{03} & -b_{02} & b_{01} \end{pmatrix}$$

- **collinearity bonds:** all c_{ij}–coordinates are zero and the matrix M has rank 1.

The presence of some kind of bond implies precise conditions on base and platform points of an n–pod. In the following, if $\varepsilon \in S^2$ is a unit vector in $\mathbb{R}^3$, we denote by π_ε the orthogonal parallel projection along ε.

Theorem 2.1. *Assume that* $\beta \in B_L$ *is an inversion/similarity bond of* L*. Then there exist directions* $l, r \in S^2$ *(depending on* β*) such that, if for* $i = 1, \ldots, n$ *we set* $q_i = \pi_l(p_i)$ *and* $Q_i = \pi_r(P_i)$*, then there is an inversion/similarity of* $\mathbb{R}^2$ *mapping* $q_1, \ldots, q_n$ *to* $Q_1, \ldots, Q_n$.

Theorem 2.2. *Assume that* $\beta \in B_L$ *is a butterfly bond of* L*. Then there exist directions* $l, r \in S^2$ *(depending on* β*) such that, up to permutation of indices* $1, \ldots, n$*, there exists* $m \leq n$ *so that* $p_1, \ldots, p_m$ *are collinear on a line parallel to* l*, and* $P_{m+1}, \ldots, P_n$ *are collinear on a line parallel to* r.

Theorem 2.3. *Assume that* $\beta \in B_L$ *is a collinearity bond of* L*. Then* $p_1, \ldots, p_n$ *are collinear or* $P_1, \ldots, P_n$ *are collinear (or both).*

The technique we adopt to show the previous results is to use left and right actions of SE_3 on X in order to find "good" representatives for the orbits of all kind of bonds, and then to perform explicit computations. Altogether, these theorems give Result 2 presented in the Introduction.

3. Möbius Photogrammetry

This Section deals with a mathematically freestanding problem, which we call *Möbius Photogrammetry.* Unlike traditional photogrammetry, which tries to recover a set of points from a finite collection of central projections, here we consider the problem of reconstructing a vector of 5 points in $\mathbb{R}^3$ starting from finitely many orthogonal parallel projections, assuming that we know them only up to Möbius transformations.

If we denote by M_5 the moduli space of 5 points in $\mathbb{P}^1_{\mathbb{C}}$, and we fix a vector $\vec{A} = (A_1, \ldots, A_5)$ of points in $\mathbb{R}^2$, it is possible to define a *photographic map*

$$f_{\vec{A}} : S^2 \longrightarrow M_5$$

associating to each unit vector $\varepsilon \in S^2$ the equivalence class of the projection along ε of the points in $\vec{A}$. Here we identify the plane $\mathbb{R}^2$, on which the points are projected, with the complex plane $\mathbb{C}$, and we embed the latter in $\mathbb{P}^1_{\mathbb{C}}$. Using the results in [1] we can write down explicitely an embedding $M_5 \subseteq \mathbb{P}^5_{\mathbb{C}}$ and the polynomials defining $f_{\vec{A}}$, and prove the following two lemmata:

Lemma 3.1. *Let $\vec{A} = (A_1, \ldots, A_5)$ be a 5–tuple of points that are not coplanar. Then the photographic map $f_{\vec{A}} : S^2 \longrightarrow M_5$ is birational to a rational curve of degree 10 or 8 in M_5.*

Lemma 3.2. *Let $\vec{A} = (A_1, \ldots, A_5)$ be a 5–tuple of planar points, not collinear. Then the photographic map $f_{\vec{A}} : S^2 \longrightarrow M_5$ is $2:1$ to a rational curve of degree 5, 4, 3, or 2 in M_5.*

Finally, taking into account the geometrical properties of M_5 (which is a Del Pezzo surface of degree 5) one can prove that the set of images under the photographic map can be used to determine, up to similarities, the vector of points we started with.

Theorem 3.3. *Let $\vec{A}$ and $\vec{B}$ be two 5–tuples of points in $\mathbb{R}^3$ such that no 4 points are collinear. Assume that $f_{\vec{A}}(S^2)$ and $f_{\vec{B}}(S^2)$ are equal as curves in M_5. If $\vec{A}$ is coplanar, then $\vec{B}$ is also coplanar and affine equivalent to $\vec{A}$. If $\vec{A}$ is not coplanar, then $\vec{B}$ is similar to $\vec{A}$.*

Now Theorem 3.3, together with the theorems of Section 2, yields Result 1 of the Introduction (for example, the hypotheses of parts (a) and (b) in Result 1 imply the existence of infinitely many inversion/similarity bonds, which in turn forces the images $f_{\vec{p}}(S^2)$ and $f_{\vec{P}}(S^2)$ to be equal).

References

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