A necessary geometric criterion for the mobility of n-pods

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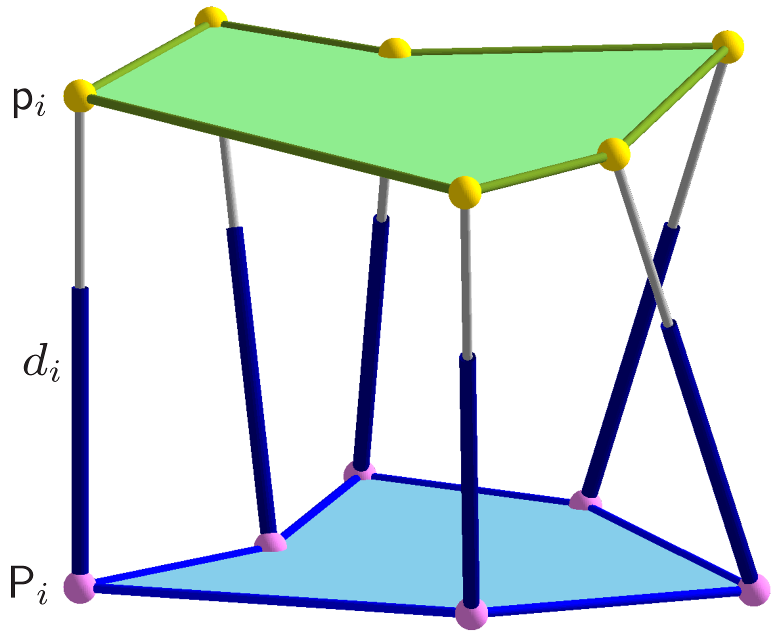
Overview

1. Introduction

n-pods are mechanical devices constituted of two rigid bodies, the base and the platform, which are connected by n rigid bodies, called legs, that are anchored via spherical joints.

An n-pod is called *mobile* if the platform can move relatively to the fixed base respecting the constraints imposed by the legs; the distances d_i between p_i and P_i are preserved.
Therefore mobile n-pods are special cases of *flexible body-bar frameworks*.

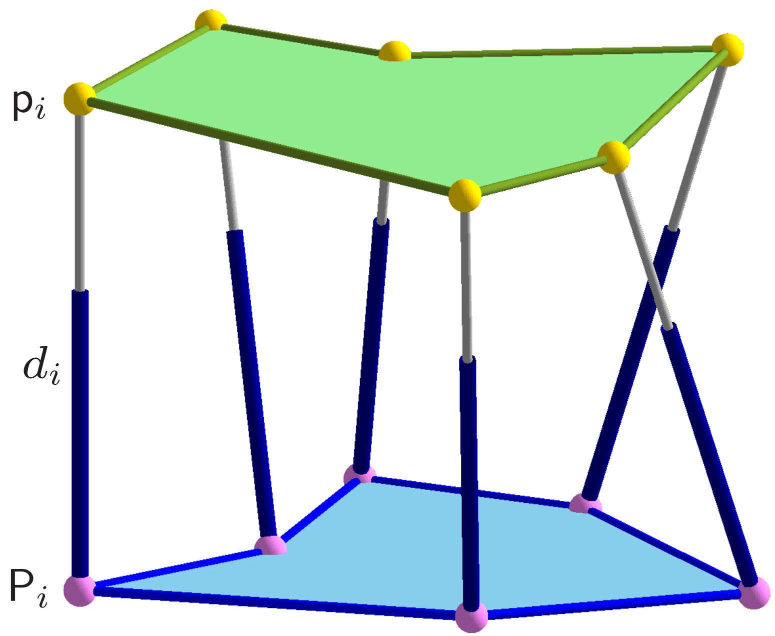


Hexapod with planar platform and base

1. Introduction

Historically the question for mobile n-pods goes back to the *Prix Vaillant* of the year 1904 posed by the French Academy of Science, which reads as follows: *"Determine and study all displacements of a rigid body in which distinct points of the body move on sphericalpaths."*

Theorem 1. MERLET [1989]
An n-pod is *infinitesimal flexible/mobile* if and only if the carrier lines of the n legs belong to a linear line complex.



Hexapod with planar platform and base

1. Introduction

Goal of the talk.
We present a necessary condition for the mobility of n-pods, which only depends on the geometry of the platform and the base, but not on their relative pose or the lengths of the n legs.

We are interested in describing which are the *self-motions* of a given n-pod, namely which direct isometries σ of $\mathbb{R}^3$ satisfy the so-called *sphere condition*:

$$\|\sigma(\mathsf{p}_i) - \mathsf{P}_i\| = d_i \qquad \text{for all } i \in \{1, \ldots, n\}$$

Remark: One can embed SE_3 as an open subset of a quadric hypersurface in $\mathbb{P}^7_{\mathbb{R}}$, called *Study quadric*. This compactification of SE_3 is extremely useful in the study of overconstrained mechanisms (e.g. Hegedüs et al [2013], Nawratil [2014a]). $\diamond$

2. Special compactification X of SE(3)

But it turns out that for n-pods the following different compactification will lead to a better comprehension of the phenomena which can arise.

Any direct isometry of $\mathbb{R}^3$ can be written as a pair $(\mathbf{M}, \mathbf{y})$, where $\mathbf{M} \in \mathrm{SO}_3$ and $\mathbf{y} \in \mathbb{R}^3$. We define

$$\mathbf{x} := -\mathbf{M}^t\mathbf{y} = -\mathbf{M}^{-1}\mathbf{y} \quad \text{and} \quad r := \langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. Moreover we denote

- the coordinate vector of p_i with respect to the moving frame by $\mathbf{p}_i = (a_i, b_i, c_i)^t$
- the coordinate vector of P_i with respect to the fixed frame by $\mathbf{P}_i = (A_i, B_i, C_i)^t$.

2. Special compactification X of SE(3)

Using this notation the sphere condition can be rewritten as

$$\begin{aligned} d_i^2 &= \langle \mathbf{M}\mathbf{p}_i + \mathbf{y} - \mathbf{P}_i, \mathbf{M}\mathbf{p}_i + \mathbf{y} - \mathbf{P}_i \rangle \\ &= \langle \mathbf{M}\mathbf{p}_i, \mathbf{M}\mathbf{p}_i \rangle + 2\langle \mathbf{M}\mathbf{p}_i, \mathbf{y} \rangle + r + \langle \mathbf{P}_i, \mathbf{P}_i \rangle - 2\langle \mathbf{M}\mathbf{p}_i, \mathbf{P}_i \rangle - 2\langle \mathbf{y}, \mathbf{P}_i \rangle \\ &= \langle \mathbf{p}_i, \mathbf{p}_i \rangle + \langle \mathbf{P}_i, \mathbf{P}_i \rangle + r + 2\langle \mathbf{p}_i, \mathbf{M}^t\mathbf{y} \rangle - 2\langle \mathbf{M}\mathbf{p}_i, \mathbf{P}_i \rangle - 2\langle \mathbf{y}, \mathbf{P}_i \rangle \\ &= \langle \mathbf{p}_i, \mathbf{p}_i \rangle + \langle \mathbf{P}_i, \mathbf{P}_i \rangle + r - 2\langle \mathbf{p}_i, \mathbf{x} \rangle - 2\langle \mathbf{y}, \mathbf{P}_i \rangle - 2\langle \mathbf{M}\mathbf{p}_i, \mathbf{P}_i \rangle. \end{aligned}$$

We consider the isometry $(\mathbf{M}, \mathbf{y})$ as a point in $\mathbb{P}^{16}_{\mathbb{R}}$ with coordinates:

$$(h : \underbrace{m_{11} : m_{12} : \ldots : m_{33}}_{\text{entries of } \mathbf{M}} : \underbrace{x_1 : x_2 : x_3}_{\text{coord. of } \mathbf{x}} : \underbrace{y_1 : y_2 : y_3}_{\text{coord. of } \mathbf{y}} : r)$$

which are abbreviated by $(h : \mathbf{M} : \mathbf{x} : \mathbf{y} : r)$, where h is a homogenizing coordinate.

2. Special compactification X of SE(3)

The group SE_3 is defined by the inequality $h \neq 0$ and the equations

$$\mathbf{M}\mathbf{M}^t = \mathbf{M}^t\mathbf{M} = h^2\mathbf{I}, \;\; \det(\mathbf{M}) = h^3,$$
$$\mathbf{M}^t\mathbf{y} + h\mathbf{x} = \mathbf{o}, \;\; \mathbf{M}\mathbf{x} + h\mathbf{y} = \mathbf{o},$$
$$\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = rh.$$

These equations define a variety X in $\mathbb{P}^{16}_{\mathbb{C}}$, whose real points satisfying $h \neq 0$ are in one to one correspondence with the elements of SE_3. A direct computer aided calculation shows that X is a projective variety of dimension 6 and degree 40.

The main feature of this choice of coordinates is that after homogenization of the sphere condition, it becomes linear in the projective coordinates of $\mathbb{P}^{16}_{\mathbb{R}}$; i.e.:

$$\left(\langle \mathbf{p}_i, \mathbf{p}_i \rangle + \langle \mathbf{P}_i, \mathbf{P}_i \rangle - d_i^2\right) h + r - 2\langle \mathbf{p}_i, \mathbf{x} \rangle - 2\langle \mathbf{y}, \mathbf{P}_i \rangle - 2\langle \mathbf{M}\mathbf{p}_i, \mathbf{P}_i \rangle \; = \; 0.$$

3. Action of SE(3) on X

The representation $(\mathbf{M}, \mathbf{y})$ of the direct isometry σ depends on the embedding ϕ of the platform into the moving space. Taking this into consideration the sphere condition reads as:

$$\|\sigma(\phi(\mathsf{p}_i)) - \mathsf{P}_i\| = d_i$$

Multiplication of σ from right by ϕ given by the direct isometry $(\mathbf{M}_R, \mathbf{y}_R)$ is sending $(h : \mathbf{M} : \mathbf{x} : \mathbf{y} : r)$ to

$$\left(hh_R : \mathbf{M}\mathbf{M}_R : \mathbf{M}_R^t\mathbf{x} + h\mathbf{x}_R : h_R\mathbf{y} + \mathbf{M}\mathbf{y}_R : h_R r + h r_R - 2\langle \mathbf{x}, \mathbf{y}_R \rangle\right)$$

where $(h_R : \mathbf{M}_R : \mathbf{x}_R : \mathbf{y}_R : r_R) \in X$ is the kinematic image of $(\mathbf{M}_R, \mathbf{y}_R)$.

3. Action of SE(3) on X

The representation $(\mathbf{M}, \mathbf{y})$ of the direct isometry σ also depends on the embedding Φ of the base into the fixed space. Taking this into consideration the sphere condition reads as:

$$\|\sigma(\mathsf{p}_i) - \Phi(\mathsf{P}_i)\| = d_i \quad \Longrightarrow \quad \left\|\Phi^{-1}(\sigma(\mathsf{p}_i)) - \mathsf{P}_i\right\| = d_i$$

Multiplication of σ from left by Φ^{-1} given by the direct isometry $(\mathbf{M}_L, \mathbf{y}_L)$ is sending $(h : \mathbf{M} : \mathbf{x} : \mathbf{y} : r)$ to

$$\left(h_L h : \mathbf{M}_L \mathbf{M} : \mathbf{M}^t \mathbf{x}_L + h_L \mathbf{x} : h \mathbf{y}_L + \mathbf{M}_L \mathbf{y} : h r_L + h_L r - 2\langle \mathbf{x}_L, \mathbf{y} \rangle\right)$$

where $(h_L : \mathbf{M}_L : \mathbf{x}_L : \mathbf{y}_L : r_L) \in X$ is the kinematic image of $(\mathbf{M}_L, \mathbf{y}_L)$.

4. Boundary of X

The boundary B of X is defined as the closed subset of X cut out by the linear equation $h = 0$. A point of B has to fulfill the following conditions:

$$\begin{array}{ccc} \mathbf{M}\mathbf{M}^t = \mathbf{M}^t\mathbf{M} = h^2\mathbf{I},\ \det(\mathbf{M}) = h^3, & & \mathbf{M}\mathbf{M}^t = \mathbf{M}^t\mathbf{M} = \mathbf{O},\ \det(\mathbf{M}) = 0, \\ \mathbf{M}^t\mathbf{y} + h\mathbf{x} = \mathbf{o},\ \mathbf{M}\mathbf{x} + h\mathbf{y} = \mathbf{o}, & \to & \mathbf{M}^t\mathbf{y} = \mathbf{o},\ \mathbf{M}\mathbf{x} = \mathbf{o}, \\ \langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = rh & & \langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 0 \end{array}$$

Due to $\mathbf{M}\mathbf{M}^t = \mathbf{M}^t\mathbf{M} = \mathbf{O}$ the vectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ with $\mathbf{M} := (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ span a totally isotropic subspace $U \in \mathbb{C}^3$ with respect to $\langle \cdot, \cdot \rangle$.

As $dim(U) + dim(U^\perp) = 3$ and $U \subset U^\perp$ has to hold we get:

$$dim(U) \leq dim(U^\perp) \ \Rightarrow \ 2dim(U) \leq dim(U) + dim(U^\perp) = 3 \ \Rightarrow \ rk(\mathbf{M}) \leq 1.$$

4. Boundary of $\mathbf{X}$

Beside the trivial case $\mathbf{M} = \mathbf{O}$ we study $\mathbf{M} \neq \mathbf{O}$ in more detail. Hence $\mathbf{M} = \mathbf{v}\mathbf{w}^t$ has to hold for two suitable non-zero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^3$, which are not unique.

$$\mathbf{M}\mathbf{M}^t = \mathbf{O} \quad \Rightarrow \quad \mathbf{v}\mathbf{w}^t\mathbf{w}\mathbf{v}^t = \mathbf{O} \quad \Rightarrow \quad \langle \mathbf{w}, \mathbf{w} \rangle = 0$$

$$\mathbf{M}\mathbf{x} = \mathbf{o} \quad \Rightarrow \quad \mathbf{v}\mathbf{w}^t\mathbf{x} = \mathbf{o} \quad \Rightarrow \quad \langle \mathbf{w}, \mathbf{x} \rangle = 0$$

together with $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ we see that $\mathbf{x}, \mathbf{w}$ span a totally isotropic subspace of $\mathbb{C}^3$
$\Rightarrow$ $\mathbf{x}$ and $\mathbf{w}$ are linearly dependent

$$\mathbf{M}^t\mathbf{M} = \mathbf{O} \quad \Rightarrow \quad \mathbf{w}\mathbf{v}^t\mathbf{v}\mathbf{w}^t = \mathbf{O} \quad \Rightarrow \quad \langle \mathbf{v}, \mathbf{v} \rangle = 0$$

$$\mathbf{M}^t\mathbf{y} = \mathbf{o} \quad \Rightarrow \quad \mathbf{w}\mathbf{v}^t\mathbf{y} = \mathbf{o} \quad \Rightarrow \quad \langle \mathbf{v}, \mathbf{y} \rangle = 0$$

together with $\langle \mathbf{y}, \mathbf{y} \rangle = 0$ we see that $\mathbf{y}, \mathbf{v}$ span a totally isotropic subspace of $\mathbb{C}^3$
$\Rightarrow$ $\mathbf{y}$ and $\mathbf{v}$ are linearly dependent

4. Boundary of X

By denoting $\mathbf{N} = r\mathbf{M} + 2\mathbf{y}\mathbf{x}^t$ (invariant under left and right translations) we can partition the boundary in the following subsets:

1. **Vertex**:= $\{(0 : \mathbf{M} : \mathbf{x} : \mathbf{y} : r) \,|\, \mathbf{M} = \mathbf{O}$ and $\mathbf{x} = \mathbf{y} = \mathbf{o}\}$ is only real point of B.
2. **Inversion points**:= $\{(0 : \mathbf{M} : \mathbf{x} : \mathbf{y} : r) \,|\, \mathbf{M} \neq \mathbf{O}$ and $\mathbf{N} \neq \mathbf{O}\}$
   By suitable left and right multiplications we can achieve the normal form
   $$\beta = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{0 : 0 : 0}_{\mathbf{x}} : \underbrace{0 : 0 : 0}_{\mathbf{y}} : r) \quad \text{with} \quad r \in \mathbb{R}_{>0}$$
3. **Butterfly points**:= $\{(0 : \mathbf{M} : \mathbf{x} : \mathbf{y} : r) \,|\, \mathbf{M} \neq \mathbf{O}$ and $\mathbf{N} = \mathbf{O}\}$
   By suitable left and right multiplications we can achieve the normal form
   $$\beta = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{0 : 0 : 0}_{\mathbf{x}} : \underbrace{0 : 0 : 0}_{\mathbf{y}} : 0)$$

4. Boundary of X

4. **Similarity points**$:= \{(0 : \mathbf{M} : \mathbf{x} : \mathbf{y} : r) \,|\, \mathbf{M} = \mathbf{O} \text{ and } \mathbf{x} \neq \mathbf{o} \neq \mathbf{y}\}$
By suitable left and right multiplications we can achieve the normal form

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{\gamma : i\gamma : 0}_{\mathbf{x}} : \underbrace{1 : i : 0}_{\mathbf{y}} : 0) \quad \text{with} \quad \gamma \in \mathbb{R}_{>0}$$

5a. **Left collinearity points**$:= \{(0 : \mathbf{M} : \mathbf{x} : \mathbf{y} : r) \,|\, \mathbf{M} = \mathbf{O} \text{ and } \mathbf{x} = \mathbf{o} \neq \mathbf{y}\}$
By suitable left and right multiplications we can achieve the normal form

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{0 : 0 : 0}_{\mathbf{x}} : \underbrace{1 : i : 0}_{\mathbf{y}} : 0)$$

5b. **Right collinearity points**$:= \{(0 : \mathbf{M} : \mathbf{x} : \mathbf{y} : r) \,|\, \mathbf{M} = \mathbf{O} \text{ and } \mathbf{x} \neq \mathbf{o} = \mathbf{y}\}$
By suitable left and right multiplications we can achieve the normal form

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{1 : i : 0}_{\mathbf{x}} : \underbrace{0 : 0 : 0}_{\mathbf{y}} : 0)$$

5. Directions associated with boundary points

Inversion and butterfly points: we have $\mathbf{M} = \mathbf{v}\mathbf{w}^t$ with $\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{w} \rangle = 0$.

Therefore $\mathbf{v}$ and $\mathbf{w}$ can be seen as points of the conic $C = \{\alpha^2 + \beta^2 + \gamma^2 = 0\}$ in $\mathbb{P}^2_{\mathbb{C}}$, which is isomorphic to S^2 due to the following construction:

$$\mathbf{c} \in C \quad \text{with} \quad \mathbf{c} = \mathbf{t} + i\mathbf{u} \quad \text{and} \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^3$$

$$\langle \mathbf{c}, \mathbf{c} \rangle = 0 \Rightarrow \langle \mathbf{t}, \mathbf{t} \rangle - \langle \mathbf{u}, \mathbf{u} \rangle + 2i \langle \mathbf{t}, \mathbf{u} \rangle = 0$$

$\mathbf{t}, \mathbf{u}$ are orthogonal vectors of equal length; w.l.o.g. we can assume $\mathbf{t}, \mathbf{u} \in S^2$

$$\iota : \ C \to S^2 : \quad \mathbf{c} \mapsto \mathbf{s} := \mathbf{t} \times \mathbf{u}$$

Conversely, let $\mathbf{s} \in S^2$ then we can find vectors $\mathbf{t}, \mathbf{u} \in S^2$ in a way that $\mathbf{t}, \mathbf{u}, \mathbf{s}$ is a right-handed Cartesian frame.

$$\iota^{-1} : \ S^2 \to C : \quad \mathbf{s} \mapsto \mathbf{c} := \mathbf{t} + i\mathbf{u}$$

5. Directions associated with boundary points

Similarity points: In this case we have $\mathbf{M} = \mathbf{O}$.

But for all boundary points the two matrices $\mathbf{M}$ and $\mathbf{y}\mathbf{x}^t$ are linear dependent, and in the case of similarity points $\mathbf{y}\mathbf{x}^t \neq \mathbf{O}$.
Moreover $\mathbf{x}$ and $\mathbf{y}$ satisfy $\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 0$. So we can associate to a similarity point the pair of elements of S^2 coming from the vectors $\mathbf{x}$ and $\mathbf{y}$.

Definition 1.
Via these identifications we can associate to every inversion, butterfly or similarity point β in B a pair $(\mathbf{l}, \mathbf{r})$ of elements of S^2, where the so-called

- *right vector* $\mathbf{r}$ equals $\iota(\mathbf{w})$ and $\iota(\mathbf{x})$, respectively, and
- *left vector* $\mathbf{l}$ equals $\iota(\mathbf{v})$ and $\iota(\mathbf{y})$, respectively.

Left/right collinear points are only associated with the left/right vector.

5. Directions associated with boundary points

Left and right multiplications of the bonds imply

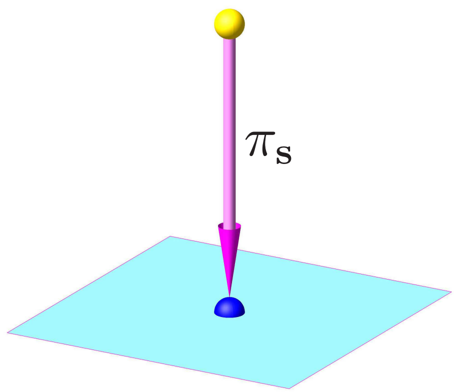
$$\underbrace{\mathbf{v}}_{\cong \mathbf{l}}\underbrace{\mathbf{w}^t}_{\cong \mathbf{r}^t} \mapsto \underbrace{\mathbf{M}_L\mathbf{v}}_{\cong \Phi(\mathbf{l})}\underbrace{\mathbf{w}^t\mathbf{M}_R}_{\cong \phi(\mathbf{r})^t} \qquad \underbrace{\mathbf{y}}_{\cong \mathbf{l}} \mapsto \underbrace{\mathbf{M}_L\mathbf{y}}_{\cong \Phi(\mathbf{l})} \qquad \underbrace{\mathbf{x}}_{\cong \mathbf{r}} \mapsto \underbrace{\mathbf{M}_R^t\mathbf{x}}_{\cong \phi(\mathbf{r})}$$

$\Rightarrow$ left/right vector is changing correspondingly to the embedding Φ/ϕ of the base/platform into the fixed/moving space.

Definition 2.
Given a unit vector $\mathbf{s} \in S^2$, we denote by

$$\pi_{\mathbf{s}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

the *orthogonal projection along* $\mathbf{s}$.



6a. Bond Theory: Basics

Definition 3.
Let Π be an n-pod, then the intersection of X with the hyperplanes defined by

$$\left(\langle \mathbf{p}_i, \mathbf{p}_i\rangle + \langle \mathbf{P}_i, \mathbf{P}_i\rangle - d_i^2\right) h + r - 2\,\langle \mathbf{p}_i, \mathbf{x}\rangle - 2\,\langle \mathbf{y}, \mathbf{P}_i\rangle - 2\,\langle \mathbf{M}\mathbf{p}_i, \mathbf{P}_i\rangle \;=\; 0$$

for $i \in \{1, \ldots, n\}$ is called the *complex configuration set* K_Π of the n-pod.

Definition 4.
Let Π be an n-pod, we define its set of *bonds* B_Π as the intersection of K_Π and the boundary B of X.

If an n-pod Π is mobile, then $\dim K_\Pi \cap (X \setminus B) \geq 1 \quad \Rightarrow \quad \dim K_\Pi \geq 1$.
Since B_Π is an hyperplane section of K_Π, the dimension decreases at most by 1
$\Rightarrow B_\Pi$ is not empty

6b. Bond Theory: Lemmata

Lemma 1.
Assume that $\beta \in B_\Pi$ is an inversion/similarity bond of Π. Let $\mathbf{l}, \mathbf{r} \in S^2$ be the left and right vector of β. Then there is an inversion/similarity of $\mathbb{R}^2$ mapping $\pi_\mathbf{r}(\mathsf{p}_1), \ldots, \pi_\mathbf{r}(\mathsf{p}_n)$ to $\pi_\mathbf{l}(\mathsf{P}_1), \ldots, \pi_\mathbf{l}(\mathsf{P}_n)$.

Conversely, let $\mathbf{l}, \mathbf{r} \in S^2$ such that the images of $(\mathsf{p}_1, \ldots, \mathsf{p}_n)$ under $\pi_\mathbf{r}$ and of $(\mathsf{P}_1, \ldots, \mathsf{P}_n)$ under $\pi_\mathbf{l}$ differ by an inversion/ similarity. Then Π has an inversion/similarity bond with left vector $\mathbf{l}$ and right vector $\mathbf{r}$.

Proof: We can apply left and right multiplications such that β is in normal form:

$$\beta = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{0 : 0 : 0}_{\mathbf{x}} : \underbrace{0 : 0 : 0}_{\mathbf{y}} : r) \quad \text{with} \quad r \in \mathbb{R}_{>0}$$

$$\beta = (0 : \overbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0} : \overbrace{\gamma : i\gamma : 0} : \overbrace{1 : i : 0} : 0) \quad \text{with} \quad \gamma \in \mathbb{R}_{>0}$$

which yields $\mathbf{l} = \mathbf{r} = (0, 0, 1)$.

6b. Bond Theory: Lemmata

$$\pi_{\mathbf{l}} : (A_i, B_i, C_i) \mapsto (A_i, B_i), \qquad \pi_{\mathbf{r}} : (a_i, b_i, c_i) \mapsto (a_i, b_i).$$

Moreover the sphere condition simplifies to:

$$r - 2\langle \mathbf{M}\mathbf{p}_i, \mathbf{P}_i \rangle = 0 \qquad r - 2\langle \mathbf{p}_i, \mathbf{x} \rangle - 2\langle \mathbf{y}, \mathbf{P}_i \rangle = 0,$$

which implies:

$$\begin{cases} a_i A_i - b_i B_i = \frac{r}{2} \\ b_i A_i + a_i B_i = 0 \end{cases} \Rightarrow a_i + ib_i = \frac{\frac{r}{2}}{A_i + iB_i} \qquad \begin{cases} A = -\gamma\, a \\ B = -\gamma\, b \end{cases}$$

This is an inversion (plus reflection with respect to the real axis) and a similarity, respectively.

The converse can be proven by going backwards in the previous arguments. $\square$

6b. Bond Theory: Lemmata

Lemma 2.
Assume that $\beta \in B_\Pi$ is a butterfly bond of Π. Let $\mathbf{l}, \mathbf{r} \in S^2$ be the left and right vector of β. Then, up to permutation of indices, there exists $m \leq n$ such that $\mathsf{p}_1, \ldots, \mathsf{p}_m$ are collinear on a line parallel to $\mathbf{r}$, and $P_{\mathsf{m}+1}, \ldots, \mathsf{P}_n$ are collinear on a line parallel to $\mathbf{l}$.

Conversely, let $\mathbf{l}, \mathbf{r} \in S^2$ such that $\mathsf{p}_1, \ldots, \mathsf{p}_m$ are collinear on a line parallel to $\mathbf{r}$, and $\mathsf{P}_{m+1}, \ldots, \mathsf{P}_n$ are collinear on a line parallel to $\mathbf{l}$. Then Π has a butterfly bond with left vector $\mathbf{l}$ and right vector $\mathbf{r}$.

Proof: Plugging the normal form of the butterfly bond

$$\beta = (0 : \underbrace{1 : i : 0 : i : -1 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{0 : 0 : 0}_{\mathbf{x}} : \underbrace{0 : 0 : 0}_{\mathbf{y}} : 0)$$

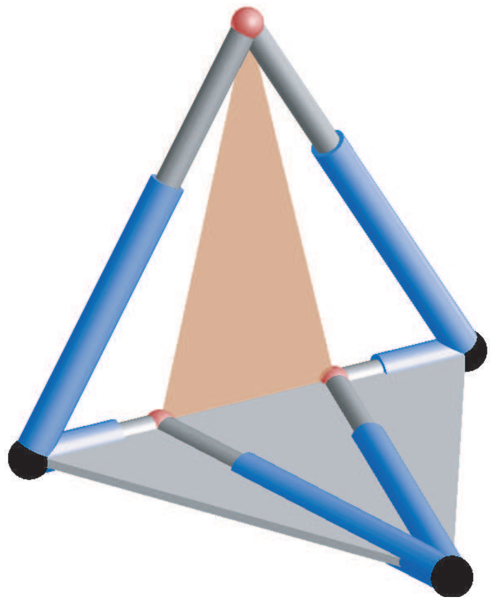
into the sphere condition implies

6b. Bond Theory: Lemmata

$$\begin{cases} a_iA_i - b_iB_i = 0 \\ a_iB_i + b_iA_i = 0 \end{cases} \Rightarrow (a_i, b_i) = (0,0) \quad \text{or} \quad (A_i, B_i) = (0,0).$$

which shows the result. By reversing the arguments we get the converse. □

Remark: The existence of a butterfly bond is already sufficient for the existence of n leg lengths such that the n-pod is mobile: If the platform is located in a way that the carrier line of $\mathsf{p}_1, \ldots, \mathsf{p}_m$ coincides with the carrier line of $\mathsf{P}_{m+1}, \ldots, \mathsf{P}_n$, then the platform can rotate freely about this line. The nomenclature goes back to KARGER [2010]. ◇



Octahedral Hexapod

6b. Bond Theory: Lemmata

Lemma 3a.
Assume that $\beta \in B_\Pi$ is a left collinearity bond of Π. Let $\mathbf{l} \in S^2$ be the left vector of β. Then $\mathsf{P}_1, \ldots, \mathsf{P}_n$ are collinear on a line parallel to $\mathbf{l}$.

Conversely, if $\mathsf{P}_1, \ldots, \mathsf{P}_n$ are collinear on a line parallel to $\mathbf{l}$, then Π has a left collinearity bond with left vector $\mathbf{l}$.

Proof: Plugging the normal form normal form of the left collinearity bond

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{0 : 0 : 0}_{\mathbf{x}} : \underbrace{1 : i : 0}_{\mathbf{y}} : 0)$$

into the sphere condition implies

$$-2(A_i + iB_i) = 0 \Rightarrow A_i = B_i = 0$$

which shows the result. By reversing the arguments we get the converse. $\square$

6b. Bond Theory: Lemmata

Lemma 3b.
Assume that $\beta \in B_\Pi$ is a right collinearity bond of Π. Let $\mathbf{r} \in S^2$ be the right vector of β. Then $\mathsf{p}_1, \ldots, \mathsf{p}_n$ are collinear on a line parallel to $\mathbf{r}$.

Conversely, if $\mathsf{p}_1, \ldots, \mathsf{p}_n$ are collinear on a line parallel to $\mathbf{r}$ then Π has a collinearity bond with right vector $\mathbf{r}$.

Proof: Plugging the normal form normal form of the left collinearity bond

$$\beta = (0 : \underbrace{0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0}_{\mathbf{M}} : \underbrace{1 : i : 0}_{\mathbf{x}} : \underbrace{0 : 0 : 0}_{\mathbf{y}} : 0)$$

into the sphere condition implies

$$-2(a_i + ib_i) = 0 \Rightarrow a_i = b_i = 0$$

which shows the result. By reversing the arguments we get the converse. □

6c. Bond Theory: Main Theorem

Finally the vertex cannot be contained in B_Π as the insertion of its coordinates into the sphere condition implies the contradiction $1 = 0$. This result and Lemmata 1, 2, 3a and 3b imply the following:

Main Theorem.
If an n-pod is mobile, then one of the following conditions holds:

- There exists at least one pair of orthogonal projections $\pi_\mathbf{l}$ and $\pi_\mathbf{r}$ such that the projections of the base points $\mathsf{P}_1, \ldots, \mathsf{P}_n$ by $\pi_\mathbf{l}$ and platform points $\mathsf{p}_1, \ldots, \mathsf{p}_n$ by $\pi_\mathbf{r}$ differ by an inversion or a similarity.

- There exists $m \leq n$ such that $\mathsf{p}_1, \ldots, \mathsf{p}_m$ are collinear and $\mathsf{P}_{m+1}, \ldots, \mathsf{P}_n$ are collinear, up to permutation of indices.

6c. Bond Theory: Corollary

Corollary.
Let Π be an n-pod with mobility 2 or higher. Then one of the following holds:

(a) there are infinitely many pair $(\mathbf{l}, \mathbf{r}) \in S^2 \times S^2$ such that the points $\pi_{\mathbf{r}}(\mathsf{p}_1), \ldots, \pi_{\mathbf{r}}(\mathsf{p}_n)$ and $\pi_{\mathbf{l}}(\mathsf{P}_1), \ldots, \pi_{\mathbf{l}}(\mathsf{P}_n)$ differ by an inversion or a similarity;

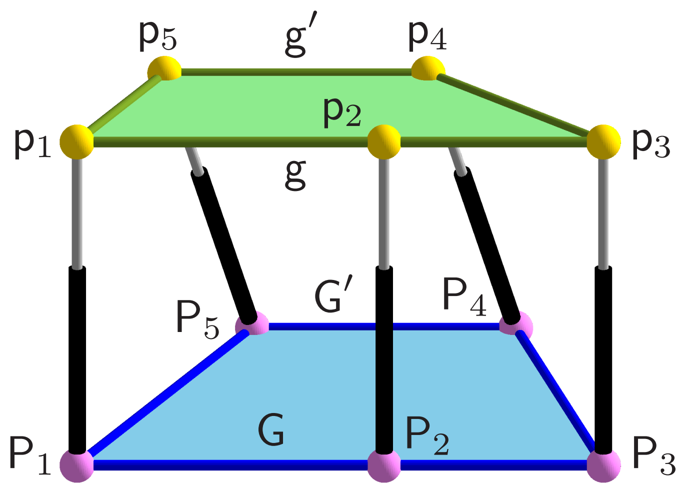
(b) there exists $m \leq n$ such that $\mathsf{p}_1, \ldots, \mathsf{p}_m$ are collinear and $\mathsf{P}_{m+1} = \ldots = \mathsf{P}_n$, up to permutation of indices and interchange between base and platform;

(c) there exists m with $1 < m < n-1$ such that, up to permutation of indices,
   - ⋆ $\mathsf{p}_1, \ldots, \mathsf{p}_m$ lie on a line g and $\mathsf{p}_{m+1}, \ldots, \mathsf{p}_n$ lie on a line $\mathsf{g}' \parallel \mathsf{g}$, and
   - ⋆ $\mathsf{P}_1, \ldots, \mathsf{P}_m$ lie on a line G and $\mathsf{P}_{m+1}, \ldots, \mathsf{P}_n$ lie on a line $\mathsf{G}' \parallel \mathsf{G}$.

Proof: Since Π has mobility at least 2, it has infinitely many bonds.

6c. Bond Theory: Corollary

- Π admits one collinearity bond $\Rightarrow$ (b) with $m = n$
- Π admits infinitely many butterfly points: There exists $m \leq n$ such that $\mathsf{p}_1, \ldots, \mathsf{p}_m$ are collinear and $\mathsf{P}_{m+1}, \ldots, \mathsf{P}_n$ lie on infinitely many lines $\Rightarrow$ (b)
- Π admits infinitely many inversion/similarity bonds:
  - ⋆ These bonds provide infinitely many different left and right vectors $\Rightarrow$ (a)
  - ⋆ Otherwise infinitely many inversion/similarity points have the same left and right vector. The fact that an inversion/similarity is completely specified if we prescribe the image of three/two points implies (c). □



5-pod of case (c) for $m = 3$

7. Outline of Pentapods with Mobility 2

As pods with mobility greater than 2 were already determined by Nawratil [2014b], we used this Corollary as the starting point for a complete classification of pentapods with mobility 2.

By means of Möbius photogrammetry given in:

Gallet M., Nawratil G., Schicho J. [2015a] *Möbius Photogrammetry. Journal of Geometry 106(3):421–442*

we were able to achieve the following

Specification of case (a).
For a pentapod with mobility 2 of case (a) one of the following holds:

(i) The platform and the base are similar.

(ii) The platform and the base are planar and affine equivalent.

7. Outline of Pentapods with Mobility 2

(iii) The following triples of points are collinear:

$$P_1, P_2, P_3, \quad P_3, P_4, P_5, \quad p_3, p_1, p_i, \quad p_3, p_j, p_k,$$

with pairwise distinct $i, j, k \in \{2, 4, 5\}$. Moreover the points $M_1, \ldots, M_5$ are pairwise distinct as well as the points $m_1, \ldots, m_5$.

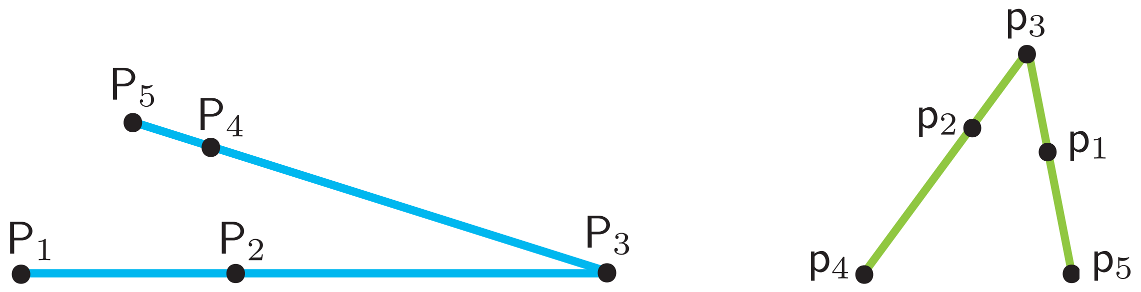


Illustration of item (iii) for $i = 5$, $j = 2$ and $k = 4$.

7. Outline of Pentapods with Mobility 2

Based on this preparatory work a full classification of pentapods with mobility 2:

- with a collinearity bond were listed in:

  NAWRATIL G., SCHICHO J. [in press] *Self-motions of pentapods with linear platform. Robotica*

- without a collinearity bond were listed in:

  NAWRATIL G., SCHICHO J. [2015] *Pentapods with Mobility 2. ASME Journal of Mechanisms and Robotics 7(3):031016*

  with exception of the case (a)(iii), which was studied in:

  NAWRATIL G., SCHICHO J. [2016] *Addendum to Pentapods with Mobility 2. arXiv: 1602.00932*

References

The presented results are contained in:

GALLET M., NAWRATIL G., SCHICHO J. [2015b] *Bond theory for pentapods and hexapods. Journal of Geometry 106(2):211–228*

The list of referred publications:

HEGEDÜS G., SCHICHO J., SCHRÖCKER H.-P. [2013] *The Theory of Bonds: A New Method for the Analysis of Linkages. Mechanism and Machine Theory 70:404–424*

KARGER A. [2010] *Self-motions of 6-3 Stewart-Gough type parallel manipulators. Advances in Robot Kinematics: Motion in Man and Machine, Springer, pp. 359–366*

MERLET J.-P. [1989] *Singular Configurations of Parallel Manipulators and Grassmann geometry. International Journal of Robotics Research 8(5):45–56*

NAWRATIL G. [2014a] *Introducing the theory of bonds for Stewart Gough platforms with self-motions. ASME Journal of Mechanisms and Robotics 6(1):011004*

NAWRATIL G. [2014b] *On Stewart Gough manipulators with multidimensional self-motions. Computer Aided Geometric Design 31(7-8):582–594*