Fundamentals of Quaternionic Kinematics in Euclidean 4-Space

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Abstract

Based on a kinematic mapping for the group SE(4) of displacements of Euclidean 4-space, we show that the mapping of basic elements (points, oriented lines, oriented planes, oriented hyperplanes, instantaneous screws) can be written compactly in terms of 2×2 quaternionic matrices. Moreover we discuss the kinematics on velocity level by investigating instantaneous screws and their geometric parameters.

Keywords: Kinematic Mapping, Euclidean 4-space, Quaternion, Quaternionic Matrix, Instantaneous Screw

1. Introduction

The elegance of the quaternion based analytical treatment of kinematics in Euclidean spaces of dimension 2 and 3 was pointed out and used by various authors (e.g. Blaschke [3], Müller [21], Ströher [29]). The quaternionic approach does not only yield a more compact notation in comparison with matrices (which also implies some computational advantages used in robotics [26, 33]), but it also provides an easier access to the geometry of motions.

Motivated by this circumstance, the author wants to extend this quaternionic kinematic to the Euclidean 4-space E^4 in the tradition of the above cited works [3, 21, 29]. As we are dealing with fundamentals of kinematics in E^4, it is clear that not all results presented in this work are totally novel (seen from the view point of linear algebra or Lie algebra), but they also require a quaternionic formulation in order to present a complete theory.

A first step in this direction was already done by the author in [23], where a kinematic mapping for the group of displacements (= orientation preserving congruence transformations) in E^4 was introduced, which can be seen as the generalization of the Blaschke-Grünwald parameters of E^2 and the Study parameters of E^3. This quaternion based kinematic parameters of E^4 are repeated in Section 2 of the paper. Moreover the notation of [23] is slightly modified in order to get more suitable formulas and representations.

We go on with a detailed study of rotations in terms of quaternions in Section 3. In Section 4 we show that the displacements of points, oriented hyperplanes, oriented lines and oriented planes can be embedded into the algebra of 2×2 quaternionic matrices, which allows a very compact and elegant notation. Finally in Section 5 we study fundamentals of the velocity analysis; namely instantaneous screws and their geometric parameters.

But before we can plunge in medias res, we have to provide some basics on quaternions as well as a literature review.

1.1. Basics on quaternions

$\mathbf{Q} := q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with $q_0, \ldots, q_3 \in \mathbb{R}$ is an element of the skew field of quaternions $\mathbb{H}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the so-called quaternion units, which are multiplied according to the following rules:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{ii} = \mathbf{jj} = \mathbf{kk} = -1$$

It can be seen within this first formula, that we write quaternions just side by side for multiplication instead of introducing an extra multiplication sign. By this way the notation gets more compact without yielding confusions, as only quaternions are printed in bold letters.

q_0 is the so-called scalar part of the quaternion and $q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ its pure part. By denoting the pure part by small letters $\mathbf{q}$, the quaternion can also be written as $\mathbf{Q} := q_0 + \mathbf{q}$. Moreover $\mathbf{Q}$ is called a pure quaternion for $q_0 = 0$ ($\Leftrightarrow$ $\mathbf{Q} = \mathbf{q}$) and a scalar quaternion for $\mathbf{q} = \mathbf{o}$ ($\Leftrightarrow$ $\mathbf{Q} = q_0$), where $\mathbf{o}$ denotes the zero pure quaternion. Therefore the scalar multiplication is just the quaternionic multiplication by a scalar quaternion. We call the scalar quaternion with scalar part 1 the identity quaternion. The zero quaternion $\mathbf{O} = 0 + \mathbf{o}$ is the only quaternion, which is a scalar one and a pure one at the same time.

The conjugated quaternion to $\mathbf{Q}$ is given by $\widetilde{\mathbf{Q}} := q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$, which implies $\widetilde{\mathbf{q}} = -\mathbf{q}$ for a pure quaternion $\mathbf{q}$. Moreover the calculation rule $(\mathbf{PQ})^\sim = \widetilde{\mathbf{Q}}\widetilde{\mathbf{P}}$ holds.

We can project any quaternion $\mathbf{Q}$ onto its pure and scalar part by the following mappings:

$$\mathbf{Q} \mapsto \frac{\mathbf{Q} - \widetilde{\mathbf{Q}}}{2} = \mathbf{q} \quad \text{and} \quad \mathbf{Q} \mapsto \frac{\mathbf{Q} + \widetilde{\mathbf{Q}}}{2} = q_0. \tag{1}$$

Moreover the multiplication of two quaternions commutes if the corresponding pure parts differ only by a scalar multiplication factor; i.e.

$$\mathbf{PQ} = \mathbf{QP} \quad \Longleftrightarrow \quad \exists (p, q) \neq (0, 0) \quad \text{with} \quad p\mathbf{p} + q\mathbf{q} = \mathbf{o}.$$

Quaternions with this property are also known as *coaxial* quaternions.

The scalar product of two quaternions, which is defined by:

$$\langle \mathbf{P}, \mathbf{Q} \rangle := \frac{\mathbf{P}\widetilde{\mathbf{Q}} + \mathbf{Q}\widetilde{\mathbf{P}}}{2} = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3,$$

induces the norm:

$$\|\mathbf{Q}\| = \sqrt{\langle \mathbf{Q}, \mathbf{Q} \rangle} = \sqrt{\mathbf{Q}\widetilde{\mathbf{Q}}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Quaternions $\mathbf{P}$ and $\mathbf{Q}$ with $\langle \mathbf{P}, \mathbf{Q} \rangle = 0$ are called *orthogonal* and quaternions with a norm of 1 are called unit-quaternions.

Moreover every quaternion $\mathbf{Q} \neq \mathbf{O}$ has a (left or right) multiplicative inverse $\mathbf{Q}^{-1}$, which is given by $\widetilde{\mathbf{Q}}\|\mathbf{Q}\|^{-2}$. Especially if $\mathbf{Q}$ is a unit-quaternion, we get $\mathbf{Q}^{-1} = \widetilde{\mathbf{Q}}$.

For the differentiation of quaternions $\mathbf{P}$ and $\mathbf{Q}$ the following rules are valid:

$$(\mathbf{P} + \mathbf{Q})^{\cdot} = \dot{\mathbf{P}} + \dot{\mathbf{Q}}, \quad (\mathbf{PQ})^{\cdot} = \dot{\mathbf{P}}\mathbf{Q} + \mathbf{P}\dot{\mathbf{Q}}, \quad \dot{\widetilde{\mathbf{Q}}} = \widetilde{\dot{\mathbf{Q}}}, \tag{2}$$

which can easily be checked by direct computations.

1.2. Literature review

It can easily be shown by applying elementary linear algebra (see also Berger [1, Chapter 9] and Bottema and Roth [5, Chapter I]) that two different poses of a rigid body in E^4 can be mapped onto each other by one of the following displacements:

1. rotation about a plane Λ with angle λ,
2. rotation about two total-orthogonal planes Λ and Γ with angles λ and γ, respectively,
3. translation,
4. composition of a rotation about a plane Λ with angle λ and a translation parallel to this plane.

Moreover it is known (e.g. [6]) that each rotation is a composition of rotations about two total-orthogonal planes. These planes (and the corresponding angles of rotation) are determined uniquely with exception of so-called (left or right) isoclinic rotations, which can be written in terms of quaternions as follows:

$$\mathbf{X} \mapsto \mathbf{X}' = \mathbf{EX}, \qquad \mathbf{X} \mapsto \mathbf{X}' = \mathbf{X}\widetilde{\mathbf{F}},$$

where $\mathbf{E}$ and $\mathbf{F}$ are unit-quaternions. These special rotations are also known as (left and right) Clifford translations. In this case every plane through the origin spanned by $\mathbf{X}$ and $\mathbf{X}'$ and its total-orthogonal plane remain fixed. Therefore every isoclinic rotation can be decomposed in ∞^2 many ways into rotations about two total-orthogonal planes (with the same angle of rotation). More details on the geometric parameters of a rotation in E^4 (and their computation) are given in Section 3.

Beside these fundamental results only few kinematic studies are known to the author (cf. Pottmann [24]), which deal explicitly with the Euclidean 4-space.

- Basics of geometric kinematics of Euclidean 4-space were already given by Bottema and Roth [5, §2 of Chapter XII] based on the representation of a displacement by an orthogonal matrix plus a translation vector.
- Wunderlich [37] studied the screw motions of E^4 .
- Vogler [32] gave three examples of 2-parametric Euclidean motions, where all points run on 2-spheres/ellipsoids.

Beside the Chapters I and II of [5], where basics of displacements and instantaneous kinematics of n-dimensional Euclidean spaces are given, the following works are to the author's knowledge:

- Müller [22] showed for even dimensions that in general a fixed and a moving polode exist, which are rolling on each other without sliding during a constrained motion.

  In the non-general case (even dimensions) and in Euclidean spaces of odd dimensions there exist at each time instant of a constrained motion a so-called axis-space, which generates the fixed and moving axoide during the motion. Note that according to Tölke [31] this axis-space can be characterized as the location of all points, which have a minimum velocity.

  As these axoids are rolling and sliding[1] upon each other during the constrained motion, we have a so-called instantaneous helical motion (Schrotung in German). The work on the axoids and the implied instantaneous helical motions was furthered by Frank [8] and finally extensively studied by Friedrich and Spallek [9].
- By means of Lie groups, Karger studied the Darboux motions of E^n [18] and a further class of special motions, which can be characterized as follows according to Frank [8]: The Frenet frames of the pole curves resp. the Frenet frames of the unit-vectors in the generating lines of the instantaneous axoids correspond to each other under the motion.
- Drabek showed in [7] that points of the moving n-dimensional space, whose i-th velocity vector[2] is collinear with the connecting line of these points and a given point, are located on an algebraic curve of degree n in the generic case.

2. Kinematic mappings of SE(2), SE(3) and SE(4)

A kinematic mapping of SE(n) is a bijective mapping between the group of displacements of E^n and a set $\mathcal{S}$ of points in a certain space. Well known examples of these mappings are the one of Blaschke [2] and Grünwald [13] for E^2 and the one of Study [30] for E^3, which are reviewed within the next two subsections.

2.1. Study mapping of SE(3)

We can embed the points X of E^3 with Cartesian coordinates (x_1, x_2, x_3) into the set of pure quaternions by the following mapping:

$$\iota_3 : \mathbb{R}^3 \to \mathbb{H} \quad \text{with} \quad (x_1, x_2, x_3) \mapsto \mathbf{x} := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}. \tag{3}$$

Classically the Study mapping is introduced by the usage of dual quaternions $\mathbb{H} + \varepsilon\mathbb{H}$, where ε is the dual unit with the property $\varepsilon^2 = 0$. An element $\mathbf{E} + \varepsilon\mathbf{T}$ of $\mathbb{H} + \varepsilon\mathbb{H}$ is called dual unit-quaternion if $\mathbf{E}$ is a unit-quaternion and following condition holds:

$$e_0t_0 + e_1t_1 + e_2t_2 + e_3t_3 = 0. \tag{4}$$

Based on the usage of dual unit-quaternions $\mathbf{E} + \varepsilon\mathbf{T}$ it can be shown (e.g. [14, Section 3.3.2.2] or [5, Chapter XIII, §8]) that the mapping of points $\mathsf{X} \in E^3$ to $\mathsf{X}' \in E^3$ induced by any element of SE(3) can be written as follows (by using ι_3 of Eq. (3)):

$$\mathbf{x} \mapsto \mathbf{x}' \quad \textit{with} \quad \mathbf{x}' := \mathbf{E}\mathbf{x}\widetilde{\mathbf{E}} + \left(\mathbf{T}\widetilde{\mathbf{E}} - \mathbf{E}\widetilde{\mathbf{T}}\right). \tag{5}$$

[1] If no sliding takes place, we have again an instantaneous rolling motion.
[2] For $i = 1$ we get the ordinary velocity vector, for $i = 2$ the acceleration vector, etc.

Remark 1. Note that $\mathbf{x}'$ is again a pure quaternion, where the first summand $\mathbf{Ex}\widetilde{\mathbf{E}}$ is the rotational component and the remaining part corresponds to a translation, for which different conventions can be found in the literature. If the expression in the brackets is multiplied by the factor $\frac{1}{2}$ we obtain for example Study's soma coordinates (cf. Study [30] or [5, Example 68]). ⋄

Moreover it can be shown that the mapping of Eq. (5) is an element of SE(3) for any dual unit-quaternion $\mathbf{E} + \varepsilon\mathbf{T}$. As both dual unit-quaternions $\pm(\mathbf{E} + \varepsilon\mathbf{T})$ correspond to the same Euclidean motion of E^3, we consider the homogeneous 8-tuple $(e_0 : \ldots : e_3 : t_0 : \ldots : t_3)$. These so-called Study parameters can be interpreted as a point of a projective 7-dimensional space P^7. Therefore there is a bijection between SE(3) and all real points $\mathcal{S}$ of P^7 located on the so-called Study quadric $\Phi \subset P^7$, which is given by Eq. (4) ($\Rightarrow$ the signature of Φ is $(4_+, 4_-, 0_0)$) and is sliced along the 3-dimensional generator-space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathbf{E}$ cannot be normalized.

2.2. *Blaschke-Grünwald mapping of SE(2)*

The Blaschke-Grünwald mapping can be obtained from the Study mapping by restricting ourselves to planar Euclidean displacements within a plane, which corresponds to a 3-dimensional generator-space of Φ. If we choose the plane $x_1 = 0$, it can easily be seen (cf. Remark 3.38 of [14]) that the corresponding generator-space of Φ is given by $e_2 = e_3 = t_0 = t_1 = 0$. Therefore there is a bijection between SE(2) and all real points $(e_0 : e_1 : t_2 : t_3)$ of a projective 3-dimensional space P^3, with exception of the points located on the line $e_0 = e_1 = 0$. The corresponding mapping reads as follows:

$$x_2'\mathbf{j} + x_3'\mathbf{k} = (e_0 + e_1\mathbf{i})(x_2\mathbf{j} + x_3\mathbf{k})(e_0 - e_1\mathbf{i}) + (t_2\mathbf{j} + t_3\mathbf{k})(e_0 - e_1\mathbf{i}) + (e_0 + e_1\mathbf{i})(t_2\mathbf{j} + t_3\mathbf{k}).$$

Remark 2. Note that the multiplication of the above equation by $-\mathbf{j}$ from the right yield the complex coordinates of a planar motion, which provide in combination with the exponential map a very compact and elegant notation for the study of planar kinematics (e.g. Wunderlich [36]). ⋄

2.3. *Klawitter-Hagemann mapping of SE(4)*

Until now the author is only aware of one explicitly given kinematic mapping of SE(4), namely the one of Klawitter and Hagemann [19]. They presented a unified concept based on Clifford algebras, for constructing kinematic mappings for certain Cayley-Klein geometries. Especially for E^2 and E^3, they demonstrated that their approach yields the Blaschke-Grünwald mapping and the Study mapping, respectively. For the latter see also Selig [28, Section 9.3]. The algebraic structure of the Study parameters (resp. Blaschke-Grünwald parameters) corresponds to the Spin group of the Clifford algebra with signature $(3_+, 0_-, 1_0)$ (resp. $(2_+, 0_-, 1_0)$).

The Spin group of the Clifford algebra with signature $(4_+, 0_-, 1_0)$ implies a mapping between displacements of SE(4) and all real points $\mathcal{S}$ of P^{15} with homogeneous coordinates $(a_0 : \ldots : a_7 : c_0 : \ldots : c_7)$, located in the intersection of nine quadrics R_i $(i = 1, \ldots, 9)$, which is additionally sliced along the quadric N. The explicit equations read as follows:

$$\begin{aligned}
&R_1 : a_2c_6 - a_3c_5 + a_4c_0 - c_1c_4 = 0, && R_2 : a_5c_0 - c_1c_7 + c_2c_5 - c_3c_6 = 0,\\
&R_3 : a_1c_5 - a_2c_7 + a_7c_0 - c_3c_4 = 0, && R_4 : a_1c_6 - a_3c_7 + a_6c_0 - c_2c_4 = 0,\\
&R_5 : a_0c_0 - a_1c_1 + a_2c_2 - a_3c_3 = 0, && R_6 : a_0c_7 - a_1a_5 - a_6c_3 + a_7c_2 = 0,\\
&R_7 : a_0c_4 - a_1a_4 + a_2a_6 - a_3a_7 = 0, && R_8 : a_0c_6 - a_3a_5 - a_4c_2 + a_6c_1 = 0,\\
&R_9 : a_0c_5 - a_2a_5 - a_4c_3 + a_7c_1 = 0, && N : a_0^2 + \ldots + a_7^2 + c_0^2 + \ldots + c_7^2 = 0.
\end{aligned}$$

A computation of the degree of $\mathcal{S}$ by the Hilbert polynomial shows that this 10-dimensional variety is of degree 12. As a consequence the Klawitter-Hagemann mapping is not suited for performing computational algebraic kinematics and kinematical geometry in E^4. Therefore we are interested in a simplified kinematic mapping of SE(4).

Remark 3. Note that the usage of Clifford algebra for the representation of motions within metric geometries was already treated by Schröder [27]. Following his approach some classical geometries of low dimension were explicitly studied by Jurk [16] and Windelberg [35]. Inter alia they already have shown that this method yields the Blaschke-Grünwald mapping for E^2 and the Study mapping for E^3, respectively. But they did not discuss E^4. ⋄

2.4. Quaternion based kinematic mapping of SE(4)

We start by embedding the points X of E^4 with Cartesian coordinates (x_0, x_1, x_2, x_3) into the set of quaternions by the mapping:

$$\iota_4 : \mathbb{R}^4 \to \mathbb{H} \quad \text{with} \quad (x_0, x_1, x_2, x_3) \mapsto \mathbf{X} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}. \tag{6}$$

Moreover we need the classical quaternion representation theorem for SO(4), which has many fathers (Euler, Cayley, Salmon, Elfrinkhof, Stringham, Bouman) according to Mebius [20] and states the following:

Theorem 1. *The mapping of points* $\mathsf{X} \in E^4$ *to* $\mathsf{X}' \in E^4$ *induced by any element of SO(4) can be written as follows (by using* ι_4 *of Eq. (6)):*

$$\mathbf{X} \mapsto \mathbf{X}' \quad \textit{with} \quad \mathbf{X}' := \mathbf{E}\mathbf{X}\widetilde{\mathbf{F}}, \tag{7}$$

where $\mathbf{E}$ *and* $\mathbf{F}$ *is a pair of unit-quaternions, which is determined uniquely up to the sign. Moreover the mapping of Eq. (7) is an element of SO(4) for any pair of unit-quaternions* $\mathbf{E}$ *and* $\mathbf{F}$.

Due to the free choice of sign in Theorem 1, the decomposition into a left unit-quaternion $\mathbf{E}$ and a right unit-quaternion $\mathbf{F}$ yields a double cover of SO(4). Therefore we consider again the homogeneous 8-tuple $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$, which can be seen as a point in P^7. Hence there is a bijection between SO(4) and all real points $\mathcal{S}$ of P^7, which are located on the quadric $\Psi \subset P^7$ given by

$$(e_0^2 + e_1^2 + e_2^2 + e_3^2) - (f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0, \tag{8}$$

($\Rightarrow$ the signature of Ψ is $(4_+, 4_-, 0_0)$) sliced along the 3-dimensional space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathbf{E}$ cannot be normalized. But this 3-space does not have a real intersection with Ψ and therefore no point of Ψ has to be removed. Note that Eq. (8) can be rewritten as $\mathbf{E}\widetilde{\mathbf{E}} - \mathbf{F}\widetilde{\mathbf{F}} = 0$, which expresses the fact that $\mathbf{F}$ is also normalized if $\mathbf{E}$ is a unit-quaternion.

Remark 4. If we identify E^3 with the hyperplane $x_0 = 0$, all points of the 3-dimensional generator-space $f_i = e_i$ for $i = 0, \ldots, 3$ ($\Leftrightarrow \mathbf{F} = \mathbf{E}$) of Ψ, map the hyperplane $x_0 = 0$ onto itself. Therefore this 3-dimensional generator-space is the well-known Euler-Rodrigues parameter space $(e_0 : \ldots : e_3)$ of SO(3). ⋄

The extension of this kinematic mapping of SO(4) with respect to translations of E^4 can be done as follows:

Theorem 2. *The mapping of points* $\mathsf{X} \in E^4$ *to* $\mathsf{X}' \in E^4$ *induced by any element of SE(4) can be written as follows (by using* ι_4 *of Eq. (6)):*

$$\mathbf{X} \mapsto \mathbf{X}' \quad \textit{with} \quad \mathbf{X}' := \mathbf{E}\mathbf{X}\widetilde{\mathbf{F}} - 2\mathbf{E}\widetilde{\mathbf{T}}. \tag{9}$$

Moreover the mapping of Eq. (9) is an element of SE(4) for any triple of quaternions $\mathbf{E}, \mathbf{F}, \mathbf{T}$*, where* $\mathbf{E}$ *and* $\mathbf{F}$ *are unit-quaternions.*

Proof: Due to Theorem 1, we only have to show that there is a bijection between the coordinates of the translation vector $(v_0, v_1, v_2, v_3)^T$ and the entries $t_0, \ldots, t_3$ of $\mathbf{T}$ for a given unit-quaternion $\mathbf{E}$. If we set $\mathbf{V} := v_0 + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ we get immediately the relation $\mathbf{T} = -\frac{1}{2}\widetilde{\mathbf{V}}\mathbf{E}$ from $\mathbf{V} = -2\mathbf{E}\widetilde{\mathbf{T}}$. □

As both triples of quaternions $\pm(\mathbf{E}, \mathbf{F}, \mathbf{T})$, where $\mathbf{E}$ and $\mathbf{F}$ are unit-quaternions, correspond to the same Euclidean motion of E^4, we consider the homogeneous 12-tuple $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3 : t_0 : \ldots : t_3)$. These 12 homogeneous motion parameters for E^4 can be interpreted as a point of a projective 11-dimensional space P^{11}. Therefore there is a bijection between SE(4) and all real points $\mathcal{S}$ of P^{11} located on the cylinder Ξ over Ψ, which is also given by Eq. (8) ($\Rightarrow$ the signature of Ξ is $(4_+, 4_-, 4_0)$) and is sliced along the 7-dimensional space $e_0 = e_1 = e_2 = e_3 = 0$, as the corresponding quaternion $\mathbf{E}$ cannot be normalized. But the real intersection of this 7-space and Ξ equals the 3-dimensional generator-space $e_0 = e_1 = e_2 = e_3 = f_0 = f_1 = f_2 = f_3 = 0$ of Ξ. Therefore only the points of this 3-space have to be removed from Ξ.

Remark 5. If we identify E^3 with the hyperplane $x_0 = 0$, all points of the 7-dimensional generator-space $f_i = e_i$ for $i = 0, \ldots, 3$ (cf. Remark 4) of Ξ, which additionally fulfill the condition that no translation is done in direction of x_0 ($\Leftrightarrow v_0 = 0$), map the hyperplane $x_0 = 0$ onto itself. As the condition $v_0 = 0$ equals the Study condition (4), the

7-dimensional generator-space of Ξ is the Study parameter space of SE(3). This shows that the Study parameters and subsequently the Blaschke-Grünwald parameters can be obtained from the 12 homogeneous motion parameters for E^4. Note that the exceptional quadric of this parameter space is given by $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 0$ and therefore it is also quasi-elliptic (cf. [12]) like the kinematic image spaces named after Study and Blaschke-Grünwald. Finally it should be mentioned that the quaternion based kinematic mapping of SE(4) allows rational motion design by constructing rational curves on hyperquadrics [11] (as for SE(3) on the Study quadric). ⋄

One can ask the question why the translational part in Theorem 2 is given by $-2\mathbf{E}\widetilde{\mathbf{T}}$ and not by any other (left or right) product of $\mathbf{T}$ or $\widetilde{\mathbf{T}}$ with $\mathbf{E}$, $\widetilde{\mathbf{E}}$, $\mathbf{F}$ or $\widetilde{\mathbf{F}}$? The reason for using $-2\mathbf{E}\widetilde{\mathbf{T}}$ is that we get the direct connection to the Study parameters (cf. Remark 5), which is quite nice. This property has also the translational part $2\mathbf{T}\widetilde{\mathbf{E}}$, but the resulting kinematic mapping would imply a quartic hypersphere condition, contrary to the one given in Theorem 2, where the hypersphere condition is quadratic (cf. [23]).

2.4.1. On the composition of displacements

After the publication of [23] the author has been referred to the work of Wilker [34] on the quaternion formalism for Möbius groups in four dimensions, which also contains the Euclidean motion group of E^4 as special case. For reasons of completeness it should be noted that within this framework the translation part is given by $\mathbf{T}\widetilde{\mathbf{F}}$. Wilker's approach shed light on the formal composition of two displacements δ_i of E^4, which are given by the triples $(\mathbf{E}_i, \mathbf{F}_i, \mathbf{T}_i)$ for $i = 1, 2$. Now the mapping $\mathsf{X}' = \delta_2(\delta_1(\mathsf{X}))$ is written as:

$$\mathbf{X}' = \mathbf{E}_2\mathbf{E}_1\mathbf{X}\widetilde{\mathbf{F}}_1\widetilde{\mathbf{F}}_2 - 2\mathbf{E}_2\mathbf{E}_1\widetilde{\mathbf{T}}_1\widetilde{\mathbf{F}}_2 - 2\mathbf{E}_2\widetilde{\mathbf{T}}_2.$$

This equals the displacement given by the triple $(\mathbf{G}, \mathbf{H}, \mathbf{U})$ with

$$\mathbf{G} := \mathbf{E}_2\mathbf{E}_1, \quad \mathbf{H} := \mathbf{F}_2\mathbf{F}_1, \quad \mathbf{U} := \mathbf{T}_2\mathbf{E}_1 + \mathbf{F}_2\mathbf{T}_1.$$

This shows together with the following equation that the composition of two displacements corresponds to the multiplication of lower triangular 2×2 quaternionic matrices (cf. Wilker [34]); i.e.:

$$\begin{pmatrix} \mathbf{E}_2 & \mathbf{O} \\ \mathbf{T}_2 & \mathbf{F}_2 \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 & \mathbf{O} \\ \mathbf{T}_1 & \mathbf{F}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{U} & \mathbf{H} \end{pmatrix}.$$

Therefore this map from SE(4) to the group of lower triangular 2×2 quaternionic matrices with unit-quaternions in the diagonal is a so-called representation (cf. [10]). Under consideration of $\mathbf{F} = \mathbf{E}$ (and the matrix representation of dual numbers) this quaternion-matrix representation shows again that SE(3) can be expressed in terms of dual unit-quaternions (cf. Section 2.1).

3. Spherical displacements in terms of quaternions

As the translational component of the displacement given in Theorem 2 can easily be handled, we restrict ourselves to the study of the rotational one in this section, which is divided into two parts. In the first one (Section 3.1) we consider a rotation given by its geometric parameters and want to determine the quaternions $\mathbf{E}$ and $\mathbf{F}$ needed for the representation given in Theorem 1. The second part (Section 3.2) deals with the inverse problem.

3.1. Rotation defined by its geometric parameters

First of all we want to study the rotation about a plane Λ through the origin about the angle λ. Let this plane be spanned by two linear independent unit-vectors, which we write as unit-quaternions $\mathbf{M}$ and $\mathbf{N}$ with $\langle \mathbf{M}, \mathbf{N} \rangle \neq \pm 1$. Then the mapping:

$$\delta_\Lambda : \mathbf{X} \mapsto \mathbf{X}' \quad \text{with} \quad \mathbf{X}' := \mathbf{N}\widetilde{\mathbf{M}}\mathbf{X}\widetilde{\mathbf{N}}\mathbf{M}$$

keeps the points of the plane Λ fixed, as

$$\mathbf{N}\widetilde{\mathbf{M}}\mathbf{M}\widetilde{\mathbf{N}}\mathbf{M} = \mathbf{M} \quad \text{and} \quad \mathbf{N}\widetilde{\mathbf{M}}\mathbf{N}\widetilde{\mathbf{N}}\mathbf{M} = \mathbf{N} \tag{10}$$

hold. Due to Theorem 1 the mapping δ_Λ is either a rotation about the plane Λ or the identity. In the following we show that it is a rotation about two times the angle $\frac{\lambda}{2}$ enclosed by $\mathbf{M}$ and $\mathbf{N}$; i.e. $\frac{\lambda}{2} = \arccos\langle \mathbf{M}, \mathbf{N}\rangle$ with $0^\circ \leq \sphericalangle(\mathbf{M}, \mathbf{N}) = \frac{\lambda}{2} \leq 180^\circ$. Without loss of generality we can choose the coordinate system in a way that:

$$\mathbf{M} = 1 \quad \text{and} \quad \mathbf{N} = \cos\tfrac{\lambda}{2} + \sin\tfrac{\lambda}{2}\mathbf{i},$$

holds. Then computation of $\mathbf{X}' = \mathbf{N}\widetilde{\mathbf{M}}\mathbf{X}\widetilde{\mathbf{N}}\mathbf{M}$ yields:

$$\mathbf{X}' = x_0 + x_i\mathbf{i} + (x_2\cos\lambda - x_3\sin\lambda)\,\mathbf{j} + (x_2\sin\lambda + x_3\cos\lambda)\,\mathbf{k},$$

which already proves the next theorem.

Theorem 3. *The mapping δ_Λ is a rotation about the plane Λ through the origin spanned by $\mathbf{M}$ and $\mathbf{N}$, where the rotation angle λ is two times the angle enclosed by $\mathbf{M}$ and $\mathbf{N}$; i.e. $0^\circ \leq \sphericalangle(\mathbf{M}, \mathbf{N}) = \frac{\lambda}{2} \leq 180^\circ$. For $\langle \mathbf{M}, \mathbf{N}\rangle = 1$ ($\Leftrightarrow \frac{\lambda}{2} = 0^\circ$) and $\langle \mathbf{M}, \mathbf{N}\rangle = -1$ ($\Leftrightarrow \frac{\lambda}{2} = 180^\circ$) the mapping δ_Λ equals the identity.*

We only have to keep in mind that there are two unit-quaternions $\mathbf{N}$ and $\overline{\mathbf{N}}$ in Λ, which enclose with $\mathbf{M}$ the angle $\frac{\lambda}{2}$. They correspond with the two different orientations of rotations about that plane. How can we identify the correct unit-quaternion $\mathbf{N}^\star$ out from $\left\{\mathbf{N}, \overline{\mathbf{N}}\right\}$? This can be done by constructing the following Cartesian right system: We identify $\mathbf{M}$ with the x_0-axis, and the x_1-axis is chosen within the plane Λ in the way that it encloses with $\mathbf{N}^\star$ an angle $< \frac{\pi}{2}$. Now we can select any unit-quaternion orthogonal to Λ as x_2-axis. This also determines the x_3-axis uniquely. As λ is positive it implies a mathematical positive rotation in the x_2, x_3-plane (x_2 is rotated in direction x_3).

Therefore we can fix the rotation about the plane Λ by orienting the plane. The oriented triangle $\mathbf{O}, \mathbf{M}, \mathbf{N}$ implies an orientation for Λ, which is denoted by $\overrightarrow{\Lambda} = \overrightarrow{\mathbf{OMN}}$. Note that the opposite oriented plane $\overleftarrow{\Lambda}$ is implied by the oriented triangle $\mathbf{O}, \mathbf{M}, \overline{\mathbf{N}}$. Moreover it should be noted that the pairs $(\mathbf{M}, \mathbf{N})$ and $(\mathbf{M}, -\mathbf{N})$ imply the same mapping δ_Λ.

Theorem 3 allows us to construct the quaternions $\mathbf{E}$ and $\mathbf{F}$ used in Theorem 1 as follows: Given are two total-orthogonal planes Λ and Γ through the origin, which are spanned by $\mathbf{M}, \mathbf{N}$ and $\mathbf{Q}, \mathbf{R}$ respectively. Under consideration of

$$\delta_\Gamma : \mathbf{X} \mapsto \mathbf{X}' \quad \text{with} \quad \mathbf{X}' := \mathbf{R}\widetilde{\mathbf{Q}}\mathbf{X}\widetilde{\mathbf{R}}\mathbf{Q}.$$

the mapping $\delta_\Gamma(\delta_\Lambda(\mathbf{X})) = \delta_\Lambda(\delta_\Gamma(\mathbf{X}))$ shows:

$$\mathbf{E} = \mathbf{R}\widetilde{\mathbf{Q}}\mathbf{N}\widetilde{\mathbf{M}} = \mathbf{N}\widetilde{\mathbf{M}}\mathbf{R}\widetilde{\mathbf{Q}}, \quad \mathbf{F} = \widetilde{\mathbf{Q}}\mathbf{R}\widetilde{\mathbf{M}}\mathbf{N} = \widetilde{\mathbf{M}}\mathbf{N}\widetilde{\mathbf{Q}}\mathbf{R}. \tag{11}$$

But we can even express $\mathbf{X}'$ in dependence of $\mathbf{M}, \mathbf{N}, \mathbf{Q}, \mathbf{R}$ in a more suitable form (less quaternion are multiplied in series; five instead of nine) than by $\mathbf{E}\mathbf{X}\widetilde{\mathbf{F}}$ with $\mathbf{E}, \mathbf{F}$ according to Eq. (11).

Theorem 4. *The image $\mathbf{X}'$ of $\mathbf{X}$ under the rotations $\delta_\Gamma(\delta_\Lambda(\mathbf{X})) = \delta_\Lambda(\delta_\Gamma(\mathbf{X}))$ determined by $\mathbf{M}, \mathbf{N}, \mathbf{Q}, \mathbf{R}$ can be written as:*

$$\mathbf{X}' = \delta_\Lambda(\mathbf{X}) + \delta_\Gamma(\mathbf{X}) - \mathbf{X} = \mathbf{N}\widetilde{\mathbf{M}}\mathbf{X}\widetilde{\mathbf{N}}\mathbf{M} + \mathbf{R}\widetilde{\mathbf{Q}}\mathbf{X}\widetilde{\mathbf{R}}\mathbf{Q} - \mathbf{X}.$$

Proof: We decompose $\mathbf{X}$ as follows:

$$\mathbf{X} = x_M\mathbf{M} + x_N\mathbf{N} + x_Q\mathbf{Q} + x_R\mathbf{R}.$$

As we rotate about total-orthogonal planes we have:

$$\mathbf{X}' = \delta_\Lambda(x_Q\mathbf{Q} + x_R\mathbf{R}) + \delta_\Gamma(x_M\mathbf{M} + x_N\mathbf{N}).$$

Moreover $\delta_\Lambda(\mathbf{X}) + \delta_\Gamma(\mathbf{X})$ yields under consideration of Eq. (10)

$$\mathbf{N}\widetilde{\mathbf{M}}(x_Q\mathbf{Q} + x_R\mathbf{R})\widetilde{\mathbf{N}}\mathbf{M} + \mathbf{R}\widetilde{\mathbf{Q}}(x_M\mathbf{M} + x_N\mathbf{N})\widetilde{\mathbf{R}}\mathbf{Q} + (x_M\mathbf{M} + x_N\mathbf{N} + x_Q\mathbf{Q} + x_R\mathbf{R}) = \mathbf{X}' + \mathbf{X},$$

which already proves the theorem. □

3.2. Geometric parameters of a rotation

In this section we study the inverse problem of the last section; i.e. we want to compute the geometric parameters of the rotation $\mathbf{E}\mathbf{X}\widetilde{\mathbf{F}}$. Using the notation:

$$\mathbf{E} = \cos\tfrac{\alpha}{2} + \sin\tfrac{\alpha}{2}\mathbf{e}_0, \qquad \mathbf{F} = \cos\tfrac{\beta}{2} + \sin\tfrac{\beta}{2}\mathbf{f}_0,$$

with unit-quaternions $\mathbf{e}_0 = \frac{\mathbf{e}}{\|\mathbf{e}\|}$ and $\mathbf{f}_0 = \frac{\mathbf{f}}{\|\mathbf{f}\|}$, this can be done as follows:

Theorem 5. *If the quaternions* $\mathbf{E}$ *and* $\mathbf{F}$ *are not coaxial, the geometric parameters of the rotation* $\mathbf{X} \mapsto \mathbf{E}\mathbf{X}\widetilde{\mathbf{F}}$ *are as follows: The total-orthogonal planes* Λ *and* Γ *through the origin are spanned by:*

$$\mathbf{M} = \frac{\mathbf{e}_0 + \mathbf{f}_0}{\sqrt{2 - \mathbf{e}_0\mathbf{f}_0 - \mathbf{f}_0\mathbf{e}_0}}, \qquad \mathbf{M}^\perp = \frac{\mathbf{e}_0\mathbf{f}_0 - 1}{\sqrt{2 - \mathbf{e}_0\mathbf{f}_0 - \mathbf{f}_0\mathbf{e}_0}}, \tag{12}$$

and

$$\mathbf{Q} = \frac{\mathbf{f}_0 - \mathbf{e}_0}{\sqrt{2 + \mathbf{e}_0\mathbf{f}_0 + \mathbf{f}_0\mathbf{e}_0}}, \qquad \mathbf{Q}^\perp = \frac{\mathbf{e}_0\mathbf{f}_0 + 1}{\sqrt{2 + \mathbf{e}_0\mathbf{f}_0 + \mathbf{f}_0\mathbf{e}_0}}, \tag{13}$$

respectively. Moreover by setting

$$\mathbf{N} = \cos\frac{\alpha+\beta}{4}\mathbf{M} + \sin\frac{\alpha+\beta}{4}\mathbf{M}^\perp, \quad \mathbf{R} = \cos\frac{\alpha-\beta}{4}\mathbf{Q} + \sin\frac{\alpha-\beta}{4}\mathbf{Q}^\perp \tag{14}$$

the relations given in Eq. (11) hold. Therefore the rotation angles about the oriented planes $\overrightarrow{\Lambda} = \overrightarrow{\mathbf{OMN}}$ *and* $\overrightarrow{\Gamma} = \overrightarrow{\mathbf{OQR}}$ *equal* $|\frac{\alpha+\beta}{2}|$ *and* $|\frac{\alpha-\beta}{2}|$*, respectively.*

Proof: A straightforward computation shows that $\mathbf{R}\widetilde{\mathbf{Q}}\mathbf{N}\widetilde{\mathbf{M}}$ equals $\mathbf{E}$ and that $\widetilde{\mathbf{Q}}\mathbf{R}\widetilde{\mathbf{M}}\mathbf{N}$ equals $\mathbf{F}$. □

Theorem 5 is a more detailed version of Theorem 9.1 of [6], which also corrects the cited theorem with respect to the orientation of the involved planes (see Appendix).

Theorem 5 does not hold for the case of coaxial quaternions, which is the content of the next theorem, for whose proof we need the following lemma:

Lemma 1. *The quaternions* $\mathbf{M}, \mathbf{M}^\perp, \mathbf{Q}, \mathbf{Q}^\perp$ *(resp.* $\mathbf{Q}, \mathbf{Q}^\perp, \mathbf{M}, \mathbf{M}^\perp$*) of Eqs. (12) and (13) form a Cartesian right system.*

Proof: As all four involved quaternions are unit-quaternions, which are pairwise orthogonal, we only have to check if the given order forms a right system. Therefore we compute the determinantsof the matrices

$$\begin{pmatrix} m_0 & m_1 & m_2 & m_3 \\ m_0^\perp & m_1^\perp & m_2^\perp & m_3^\perp \\ q_0 & q_1 & q_2 & q_3 \\ q_0^\perp & q_1^\perp & q_2^\perp & q_3^\perp \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ q_0^\perp & q_1^\perp & q_2^\perp & q_3^\perp \\ m_0 & m_1 & m_2 & m_3 \\ m_0^\perp & m_1^\perp & m_2^\perp & m_3^\perp \end{pmatrix},$$

respectively, which shows that they are equal to 1. □

Theorem 6. *Assume that the quaternions* $\mathbf{E}$ *and* $\mathbf{F}$ *of the rotation* $\mathbf{X} \mapsto \mathbf{E}\mathbf{X}\widetilde{\mathbf{F}}$ *are coaxial. If* $\mathbf{e}_0 = \mathbf{f}_0 = \mathbf{o}$ *holds the mapping equals either the identity or the central inversion with respect to the origin. In both cases all linear subspaces through the origin are fixed. For the remaining cases the geometric parameters of the rotation are as follows:*

1. $\mathbf{e}_0 = \mathbf{f}_0 \neq \mathbf{o}$ *or* $\mathbf{f}_0 = \mathbf{o} \neq \mathbf{e}_0$*: In this case the plane* Λ *through the origin is spanned by* $\mathbf{M} = \mathbf{e}_0$ *and* $\mathbf{M}^\perp = -1$*. Now* $\mathbf{Q}$ *denotes any unit-quaternion orthogonal to* Λ *and the unit-quaternion* $\mathbf{Q}^\perp$ *is chosen in a way that* $\mathbf{M}, \mathbf{M}^\perp, \mathbf{Q}, \mathbf{Q}^\perp$ *forms a Cartesian right system.*

2. $\mathbf{e}_0 = -\mathbf{f}_0 \neq \mathbf{o}$ *or* $\mathbf{e}_0 = \mathbf{o} \neq \mathbf{f}_0$*: In this case the plane* Γ *through the origin is spanned by* $\mathbf{Q} = \mathbf{f}_0$ *and* $\mathbf{Q}^\perp = 1$*. Now* $\mathbf{M}$ *denotes any unit-quaternion orthogonal to* Γ *and the unit-quaternion* $\mathbf{M}^\perp$ *is chosen in a way that* $\mathbf{Q}, \mathbf{Q}^\perp, \mathbf{M}, \mathbf{M}^\perp$ *forms a Cartesian right system.*

With respect to these points $\mathbf{M}, \mathbf{M}^{\perp}, \mathbf{Q}, \mathbf{Q}^{\perp}$ *the quaternions* $\mathbf{N}$ *and* $\mathbf{R}$ *of Eq. (14) fulfill the relations given in Eq. (11). Therefore the rotation angles about the oriented planes* $\overrightarrow{\Lambda} = \overrightarrow{\mathbf{OMN}}$ *and* $\overrightarrow{\Gamma} = \overrightarrow{\mathbf{OQR}}$ *equal* $|\frac{\alpha+\beta}{2}|$ *and* $|\frac{\alpha-\beta}{2}|$*, respectively.*

Proof: For these special cases we only have to insert into the formula of the general case, which shows for item 1 (resp. item 2) that Λ (resp. Γ) is spanned by $\mathbf{M} = \mathbf{e}_0$ and $\mathbf{M}^{\perp} = -1$ (resp. $\mathbf{Q} = \mathbf{f}_0$ and $\mathbf{Q}^{\perp} = 1$).

The general formula cannot be used for the remaining plane Γ (resp. Λ) as it either collapse into the origin or coincides with Λ (resp. Γ).

Due to Lemma 1 and the usage of Cartesian right systems within the formulation of Theorem 6 the given geometric interpretation of the special cases is the same as in Theorem 5. □

Note that the cases $\mathbf{f}_0 = \mathbf{o} \neq \mathbf{e}_0$ and $\mathbf{e}_0 = \mathbf{o} \neq \mathbf{f}_0$ yield isoclinic rotations. Therefore the above theorem only notes one pair of total-orthogonal planes together with their angle of rotation.

Finally it should be noted that the rotation angles of Theorems 5 and 6 with respect to the oriented planes $\overrightarrow{\Lambda} = \overrightarrow{\mathbf{OMM}^{\perp}}$ and $\overrightarrow{\Gamma} = \overrightarrow{\mathbf{OQQ}^{\perp}}$ are given by $\frac{\alpha+\beta}{2}$ and $\frac{\alpha-\beta}{2}$, respectively.

4. Representation of displacements of basic geometric elements

In the following we want to embed the quaternion $\mathbf{X}$ of a point $\mathsf{X} \in E^4$ into a 2×2 quaternionic matrix in a way that its multiplication with quaternionic matrices (see Section 2.4.1) gives the point coordinates $\mathbf{X}'$; i.e. an analogue to the 3-dimensional case, where we can embed $\mathbf{x}$ and $\mathbf{x}'$, respectively, into the set of dual unit-quaternions in a way that:

$$1 + \varepsilon\mathbf{x}' = (\mathbf{E} + \varepsilon\mathbf{T})(1 + \varepsilon\mathbf{x})\left(\widetilde{\mathbf{E}} - \varepsilon\widetilde{\mathbf{T}}\right)$$

holds. By introducing the following notation

$$\underline{\mathbf{D}} = \begin{pmatrix} \mathbf{E} & \mathbf{O} \\ \mathbf{T} & \mathbf{F} \end{pmatrix}, \qquad \widetilde{\underline{\mathbf{D}}}^T = \begin{pmatrix} \widetilde{\mathbf{E}} & \widetilde{\mathbf{T}} \\ \mathbf{O} & \widetilde{\mathbf{F}} \end{pmatrix}, \qquad \underline{\mathbf{X}}' = \begin{pmatrix} -1 & \mathbf{X}' \\ \mathbf{O} & 1 \end{pmatrix}, \qquad \underline{\mathbf{X}} = \begin{pmatrix} -1 & \mathbf{X} \\ \mathbf{O} & 1 \end{pmatrix}$$

and under consideration that $\widetilde{\underline{\mathbf{D}}}^{-T}$ with

$$\widetilde{\underline{\mathbf{D}}}^{-T} = \begin{pmatrix} \mathbf{E} & -\mathbf{E}\widetilde{\mathbf{T}}\mathbf{F} \\ \mathbf{O} & \mathbf{F} \end{pmatrix}$$

denotes the (left or right) multiplicative inverse of $\widetilde{\underline{\mathbf{D}}}^T$, this can be done as follows:

Theorem 7. *The mapping of points* $\mathsf{X} \in E^4$ *to* $\mathsf{X}' \in E^4$ *induced by any element of SE(4) can be written as follows:*

$$\underline{\mathbf{X}} \mapsto \underline{\mathbf{X}}' \quad \textit{with} \quad \underline{\mathbf{X}}' := \widetilde{\underline{\mathbf{D}}}^{-T}\underline{\mathbf{X}}\,\widetilde{\underline{\mathbf{D}}}^T.$$

It is also possible to formulate the displacements of lines and planes of E^3 within the dual quaternion calculus (e.g. Blaschke [3]). Therefore also the displacements of lines of E^2 can be formulated within the Blaschke-Grünwald parameters (e.g. [2, 4, 13]).

In the following we show that the displacement of lines, planes and hyperplanes in E^4 can also be written by 2×2 quaternionic matrices.

4.1. Hyperplanes

All points $\mathsf{X} \in E^4$ with coordinates (x_0, x_1, x_2, x_3) located in a hyperplane (3-space) fulfill a linear equation, which can be written in the Hesse normal form as

$$x_0w_0 + x_1w_1 + x_2w_2 + x_3w_3 + w = 0 \quad \text{with} \quad w_0^2 + w_1^2 + w_2^2 + w_3^2 = 1.$$

Thus a hyperplane can be fixed by a unit-quaternion $\mathbf{W} = w_0 + w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ and a real number w. Therefore $-w$ gives the oriented distance of the footpoint on the hyperplane (with respect to the origin) to the origin with respect to the direction of $\mathbf{W}$. Applying a rotation about the origin the footpoint has still the distance $-w$, but now in direction of $\mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}$. This distance is only changed by the component of the translational vector, which is orthogonal to the rotated hyperplane; i.e. $\langle -2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}\rangle$. Summed up we have:

$$\mathbf{W} \mapsto \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}} \quad \text{and} \quad w \mapsto w + \langle 2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}\rangle.$$

The scalar product can be simplified as follows, where the multiplications from the left and right are written on the respective sides:

$$\begin{aligned}
\widetilde{\mathbf{E}} \mid \qquad & \langle 2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}\rangle = \mathbf{E}\left(\widetilde{\mathbf{T}}\mathbf{F}\widetilde{\mathbf{W}} + \mathbf{W}\widetilde{\mathbf{F}}\mathbf{T}\right)\widetilde{\mathbf{E}} & \mid \mathbf{E} \\
\widetilde{\mathbf{T}}^{-1} \mid \qquad & \langle 2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}\rangle = \widetilde{\mathbf{T}}\mathbf{F}\widetilde{\mathbf{W}} + \mathbf{W}\widetilde{\mathbf{F}}\mathbf{T} & \mid \mathbf{T}^{-1} \\
& \langle 2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}\rangle ||\mathbf{T}||^{-2} = \mathbf{F}\widetilde{\mathbf{W}}\mathbf{T}^{-1} + \widetilde{\mathbf{T}}^{-1}\mathbf{W}\widetilde{\mathbf{F}} &
\end{aligned}$$

Multiplying both sides with the scalar $||\mathbf{T}||^2 = \mathbf{T}\widetilde{\mathbf{T}}$ yields:

$$\langle 2\mathbf{E}\widetilde{\mathbf{T}}, \mathbf{E}\mathbf{W}\widetilde{\mathbf{F}}\rangle = \mathbf{F}\widetilde{\mathbf{W}}\widetilde{\mathbf{T}} + \mathbf{T}\mathbf{W}\widetilde{\mathbf{F}}. \tag{15}$$

Having in mind that $(\mathbf{W}, w)$ also assigns an orientation we can state the following theorem under consideration of the notation:

$$\underline{\mathbf{W}}' = \begin{pmatrix} \mathbf{O} & \mathbf{W}' \\ \widetilde{\mathbf{W}}' & w' \end{pmatrix}, \qquad \underline{\mathbf{W}} = \begin{pmatrix} \mathbf{O} & \mathbf{W} \\ \widetilde{\mathbf{W}} & w \end{pmatrix}.$$

Theorem 8. *The mapping of oriented hyperplanes* $(\mathbf{W}, w)$ *of* E^4 *to oriented hyperplanes* $(\mathbf{W}', w')$ *of* E^4 *induced by any element of SE(4) can be written as follows:*

$$\underline{\mathbf{W}} \mapsto \underline{\mathbf{W}}' \quad \textit{with} \quad \underline{\mathbf{W}}' := \underline{\mathbf{D}}\,\underline{\mathbf{W}}\,\underline{\widetilde{\mathbf{D}}}^T.$$

4.2. Lines

Now we discuss the set of oriented lines of E^4. Geometrically we can characterize a line by its footpoint $\mathbf{C}$ with respect to the origin and by its direction, which can be written as a unit-quaternion $\mathbf{Y}$. Clearly this direction is transformed by an arbitrary displacement into $\mathbf{Y}' = \mathbf{E}\mathbf{Y}\widetilde{\mathbf{F}}$. Now it only remains to calculate the footpoint $\mathbf{C}'$ of the displaced line, which is composed of the rotated footpoint $\mathbf{E}\mathbf{C}\widetilde{\mathbf{F}}$ plus the component of the translational vector orthogonal to $\mathbf{Y}'$. Under consideration of Eq. (15) the latter can be written as

$$-2\mathbf{E}\widetilde{\mathbf{T}} + \mathbf{E}\mathbf{Y}\widetilde{\mathbf{F}}\left(\mathbf{F}\widetilde{\mathbf{Y}}\widetilde{\mathbf{T}} + \mathbf{T}\mathbf{Y}\widetilde{\mathbf{F}}\right),$$

which yields:

$$\mathbf{C}' = \mathbf{E}\mathbf{C}\widetilde{\mathbf{F}} - \mathbf{E}\widetilde{\mathbf{T}} + \mathbf{E}\mathbf{Y}\widetilde{\mathbf{F}}\mathbf{T}\mathbf{Y}\widetilde{\mathbf{F}}.$$

Due to the last term we do not represent the line by the pair $(\mathbf{Y}, \mathbf{C})$ but by $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ as the following holds:

$$\widetilde{\mathbf{Y}}'\mathbf{C}' = \mathbf{F}\widetilde{\mathbf{Y}}\mathbf{C}\widetilde{\mathbf{F}} - \mathbf{F}\widetilde{\mathbf{Y}}\widetilde{\mathbf{T}} + \mathbf{T}\mathbf{Y}\widetilde{\mathbf{F}}.$$

Note that in the 3-dimensional case $\widetilde{\mathbf{Y}}\mathbf{C}$ equals the crossproduct of the footpoint and the direction vector. Therefore $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ is the 4-dimensional analogue of the spear coordinates (oriented line coordinates) of E^3. As $\widetilde{\mathbf{Y}}\mathbf{C}$ is a pure quaternion, the spear coordinates $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ of E^4 have 7 entries. Moreover the expression $\widetilde{\mathbf{Y}}\mathbf{C}$ can be computed from any point $\mathbf{X} = \mathbf{C} + \xi\mathbf{Y}$ of the oriented line as follows:

$$\frac{\widetilde{\mathbf{Y}}\mathbf{X} - \widetilde{\mathbf{X}}\mathbf{Y}}{2} = \frac{\widetilde{\mathbf{Y}}\mathbf{C} + \xi - \widetilde{\mathbf{C}}\mathbf{Y} - \xi}{2} = \widetilde{\mathbf{Y}}\mathbf{C}. \tag{16}$$

By introducing the following notation:

$$\underline{\mathbf{Y}}' = \begin{pmatrix} \mathbf{O} & \mathbf{Y}' \\ -\widetilde{\mathbf{Y}}' & \widetilde{\mathbf{Y}}'\mathbf{C}' \end{pmatrix}, \qquad \underline{\mathbf{Y}} = \begin{pmatrix} \mathbf{O} & \mathbf{Y} \\ -\widetilde{\mathbf{Y}} & \widetilde{\mathbf{Y}}\mathbf{C} \end{pmatrix},$$

we can sum up our results in the next theorem:

Theorem 9. *The mapping of oriented lines* $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$ *of* E^4 *to oriented lines* $(\mathbf{Y}', \widetilde{\mathbf{Y}}'\mathbf{C}')$ *of* E^4 *induced by any element of SE(4) can be written as follows:*

$$\underline{\mathbf{Y}} \mapsto \underline{\mathbf{Y}}' \quad \textit{with} \quad \underline{\mathbf{Y}}' := \underline{\mathbf{D}}\,\underline{\mathbf{Y}}\,\underline{\widetilde{\mathbf{D}}}^T.$$

4.3. Planes

We describe a finite plane by a finite point $\mathbf{X}$ and two unit-vectors, which are orthogonal to each other. The latter can be written as unit-quaternions by $\mathbf{Y}$ and $\mathbf{Z}$, respectively. Instead of these two directions one can compute the oriented Plücker coordinates $(\bar{\mathbf{l}}, \widehat{\mathbf{l}})$ of the planes ideal line with respect to the ideal 3-space according to Müller [21, §6] as

$$\bar{\mathbf{l}} := \frac{\mathbf{Z}\widetilde{\mathbf{Y}} + \widetilde{\mathbf{Y}}\mathbf{Z}}{2}, \quad \widehat{\mathbf{l}} := \frac{\mathbf{Z}\widetilde{\mathbf{Y}} - \widetilde{\mathbf{Y}}\mathbf{Z}}{2}.$$

Note that $(\bar{\mathbf{l}}, \widehat{\mathbf{l}})$ expresses the ideal line oriented from the ideal point in direction $\mathbf{Y}$ to the ideal point in direction $\mathbf{Z}$. This can be proven by direct computation as

$$\bar{\mathbf{l}} = l_{01}\mathbf{i} + l_{02}\mathbf{j} + l_{03}\mathbf{k}, \qquad \widehat{\mathbf{l}} = l_{23}\mathbf{i} + l_{31}\mathbf{j} + l_{12}\mathbf{k},$$

hold, where $l_{ij} := y_i z_j - z_j y_i$ are the Plücker coordinates (cf. [25, Section 2.1.1]).

Moreover the ideal 3-space is the elliptic space described in [21], where an oriented line can alternatively be described by the left and right direction vector $\mathbf{l}_+$ and $\mathbf{l}_-$, which read as:

$$\mathbf{l}_+ = \mathbf{Z}\widetilde{\mathbf{Y}}, \qquad \mathbf{l}_- = \widetilde{\mathbf{Y}}\mathbf{Z}.$$

$\mathbf{l}_+$ and $\mathbf{l}_-$ can be seen as points on the so-called left and right unit-sphere, respectively. Now a displacement of E^4 implies a displacement within the elliptic ideal 3-space of the form $\mathbf{E}\mathbf{X}\widetilde{\mathbf{F}}$. Then $\mathbf{l}_+$ and $\mathbf{l}_-$ are transformed by:

$$\mathbf{l}_+ \mapsto \mathbf{E}\mathbf{l}_+\widetilde{\mathbf{E}}, \qquad \mathbf{l}_- \mapsto \mathbf{F}\mathbf{l}_-\widetilde{\mathbf{F}}, \tag{17}$$

i.e. which are rotations of the left and right unit-sphere, respectively.

Clearly one can compute the Grassmann coordinates p_{ijk} of the plane (cf. [25, Section 2.2]) based on these Plücker coordinates; i.e. we have to compute the wedge product of these Plücker coordinates with the homogeneous point coordinates $(x_0 : x_1 : x_2 : x_3 : 1)$ of any finite point X of the plane. Computation shows that $p_{ij4} = l_{ij}$ holds and that the remaining four Grassmann coordinates $p_{321}, p_{012}, p_{031}, p_{023}$ read as follows:

$$p_{321} = -x_1 l_{23} - x_2 l_{31} - x_3 l_{12}, \qquad p_{023} = x_3 l_{02} - x_2 l_{03} + x_0 l_{23}, \tag{18}$$

$$p_{031} = x_1 l_{03} - x_3 l_{01} + x_0 l_{31}, \qquad p_{012} = x_2 l_{01} - x_1 l_{02} + x_0 l_{12}, \tag{19}$$

which can be expressed within the quaternionic formulation as:

$$\begin{aligned} \mathbf{L} &: = p_{321} + p_{023}\mathbf{i} + p_{031}\mathbf{j} + p_{012}\mathbf{k} \\ &= \frac{\mathbf{Z}\widetilde{\mathbf{Y}}\mathbf{X} - \mathbf{X}\widetilde{\mathbf{Y}}\mathbf{Z}}{2} = \frac{\mathbf{l}_+\mathbf{X} - \mathbf{X}\mathbf{l}_-}{2}. \end{aligned} \tag{20}$$

Therefore the ten Grassmann plane coordinates can abstractly be represented by the quaternionic triple $(\bar{\mathbf{l}} : \widehat{\mathbf{l}} : \mathbf{L})$. In order to avoid the loss of the information on the plane's orientation, we can use normalized Grassmann coordinates, where the normalization is done with respect to the Plücker coordinates of the ideal line. As already $l_{01}^2 + l_{02}^2 + l_{03}^2 + l_{23}^2 + l_{31}^2 + l_{12}^2 = 1$ holds, the normalized Grassmann coordinates of the plane can be written as $(\bar{\mathbf{l}}, \widehat{\mathbf{l}}, \mathbf{L})$.

Under a displacement the quaternion $\mathbf{L}$ is transformed to $\mathbf{L}'$ with:

$$\begin{aligned} 2\mathbf{L}' &= \mathbf{Z}'\widetilde{\mathbf{Y}}'\mathbf{X}' - \mathbf{X}'\widetilde{\mathbf{Y}}'\mathbf{Z}' \\ &= \mathbf{E}\mathbf{Z}\widetilde{\mathbf{Y}}\mathbf{X}\widetilde{\mathbf{F}} - 2\mathbf{E}\mathbf{Z}\widetilde{\mathbf{Y}}\widetilde{\mathbf{T}} - \mathbf{E}\mathbf{X}\widetilde{\mathbf{Y}}\mathbf{Z}\widetilde{\mathbf{F}} + 2\mathbf{E}\widetilde{\mathbf{T}}\mathbf{F}\widetilde{\mathbf{Y}}\mathbf{Z}\widetilde{\mathbf{F}}. \end{aligned}$$

Instead of the triple $(\bar{\mathbf{l}}, \widehat{\mathbf{l}}, \mathbf{L})$, we can also use the representation $(\bar{\mathbf{l}} + \widehat{\mathbf{l}}, \bar{\mathbf{l}} - \widehat{\mathbf{l}}, \mathbf{L}) = (\mathbf{l}_+, \mathbf{l}_-, \mathbf{L})$. This is the most suitable form for our purpose, as we can state the following theorem under consideration of the notation:

$$\underline{\mathbf{L}}' = \begin{pmatrix} -\mathbf{l}'_+ & \mathbf{L}' \\ \mathbf{O} & -\mathbf{l}'_- \end{pmatrix}, \qquad \underline{\mathbf{L}} = \begin{pmatrix} -\mathbf{l}_+ & \mathbf{L} \\ \mathbf{O} & -\mathbf{l}_- \end{pmatrix}.$$

Theorem 10. *The mapping of oriented planes* $(\mathbf{l}_+, \mathbf{l}_-, \mathbf{L})$ *of* E^4 *to oriented planes* $(\mathbf{l}'_+, \mathbf{l}'_-, \mathbf{L}')$ *of* E^4 *induced by any element of SE(4) can be written as follows:*

$$\underline{\mathbf{L}} \mapsto \underline{\mathbf{L}}' \quad \textit{with} \quad \underline{\mathbf{L}}' := \widetilde{\underline{\mathbf{D}}}^{-T}\, \underline{\mathbf{L}}\, \widetilde{\underline{\mathbf{D}}}^{T}.$$

Within this section we showed that the displacements of the basic geometric elements can be treated in a unified way using 2×2 quaternionic matrices. As for the differentiation of these matrices $\underline{\mathbf{P}}$ and $\underline{\mathbf{Q}}$ the following hold:

$$(\underline{\mathbf{P}} + \underline{\mathbf{Q}})^{\cdot} = \dot{\underline{\mathbf{P}}} + \dot{\underline{\mathbf{Q}}}, \quad (\underline{\mathbf{P}}\underline{\mathbf{Q}})^{\cdot} = \dot{\underline{\mathbf{P}}}\underline{\mathbf{Q}} + \underline{\mathbf{P}}\dot{\underline{\mathbf{Q}}}, \quad \dot{\widetilde{\underline{\mathbf{Q}}}} = \widetilde{\dot{\underline{\mathbf{Q}}}}, \tag{21}$$

this notation is also suited for writing differential geometric properties in a very compact way, which is also demonstrated in Section 5.

5. Instantaneous kinematics

Now $\mathbf{X}$ contains the coordinates of X with respect to the moving coordinate frame C and $\mathbf{X}^{\oplus}_{\tau}$ denotes the coordinates of X with respect to the fixed frame $C^{\oplus}$ in dependency of the time τ of the constrained motion. According to Eq. (9) the following relation holds:

$$\mathbf{X}^{\oplus}_{\tau} = \mathbf{E}_{\tau}\mathbf{X}\widetilde{\mathbf{F}}_{\tau} - 2\mathbf{E}_{\tau}\widetilde{\mathbf{T}}_{\tau}, \tag{22}$$

where $\mathbf{E}_{\tau}$, $\mathbf{F}_{\tau}$ and $\mathbf{T}_{\tau}$ are functions of the time τ. Eq. (22) can be rewritten in terms of 2×2 quaternionic matrices (cf. Section 4) as follows:

$$\underline{\mathbf{X}}^{\oplus}_{\tau} = \widetilde{\underline{\mathbf{D}}}_{\tau}^{-T}\, \underline{\mathbf{X}}\, \widetilde{\underline{\mathbf{D}}}_{\tau}^{T}.$$

W.l.o.g. we can change the fixed frame from the old $C^{\oplus}$ into the new one $C^{\otimes}$ in a way that at the time instance $\tau = *$ the moving frame C and $C^{\otimes}$ coincide. This is achieved by the transformation:

$$\underline{\mathbf{X}}^{\otimes}_{\tau} = \widetilde{\underline{\mathbf{D}}}_{*}^{T}\, \underline{\mathbf{X}}^{\oplus}_{\tau}\, \widetilde{\underline{\mathbf{D}}}_{*}^{-T}.$$

By introducing the notation $\underline{\mathbf{B}}_{\tau} := \underline{\mathbf{D}}_{*}^{-1}\underline{\mathbf{D}}_{\tau}$ with

$$\underline{\mathbf{B}}_{\tau} = \begin{pmatrix} \mathbf{G}_{\tau} & \mathbf{O} \\ \mathbf{U}_{\tau} & \mathbf{H}_{\tau} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{E}}_{*}\mathbf{E}_{\tau} & \mathbf{O} \\ -\widetilde{\mathbf{F}}_{*}\mathbf{T}_{*}\widetilde{\mathbf{E}}_{*}\mathbf{E}_{\tau} + \widetilde{\mathbf{F}}_{*}\mathbf{T}_{\tau} & \widetilde{\mathbf{F}}_{*}\mathbf{F}_{\tau} \end{pmatrix}$$

the constrained motion with respect to the system $C^{\otimes}$ is written as:

$$\underline{\mathbf{X}}^{\otimes}_{\tau} = \widetilde{\underline{\mathbf{B}}}_{\tau}^{-T}\, \underline{\mathbf{X}}\, \widetilde{\underline{\mathbf{B}}}_{\tau}^{T} \quad \Longleftrightarrow \quad \mathbf{X}^{\otimes}_{\tau} = \mathbf{G}_{\tau}\mathbf{X}\widetilde{\mathbf{H}}_{\tau} - 2\mathbf{G}_{\tau}\widetilde{\mathbf{U}}_{\tau}. \tag{23}$$

Note that $\widetilde{\mathbf{B}}_{\tau}$ evaluated at $\tau = *$ equal the 2×2 identity matrix. The advantage of this coordinate transformation is that the geometric properties can be studied in a more compact way.

5.1. Velocity quaternion and instantaneous screw

According to the calculation rules for the differentiation of quaternions (see Eq. (2)) the time derivative of the normalizing condition $\mathbf{G}_{\tau}\widetilde{\mathbf{G}}_{\tau} = 1$ and the equation $\mathbf{G}_{\tau}\widetilde{\mathbf{G}}_{\tau} - \mathbf{H}_{\tau}\widetilde{\mathbf{H}}_{\tau} = 0$ of the cylinder Ξ with respect to τ yields:

$$\dot{\mathbf{G}}_{\tau}\widetilde{\mathbf{G}}_{\tau} + \mathbf{G}_{\tau}\dot{\widetilde{\mathbf{G}}}_{\tau} = 0 \quad \text{and} \quad \dot{\mathbf{G}}_{\tau}\widetilde{\mathbf{G}}_{\tau} + \mathbf{G}_{\tau}\dot{\widetilde{\mathbf{G}}}_{\tau} - \dot{\mathbf{H}}_{\tau}\widetilde{\mathbf{H}}_{\tau} - \mathbf{H}_{\tau}\dot{\widetilde{\mathbf{H}}}_{\tau} = 0,$$

respectively, where the superior dot denotes the time derivative. Evaluation of these formulas at $\tau = *$ implies $\dot{g}_0(*) = \dot{h}_0(*) = 0$ ($\Rightarrow \dot{\mathbf{G}}_{*} = \dot{\mathbf{g}}_{*}$ and $\dot{\mathbf{H}}_{*} = \dot{\mathbf{h}}_{*}$). Moreover by differentiation of Eq. (23) according to Eq. (21) and Eq. (2), respectively, we get:

$$\dot{\underline{\mathbf{X}}}^{\otimes}_{\tau} = \dot{\widetilde{\underline{\mathbf{B}}}}_{\tau}^{-T}\, \underline{\mathbf{X}}\, \widetilde{\underline{\mathbf{B}}}_{\tau}^{T} + \widetilde{\underline{\mathbf{B}}}_{\tau}^{-T}\, \underline{\mathbf{X}}\, \dot{\widetilde{\underline{\mathbf{B}}}}_{\tau}^{T} \quad \Longleftrightarrow \quad \dot{\mathbf{X}}^{\otimes}_{\tau} = \dot{\mathbf{G}}_{\tau}\mathbf{X}\widetilde{\mathbf{H}}_{\tau} + \mathbf{G}_{\tau}\mathbf{X}\dot{\widetilde{\mathbf{H}}}_{\tau} - 2\dot{\mathbf{G}}_{\tau}\widetilde{\mathbf{U}}_{\tau} - 2\mathbf{G}_{\tau}\dot{\widetilde{\mathbf{U}}}_{\tau}.$$

Its evaluation at time instance $\tau = *$ yields:

$$\dot{\mathbf{X}}^{\otimes}_{*} = \dot{\mathbf{g}}_{*}\mathbf{X} - \mathbf{X}\dot{\mathbf{h}}_{*} - 2\dot{\widetilde{\mathbf{U}}}_{*}, \tag{24}$$

which we call the velocity quaternion of X implied by the constrained motion at the time instance $\tau = *$ with respect to the fixed coordinate system $C^{\otimes}$. Its norm gives the corresponding velocity.

Remark 6. Note that in a similar way also the acceleration/lurch/snap/crackle/pop/lock/drop/... quaternions can be computed, but they are not of interest for the remainder of the paper. Moreover we can also compute in an analogous way the instantaneous motion of an oriented line, plane and hyperplane as $\underline{\dot{\mathbf{Y}}}_*^{\otimes}$, $\underline{\dot{\mathbf{L}}}_*^{\otimes}$ and $\underline{\dot{\mathbf{W}}}_*^{\otimes}$, respectively, which is left to the reader. ⋄

It can easily be checked that the affine mapping $\mathbf{X} \mapsto \dot{\mathbf{X}}_*^{\otimes}$ of Eq. (24) is singular if and only if $\dot{\mathbf{g}}_*\dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\dot{\mathbf{h}}_* = 0$ holds, which implies the following notation.

Definition 1. *The triple* $(\dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*)$ *is called the instantaneous screw* $\$_*^{\otimes}$ *of the motion* $(\mathbf{G}_\tau, \mathbf{H}_\tau, \mathbf{U}_\tau)$ *at time instance* $\tau = *$ *with respect to the fixed coordinate system* $C^{\otimes}$. *The instantaneous screw* $\$_*^{\otimes}$ *is called singular if* $\dot{\mathbf{g}}_*\dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\dot{\mathbf{h}}_* = 0$ *holds; otherwise regular.*

Remark 7. Note that in the singular case $\$_*^{\otimes}$ is located on Ξ given by Eq. (8). Moreover it should be noted that the linear space of instantaneous screws is nothing but the Lie algebra $\mathfrak{se}_4$ (e.g. [28, Section 4]). ⋄

As $\dot{\mathbf{X}}_*^{\otimes}$ represents a vector (and no point) its transformation into the initial fixed coordinate system $C^{\oplus}$ reads as follows:

$$\dot{\mathbf{X}}_*^{\oplus} = \mathbf{E}_*\dot{\mathbf{X}}_*^{\otimes}\widetilde{\mathbf{F}}_*. \tag{25}$$

In contrast to this, the instantaneous screw $\$_*^{\otimes}$ is transformed in the following way into $\$_*^{\oplus} = (\dot{\mathbf{g}}_*^{\oplus}, \dot{\mathbf{h}}_*^{\oplus}, \dot{\mathbf{U}}_*^{\oplus})$:

$$\underline{\$}_*^{\oplus} = \underline{\widetilde{\mathbf{D}}}_*^{-T}\,\underline{\$}_*^{\otimes}\,\underline{\widetilde{\mathbf{D}}}_*^{T}$$

with

$$\underline{\$}_*^{\oplus} = \begin{pmatrix} -\dot{\mathbf{g}}_*^{\oplus} & \dot{\widetilde{\mathbf{U}}}_*^{\oplus} \\ \mathbf{O} & \dot{\mathbf{h}}_*^{\oplus} \end{pmatrix}, \qquad \underline{\$}_*^{\otimes} = \begin{pmatrix} -\dot{\mathbf{g}}_* & \dot{\widetilde{\mathbf{U}}}_* \\ \mathbf{O} & \dot{\mathbf{h}}_* \end{pmatrix}.$$

This can easily be proven by showing that the expression of Eq. (25) equals the corresponding expression of Eq. (24), which reads as:

$$\dot{\mathbf{X}}_*^{\oplus} = \dot{\mathbf{g}}_*^{\oplus}\mathbf{X}_*^{\oplus} - \mathbf{X}_*^{\oplus}\dot{\mathbf{h}}_*^{\oplus} - 2\dot{\widetilde{\mathbf{U}}}_*^{\oplus}.$$

Therefore also the displacement of screws can be embedded into the algebra of 2×2 quaternionic matrices, which is the content of the next theorem:

Theorem 11. *The mapping of an instantaneous screw* \$ *of* E^4 *to an instantaneous screw* \$′ *of* E^4 *induced by any element of SE(4) can be written as follows:*

$$\underline{\$} \mapsto \underline{\$}' \quad \textit{with} \quad \underline{\$}' := \underline{\widetilde{\mathbf{D}}}^{-T}\,\underline{\$}\,\underline{\widetilde{\mathbf{D}}}^{T}.$$

5.2. Quaternionic characterization of instantaneous screws

Within the following two subsections we compute the geometric criteria for the classification of instantaneous screws in terms of quaternions.

5.2.1. Instantaneously fixed ideal lines and angular velocities

We are interested in those ideal lines, which are instantaneously fixed with respect to a given instantaneous screw $\$_*^{\otimes} = (\dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*)$. We describe an oriented ideal line by its left and right direction vector $\mathbf{l}_+$ and $\mathbf{l}_-$ (cf. Section 4.3). According to Eq. (17) a constrained motion implies:

$$\mathbf{l}_{+,\tau}^{\otimes} = \mathbf{G}_\tau\mathbf{l}_+\widetilde{\mathbf{G}}_\tau, \qquad \mathbf{l}_{-,\tau}^{\otimes} = \mathbf{H}_\tau\mathbf{l}_-\widetilde{\mathbf{H}}_\tau.$$

Differentiation with respect to τ and evaluation at time instance $\tau = *$ yields:

$$\dot{\mathbf{l}}_{+,*}^{\otimes} = \dot{\mathbf{g}}_*\mathbf{l}_+ - \mathbf{l}_+\dot{\mathbf{g}}_* \qquad \dot{\mathbf{l}}_{-,*}^{\otimes} = \dot{\mathbf{h}}_*\mathbf{l}_- - \mathbf{l}_-\dot{\mathbf{h}}_*. \tag{26}$$

The ideal lines which are instantaneously fixed have to fulfill $\dot{\mathbf{l}}_{+,*}^{\otimes} = \dot{\mathbf{l}}_{-,*}^{\otimes} = \mathbf{o}$ and $\|\mathbf{l}_+\| = \|\mathbf{l}_-\| = 1$. We have to distinguish the following cases:

1. $\dot{\mathbf{g}}_* \neq \mathbf{o} \neq \dot{\mathbf{h}}_*$: The solutions for $\mathbf{l}_+$ and $\mathbf{l}_-$ are

$$\pm\dot{\mathbf{g}}_0 := \pm\frac{\dot{\mathbf{g}}_*}{\|\dot{\mathbf{g}}_*\|}, \qquad \pm\dot{\mathbf{h}}_0 := \pm\frac{\dot{\mathbf{h}}_*}{\|\dot{\mathbf{h}}_*\|}, \tag{27}$$

respectively. We orient these lines as follows:

$$\mathsf{L}_1: \ (\mathbf{l}_+, \mathbf{l}_-) = (\dot{\mathbf{g}}_0, \dot{\mathbf{h}}_0), \qquad \mathsf{L}_2: \ (\mathbf{l}_+, \mathbf{l}_-) = (\dot{\mathbf{g}}_0, -\dot{\mathbf{h}}_0), \tag{28}$$

which are the ideal lines of the instantaneous planes of rotation Ω_1 and Ω_2, respectively.

2. $\dot{\mathbf{g}}_* \neq \mathbf{o} = \dot{\mathbf{h}}_*$: According to Eq. (26) we can take any pure unit-quaternion for $\mathbf{l}_-$. Therefore there exists a 2-dimensional set of instantaneously fixed ideal lines (⇒ instantaneously isoclinic left rotation). For the special choice $\mathbf{l}_- = \pm\dot{\mathbf{g}}_0$ we get the following oriented lines:

$$\mathsf{L}_1: \ (\mathbf{l}_+, \mathbf{l}_-) = (\dot{\mathbf{g}}_0, \dot{\mathbf{g}}_0), \qquad \mathsf{L}_2: \ (\mathbf{l}_+, \mathbf{l}_-) = (\dot{\mathbf{g}}_0, -\dot{\mathbf{g}}_0). \tag{29}$$

3. $\dot{\mathbf{g}}_* = \mathbf{o} \neq \dot{\mathbf{h}}_*$: In this case we have an instantaneously isoclinic right rotation. Analogous considerations for the special choice $\mathbf{l}_+ = \pm\dot{\mathbf{h}}_0$ yield the following oriented lines:

$$\mathsf{L}_1: \ (\mathbf{l}_+, \mathbf{l}_-) = (\dot{\mathbf{h}}_0, \dot{\mathbf{h}}_0), \qquad \mathsf{L}_2: \ (\mathbf{l}_+, \mathbf{l}_-) = (\dot{\mathbf{h}}_0, -\dot{\mathbf{h}}_0). \tag{30}$$

4. $\dot{\mathbf{g}}_* = \mathbf{o} = \dot{\mathbf{h}}_*$: Each line of the ideal 3-space remains fixed which corresponds to an instantaneous translation or standstill.

Lemma 2. *The angular velocity ω_1 (resp. ω_2) of the instantaneous rotation about the plane Ω_1 (resp. Ω_2), whose orientation is implied by the one of its ideal line L_1 (resp. L_2) given in Eqs. (28-30), equals $\|\dot{\mathbf{g}}_*\|+\|\dot{\mathbf{h}}_*\|$ (resp. $\|\dot{\mathbf{g}}_*\|-\|\dot{\mathbf{h}}_*\|$).*

Proof: We only prove the case $\dot{\mathbf{g}}_* \neq \mathbf{o} \neq \dot{\mathbf{h}}_*$ as the isoclinic cases can be done in a similar fashion. We choose a special coordinate system in a way that $\overrightarrow{\Omega_1}$ equals the positively oriented x_0x_1-plane and that $\overrightarrow{\Omega_2}$ equals the positively oriented x_2x_3-plane. Therefore $\dot{\mathbf{g}}_0 = \dot{\mathbf{h}}_0 = \mathbf{i}$ holds.

In order to identify the angular velocity of the rotations about the plane Ω_1 we can compute for example the velocity of the point $\mathbf{j}$ as:

$$\dot{\mathbf{g}}_*\mathbf{j} - \mathbf{j}\dot{\mathbf{h}}_* = \|\dot{\mathbf{g}}_*\|\,\dot{\mathbf{g}}_0\,\mathbf{j} - \mathbf{j}\dot{\mathbf{h}}_0\,\|\dot{\mathbf{h}}_*\| = \left(\|\dot{\mathbf{g}}_*\| + \|\dot{\mathbf{h}}_*\|\right)\mathbf{k}. \tag{31}$$

Otherwise we can consider a continuous rotation of the point $\mathbf{j}$ around the plane Ω_1 with the rotation angle $\varphi(\tau) \geq 0$, which reads as follows according to Section 3.1:

$$\left(\cos\tfrac{\varphi}{2} + \sin\tfrac{\varphi}{2}\mathbf{i}\right)\mathbf{j}\left(\cos\tfrac{\varphi}{2} - \sin\tfrac{\varphi}{2}\mathbf{i}\right) = \cos\varphi\mathbf{i} + \sin\varphi\mathbf{k}.$$

Differentiation with respect to τ and evaluation at $\tau = *$ yields $\dot{\varphi}\mathbf{k}$ as $\varphi(*) = 0$ holds. Comparing this formula with Eq. (31) shows that the angular velocity $\omega_1 = \dot{\varphi}$ of the instantaneous rotation about the oriented plane $\overrightarrow{\Omega_1}$ equals $\|\dot{\mathbf{g}}_*\| + \|\dot{\mathbf{h}}_*\|$.

Analogue considerations can be done for the point, which is represented by the identity quaternion, with respect to the plane Ω_2. □

5.2.2. Velocity pole and axis-plane

All points of the moving system, which have instantaneously a zero velocity, constitute the so-called velocity pole.[3] Therefore we want to compute the point $\mathbf{P}$ with $\dot{\mathbf{P}}^{\otimes}_* = \mathbf{O}$ according to Eq. (24). The solution of this quaternionic linear equation reads as follows (cf. [15, Section "General solution to linear problems in quaternion variables"]):

$$\mathbf{P} = \frac{2}{\dot{\mathbf{g}}_*\dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\dot{\mathbf{h}}_*}\left(\dot{\mathbf{g}}_*\tilde{\dot{\mathbf{U}}}_* + \tilde{\dot{\mathbf{U}}}_*\dot{\mathbf{h}}_*\right), \tag{32}$$

[3]Note that only the velocity pole has a geometric meaning, as poles of higher derivatives (acceleration/lurch/snap/... pole) depend on the parametrization of the motion.

if the instantaneous screw is regular.

The singular case implies that $\omega_2 = 0$ holds (cf. Lemma 2). If we do not have an instantaneous translation or standstill, the instantaneous motion is a rotation about a plane, which can additionally be composed with a translation parallel to this plane. Note that this plane is the axis-space mentioned in the review of known results given in Section 1.2.

In order to fix this axis-plane, we only have to compute one point of it; preferable the footpoint A with respect to the origin of the fixed frame. In the following we show how this can be done in a pure quaternionic way.

It can easily be checked[4] that the ideal points of the directions determined by the following two orthogonal unit-quaternions:

$$\mathbf{S} = \frac{\dot{\mathbf{h}}_0 - \dot{\mathbf{g}}_0}{\sqrt{2 + \dot{\mathbf{g}}_0\dot{\mathbf{h}}_0 + \dot{\mathbf{h}}_0\dot{\mathbf{g}}_0}}, \qquad \mathbf{S}^{\perp} = \frac{\dot{\mathbf{g}}_0\dot{\mathbf{h}}_0 + 1}{\sqrt{2 + \dot{\mathbf{g}}_0\dot{\mathbf{h}}_0 + \dot{\mathbf{h}}_0\dot{\mathbf{g}}_0}},$$

are located on the ideal line L_2 of Eq. (28). Therefore we can set:

$$\mathbf{A} = \varsigma\mathbf{S} + \varsigma^{\perp}\mathbf{S}^{\perp}. \tag{33}$$

Now we decompose $\dot{\widetilde{\mathbf{U}}}_*$ in a component $\dot{\widetilde{\mathbf{U}}}_*^{\|}$ parallel to $\mathbf{\Omega}_1$ and a component $\dot{\widetilde{\mathbf{U}}}_*^{\perp}$ orthogonal to it with

$$\dot{\widetilde{\mathbf{U}}}_*^{\perp} = \langle\dot{\widetilde{\mathbf{U}}}_*, \mathbf{S}\rangle\mathbf{S} + \langle\dot{\widetilde{\mathbf{U}}}_*, \mathbf{S}^{\perp}\rangle\mathbf{S}^{\perp}.$$

Then $\mathbf{A}$ can be computed as the point, which has zero velocity with respect to the instantaneous screw $(\dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*^{\perp})$. Under consideration that

$$\dot{\mathbf{g}}_* = \|\dot{\mathbf{g}}_*\|\mathbf{S}^{\perp}\widetilde{\mathbf{S}}, \qquad \dot{\mathbf{h}}_* = -\|\dot{\mathbf{h}}_*\|\widetilde{\mathbf{S}}\mathbf{S}^{\perp}$$

hold, which can be checked by direct computations, the corresponding condition (cf. Eq. (24)) can be written as:

$$\|\dot{\mathbf{g}}_*\|\mathbf{S}^{\perp}\widetilde{\mathbf{S}}\left(\varsigma\mathbf{S} + \varsigma^{\perp}\mathbf{S}^{\perp}\right) + \left(\varsigma\mathbf{S} + \varsigma^{\perp}\mathbf{S}^{\perp}\right)\|\dot{\mathbf{h}}_*\|\widetilde{\mathbf{S}}\mathbf{S}^{\perp} = 2\dot{\widetilde{\mathbf{U}}}_*^{\perp}$$

which simplifies to

$$\omega_1\left(\varsigma\mathbf{S}^{\perp} + \varsigma^{\perp}\mathbf{S}^{\perp}\widetilde{\mathbf{S}}\mathbf{S}^{\perp}\right) = 2\dot{\widetilde{\mathbf{U}}}_*^{\perp}.$$

As we have no instantaneous standstill or translation we can divide by ω_1. Moreover we multiply by $\widetilde{\mathbf{S}}^{\perp}$ from the right side which yields:

$$\varsigma + \varsigma^{\perp}\dot{\mathbf{g}}_0 = 2\omega_1^{-1}\dot{\widetilde{\mathbf{U}}}_*^{\perp}\widetilde{\mathbf{S}}^{\perp}.$$

Decomposition of this equation into the scalar part and pure part according to Eq. (1) implies the desired formulas:

$$\varsigma = \omega_1^{-1}\left(\dot{\widetilde{\mathbf{U}}}_*^{\perp}\widetilde{\mathbf{S}}^{\perp} + \mathbf{S}^{\perp}\dot{\mathbf{U}}_*^{\perp}\right), \qquad \varsigma^{\perp} = -\omega_1^{-1}\dot{\mathbf{g}}_0\left(\dot{\widetilde{\mathbf{U}}}_*^{\perp}\widetilde{\mathbf{S}}^{\perp} - \mathbf{S}^{\perp}\dot{\mathbf{U}}_*^{\perp}\right). \tag{34}$$

Now we are able to sum up the results of Section 5.2 within the next theorem.

Theorem 12. *For an instantaneous screw $\$_*^{\otimes} = (\dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*)$ of Definition 1, which differs from the instantaneous standstill $(\mathbf{o}, \mathbf{o}, \mathbf{O})$, we can distinguish the following cases, where the oriented planes are given by the triple $(\mathbf{l}_+, \mathbf{l}_-, \mathbf{L})$ according to Section 4.3:*

1. *$\$_*^{\otimes}$ is regular:*

   (a) *$\dot{\mathbf{g}}_* \neq \mathbf{o} \neq \dot{\mathbf{h}}_*$: Instantaneously there is a rotation with angular velocities $\omega_{1,2} = \|\dot{\mathbf{g}}_*\| \pm \|\dot{\mathbf{h}}_*\|$ about the total-orthogonal planes*

   $$\left(\dot{\mathbf{g}}_0, \pm\dot{\mathbf{h}}_0, \tfrac{1}{2}(\dot{\mathbf{g}}_0\mathbf{P} \mp \mathbf{P}\dot{\mathbf{h}}_0)\right)$$

   with $\mathbf{P}$ of Eq. (32) and $\dot{\mathbf{g}}_0$ and $\dot{\mathbf{h}}_0$ according to Eq. (27).

[4]One only has to verify that $\mathbf{L}$ given in Eq. (20) equals the zero quaternions.

(b) $\dot{\mathbf{h}}_* = \mathbf{o}$*: Instantaneously there is an isoclinic left rotation. One pair of total-orthogonal rotation planes with angular velocities* $\omega_{1,2} = \|\dot{\mathbf{g}}_*\|$ *is given by*

$$\left(\dot{\mathbf{g}}_0, \pm\dot{\mathbf{g}}_0, \tfrac{1}{2}(\dot{\mathbf{g}}_0\mathbf{P} \mp \mathbf{P}\dot{\mathbf{g}}_0)\right).$$

(c) $\dot{\mathbf{g}}_* = \mathbf{o}$*: Instantaneously there is an isoclinic right rotation. One pair of total-orthogonal rotation planes with angular velocities* $\omega_{1,2} = \pm\|\dot{\mathbf{h}}_*\|$ *is given by*

$$\left(\dot{\mathbf{h}}_0, \pm\dot{\mathbf{h}}_0, \tfrac{1}{2}(\dot{\mathbf{h}}_0\mathbf{P} \mp \mathbf{P}\dot{\mathbf{h}}_0)\right).$$

2. $\$^{\otimes}_*$ *is singular:*

   (a) $\dot{\mathbf{g}}_* = \dot{\mathbf{h}}_* = \mathbf{o}$*: There is an instantaneous translation given by* $-2\dot{\widetilde{\mathbf{U}}}_*$.

   (b) $\dot{\mathbf{g}}_* \neq \mathbf{o} \neq \dot{\mathbf{h}}_*$ *and* $\dot{\widetilde{\mathbf{U}}}_*^{||} = \mathbf{O}$*: Instantaneously there is a rotation with angular velocity* $\omega_1 = \|\dot{\mathbf{g}}_*\| + \|\dot{\mathbf{h}}_*\|$ *about the plane*

$$\left(\dot{\mathbf{g}}_0, \dot{\mathbf{h}}_0, \tfrac{1}{2}(\dot{\mathbf{g}}_0\mathbf{A} - \mathbf{A}\dot{\mathbf{h}}_0)\right) \tag{35}$$

   where $\mathbf{A}$ *is given by Eqs. (33) and (34).*

   (c) $\dot{\mathbf{g}}_* \neq \mathbf{o} \neq \dot{\mathbf{h}}_*$ *and* $\dot{\widetilde{\mathbf{U}}}_*^{||} \neq \mathbf{O}$*: Instantaneously there is a composition of a rotation with angular velocity* $\omega_1 = \|\dot{\mathbf{g}}_*\| + \|\dot{\mathbf{h}}_*\|$ *about the plane given in Eq. (35) and a translation* $-2\dot{\widetilde{\mathbf{U}}}_*^{||}$.

Remark 8. If the instantaneous screw $\$^{\otimes}_*$ is regular we can define its pitch by $\frac{\omega_2}{\omega_1}$. As a consequence of a result of Wunderlich [37, §10] the singular instantaneous screws $\$^{\otimes}_*$ can be seen as an extension of the instantaneous screws of E^3 to E^4. Therefore we can assign to the singular instantaneous screws the screw parameter of the corresponding instantaneous screws in E^3, which yield for item 2a and item 2b the value ∞ and 0, respectively. For item 2c we get $\pm 2\|\dot{\widetilde{\mathbf{U}}}_*^{||}\|\omega_1^{-1}$. ◇

6. Conclusion

Within this paper we showed that the action of SE(4) on the basic geometric elements in E^4 leads to a quaternion-matrix representation of SE(4). The odd-dimensional elements (oriented hyperplanes, oriented lines) of E^4 induce the representation $\underline{*} \mapsto \underline{\mathbf{D}}\,\underline{*}\,\underline{\widetilde{\mathbf{D}}}^T$ (cf. Theorems 8 and 9) and the even-dimensional elements (points, oriented planes) of E^4 yield the dual (contragredient) representation as $\underline{*} \mapsto \underline{\widetilde{\mathbf{D}}}^{-T}\,\underline{*}\,\underline{\widetilde{\mathbf{D}}}^T$ (cf. Theorems 7 and 10) holds. In the latter way also the instantaneous screws are displaced (cf. Theorem 11), which are studied in detail in Section 5 (cf. Theorems 12).

Finally it should be noted that the algebra of 2×2 quaternionic matrices is isomorphic to the Clifford algebra with signature $(1_+, 3_-, 0_0)$, which shows again the difference to the Klawitter-Hagemann construction (cf. Section 2.3) based on the Spin group of the Clifford algebra with signature $(4_+, 0_-, 1_0)$.

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Appendix

Adopted to our notation Theorem 9.1 of Coxeter [6] reads as follows:

Theorem 9.1 of [6]. *The general displacement* $\mathbf{X} \mapsto \mathbf{E}\mathbf{X}\widetilde{\mathbf{F}}$ *is the double rotation through angles* $\alpha \pm \beta$ *about planes* $\mathbf{O}, \mathbf{e}_0 \pm \mathbf{f}_0, 1 \mp \mathbf{e}_0\mathbf{f}_0$.

Analogously to Coxeter's notation of vectors, the order of the above points also imply an orientation for these planes. Therefore we can set:

$$\mathbf{M} = \frac{\mathbf{e}_0 + \mathbf{f}_0}{\sqrt{2 - \mathbf{e}_0\mathbf{f}_0 - \mathbf{f}_0\mathbf{e}_0}}, \qquad \mathbf{M}^\perp = \frac{1 - \mathbf{e}_0\mathbf{f}_0}{\sqrt{2 - \mathbf{e}_0\mathbf{f}_0 - \mathbf{f}_0\mathbf{e}_0}}, \tag{36}$$

and

$$\mathbf{Q} = \frac{\mathbf{e}_0 - \mathbf{f}_0}{\sqrt{2 + \mathbf{e}_0\mathbf{f}_0 + \mathbf{f}_0\mathbf{e}_0}}, \qquad \mathbf{Q}^\perp = \frac{1 + \mathbf{e}_0\mathbf{f}_0}{\sqrt{2 + \mathbf{e}_0\mathbf{f}_0 + \mathbf{f}_0\mathbf{e}_0}}, \tag{37}$$

respectively. Moreover with the formulas of $\mathbf{N}$ and $\mathbf{R}$ given in Eq. (14) with respect to $\mathbf{M}, \mathbf{M}^\perp, \mathbf{Q}, \mathbf{Q}^\perp$ of Eqs. (36) and (37), we can compute $\mathbf{R}\widetilde{\mathbf{Q}}\mathbf{N}\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{Q}}\mathbf{R}\widetilde{\mathbf{M}}\mathbf{N}$, respectively. We get $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{F}}$ instead of $\mathbf{E}$ and $\mathbf{F}$ (cf. Eq. (11)), which shows that the orientation of the planes is not correct.