# On the line-symmetry of self-motions of linear pentapods

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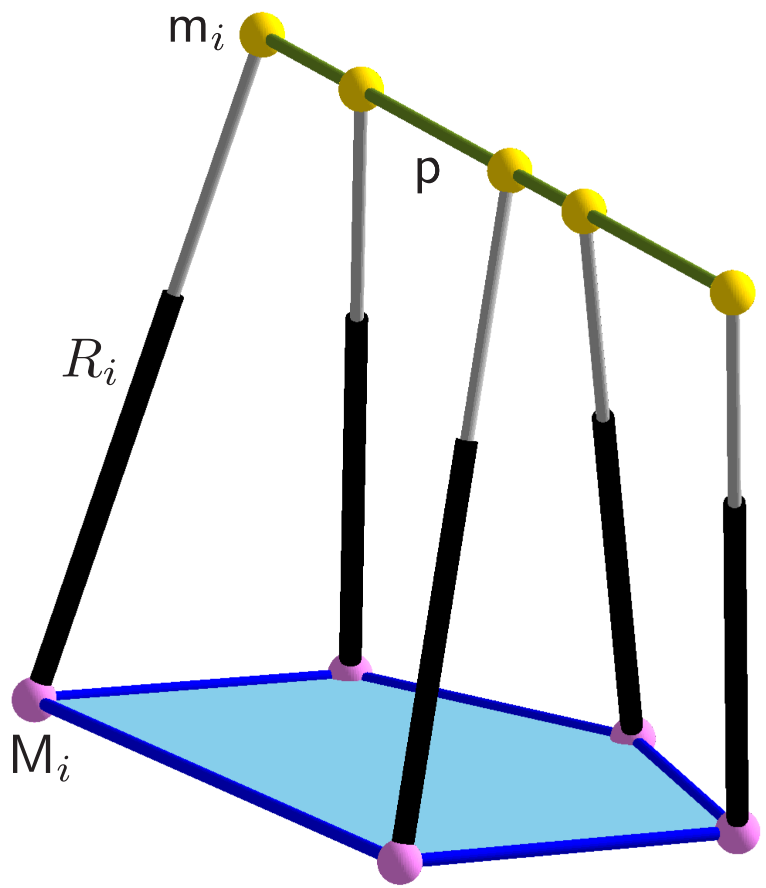
# Overview

# 1. Introduction

The geometry of a linear pentapod (LP) is given by the five base anchor points $\mathsf{M}_i$ in the fixed system and by the five collinear platform anchor points $\mathsf{m}_i$ in the moving system (for $i = 1, \ldots, 5$).

$\mathsf{M}_i$ and $\mathsf{m}_i$ are connected with a S<u>P</u>S leg.

If the geometry of the LP is given as well as the lengths $R_1, \ldots, R_5$, then it has generically mobility 1, which corresponds to the rotation about the carrier line p of the five platform anchor points.



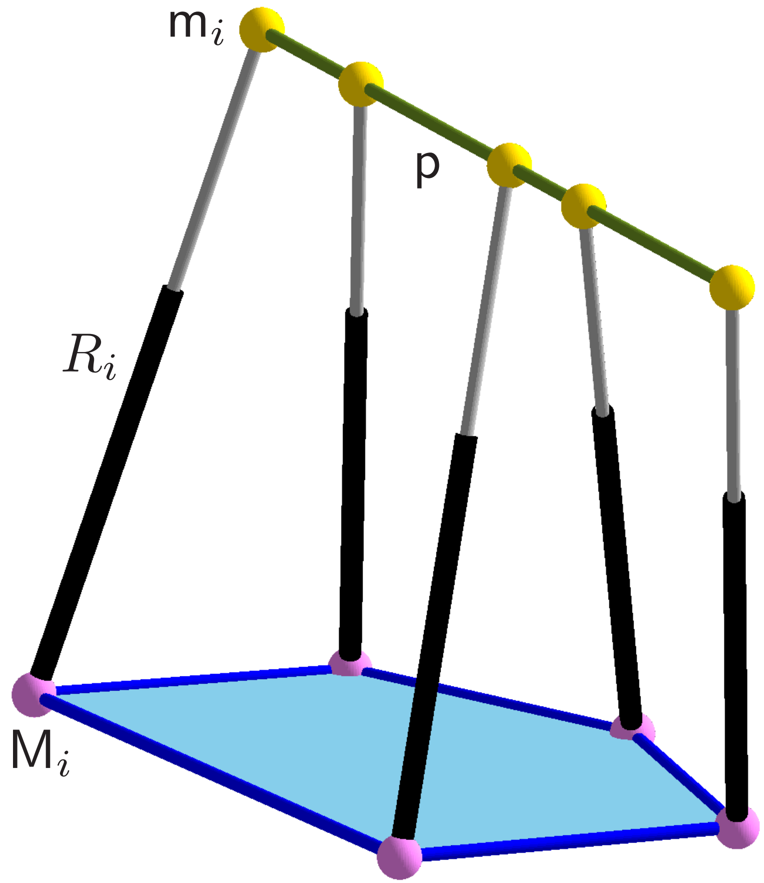
# 1. Introduction

As this rotational motion is irrelevant for applications with axial symmetry

- 5-axis milling,
- spray-based painting,
- laser cutting,
- spot-welding, ...

these robots are of great practical interest.

**Definition.**
Any additional uncontrollable mobility beside the rotational motion about p is referred as self-motion of the LP.



# 2. Review on self-motions of LPs

Self-motions of LPs represent interesting solutions to the still unsolved

**Borel-Bricard problem.**
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.

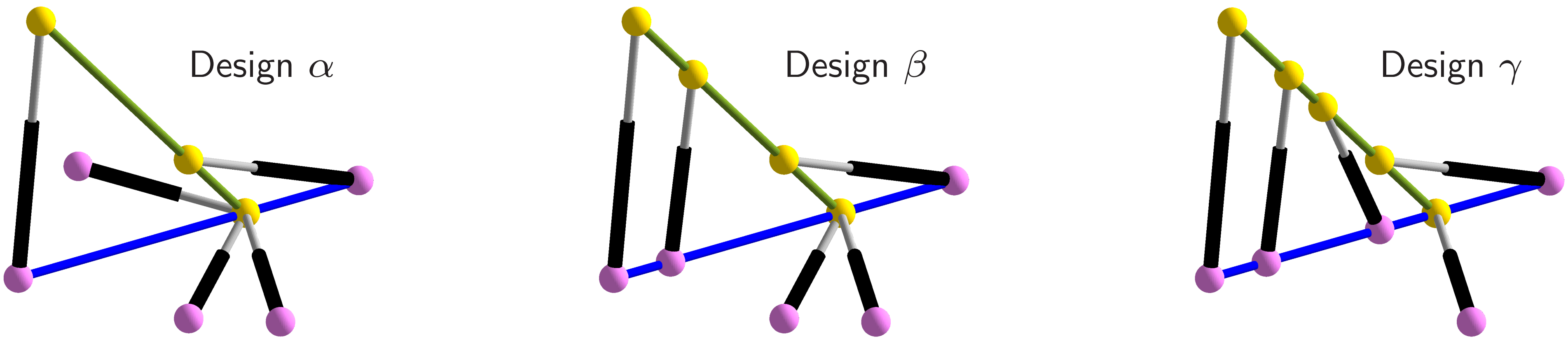
For five collinear points the Borel-Bricard problem was studied by:

- Darboux [5]
- Mannheim [6]
- Duporcq [7] (see also Bricard [3])

A contemporary and accurate reexamination of these old results, which also takes the coincidence of platform anchor points into account, was done by Nawratil & Schicho [1] yielding a full classification of LPs with self-motions.

# 2. Review on self-motions of LPs

Beside
- architecturally singular LPs (see Corollary 1 of [1])
- LPs with circular translational self-motions (see Theorem 1 of [1])
- LPs with pure rotational self-motions (Designs $\alpha$, $\beta$, $\gamma$ of [1])




there only remain the following designs:

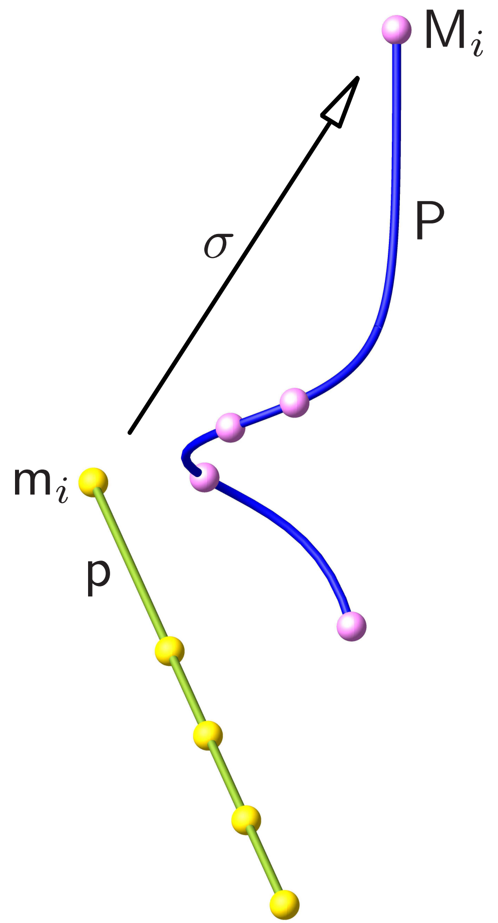
# 2. Review on self-motions of LPs

Under a self-motion each point of the line p has a spherical (or planar) trajectory.

The locus of the corresponding sphere centers is a

**Straight Cubic Circle P.**
This is a space curve of degree 3, which intersects the ideal plane in one real point W and two conjugate complex points, where the latter ones are the cyclic points I and J of a plane orthogonal to the direction of W.

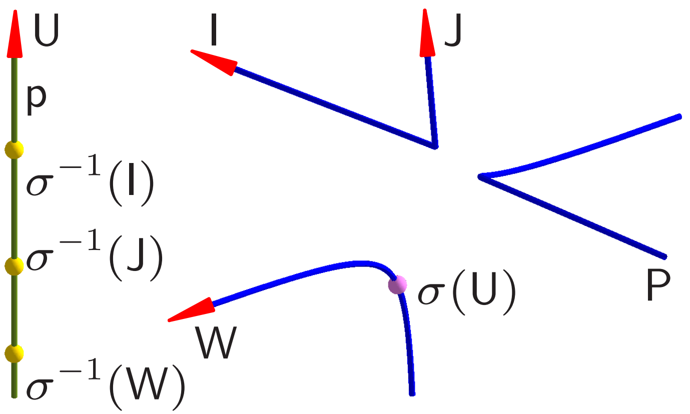
The mapping from p to P is named $\sigma$.



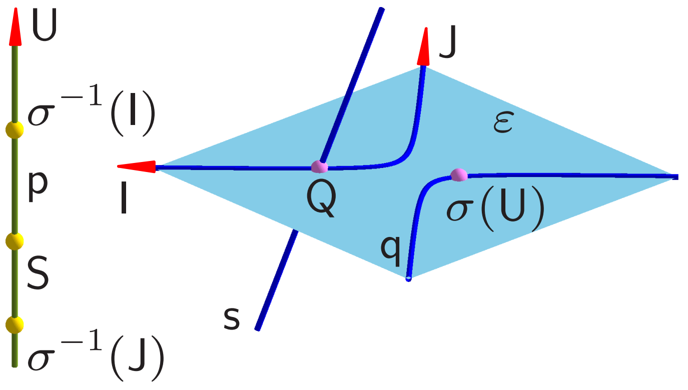
# 2. Review on self-motions of LPs

The following subcases can be distinguished:

- P is irreducible:
  - Type 5 (according to [1]):
    $\sigma$ maps the ideal point U of p to W.
  - Type 1 (according to [1]):
    $\sigma$ maps U to a finite point of P.




- Type 2 (according to [1]):
  P splits up into a circle q and a line s, which is orthogonal to the carrier plane $\varepsilon$ of q and intersects q in a point Q. Moreover $\sigma$ maps U to a point on $q \setminus \{Q\}$.



# 3. Line-symmetric self-motions of LPs

KRAMES [4,10] studied 1-parametric motions obtained by reflecting the moving system in the generators of a ruled surface (*basic surface*) of the fixed system.

These so-called *line-symmetric motions* were also studied by BOTTEMA & ROTH [8], who gave an intuitive algebraic characterization in terms of Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ fulfilling $\Psi : e_0 f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3 = 0$.

There always exists a moving frame (in dependence of a given fixed frame) in a way that $e_0 = f_0 = 0$ holds for a line-symmetric motion. Then $(e_1 : e_2 : e_3 : f_1 : f_2 : f_3)$ are the Plücker coordinates of the generators of the basic surface.

⋆ Rotational and circular translational self-motions are trivially line-symmetric.
⋆ Self-motions of Type 5 are also line-symmetric (cf. KRAMES [4]).

**Question.**
Can all Type 1 & 2 self-motions of LPs be generated by line-symmetric motions?

# 4. On the line-symmetry of Type 1 & 2 self-motions

For computations we select special pairs of anchor points (incl. special fixed frame):

| $i$ | $\mathsf{M}_i \in \mathsf{P}$ | $\mathsf{m}_i \in \mathsf{p}$ | Condition | Leg Parameter |
|---|---|---|---|---|
| 1 | $(1 : A : 0 : C) \quad A \neq 0$ | $\sigma^{-1}(\mathsf{M}_1)$ | Sphere $\Lambda_1$ | $R_1$ |
| 2 | $\mathsf{I} = (0 : 1 : i : 0)$ | $\sigma^{-1}(\mathsf{I})$ | Darboux $\Omega_2$ | $p_2$ |
| 3 | $\mathsf{J} = (0 : 1 : -i : 0)$ | $\sigma^{-1}(\mathsf{J})$ | Darboux $\Omega_3$ | $p_3$ |
| 4 | $\mathsf{W} = (0 : 0 : 0 : 1)$ | $\sigma^{-1}(\mathsf{W})$ | Darboux $\Omega_4$ | $p_4$ |
| 5 | $\sigma(\mathsf{U}) = (1 : 0 : 0 : 0)$ | $\mathsf{U}$ | Mannheim $\Pi_5$ | $p_5$ |

The pose of p with respect to moving frame is parametrized as follows:

$$\mathbf{m}_i = \mathbf{n} + (a_i - a_r)\mathbf{d} \quad \text{for } i = 1, \ldots, 4 \text{ with } \begin{cases} a_1 = 0 \\ a_2 = a_r + i a_c \\ a_3 = a_r - i a_c \\ a_4 \in \mathbb{R} \end{cases} \quad a_r \in \mathbb{R}, a_c \in \mathbb{R}^*$$

# 4. On the line-symmetry of Type 1 & 2 self-motions

$m_5$ is the ideal point in direction of the unit-vector $\mathbf{d} = (d_1, d_2, d_3)^T$, which obtains the rational homogeneous parametrization of the unit-sphere, i.e.

$$d_1 = \frac{2h_0h_1}{h_0^2+h_1^2+h_2^2}, \quad d_2 = \frac{2h_0h_2}{h_0^2+h_1^2+h_2^2}, \quad d_3 = \frac{h_1^2+h_2^2-h_0^2}{h_0^2+h_1^2+h_2^2}.$$

According to [1] the leg-parameters $R_1, p_2, \ldots, p_5$ have to fulfill the following necessary and sufficient conditions for the self-mobility (over $\mathbb{C}$):

$$p_2 = \frac{Aa_3v}{(a_3-a_4)^2}, \qquad p_3 = \frac{Aa_2v}{(a_2-a_4)^2}, \qquad p_4 = -\frac{Ca_4v}{(a_2-a_4)(a_3-a_4)},$$

$$(a_2 - a_4)^2(a_3 - a_4)^2 \left[2wp_5 - vR_1{}^2 - (2w - va_4)a_4\right] + vw^2(A^2 + C^2) = 0, \qquad (\star)$$

with $v := a_2 + a_3 - 2a_4$ and $w := a_2a_3 - a_4^2$.

**Remark:** Due to $(\star)$ LPs of Type 1 & 2 have a 1-dim set of self-motions. $\diamond$

# 4. On the line-symmetry of Type 1 & 2 self-motions

**Main Theorem.**
Each self-motion of a LP of Type 1 or 2 can be generated by a 1-dim set of line-symmetric motions. For the special case $p_5 = a_4 = a_r$ this set is even 2-dim.

**Corollary:** The self-motions of non-architecturally singular LPs are line-symmetric.

**Proof:** We can discuss Type 1 and Type 2 at the same time, just having in mind that $a_4 \neq 0 \neq C$ has to hold for Type 1 and $a_4 = 0 = C$ for Type 2 (cf. [1]).

We are looking for the pose of p (determined by $\mathbf{n}$ and $\mathbf{d}$) in a way that for the self-motion $e_0 = f_0 = 0$ holds.

W.l.o.g. we can set $e_0 = 0$ as any two directions $\mathbf{d}$ of p can be transformed into each other by a half-turn about their enclosed bisecting line. Note that this line is not uniquely determined if and only if the two directions are antipodal.

# 4. On the line-symmetry of Type 1 & 2 self-motions

$\Psi, \Omega_2, \Omega_3, \Omega_4, \Pi_5$ are homogeneous quadratic in the Study parameters and especially linear in $f_0, \ldots, f_3$. W.l.o.g. we can solve $\Psi, \Omega_2, \Omega_3, \Omega_4$ for $f_0, f_1, f_2, f_3$.

The numerator of $\begin{cases} f_0 \\ \Pi_5 \end{cases}$ yields a homogeneous $\begin{cases} \text{cubic expression } F(e_1, e_2, e_3) \\ \text{quartic expression } G(e_1, e_2, e_3) \end{cases}$

**General Case** $(a_4 \neq a_r)$**:** The condition $G = 0$ already expresses the self-motion as $G$ equals $\Lambda_1$ if we solve $(\star)$ for $R_1$.

$G$ has to split into $F$ and a homogeneous linear factor $L$ in $e_1, e_2, e_3$.

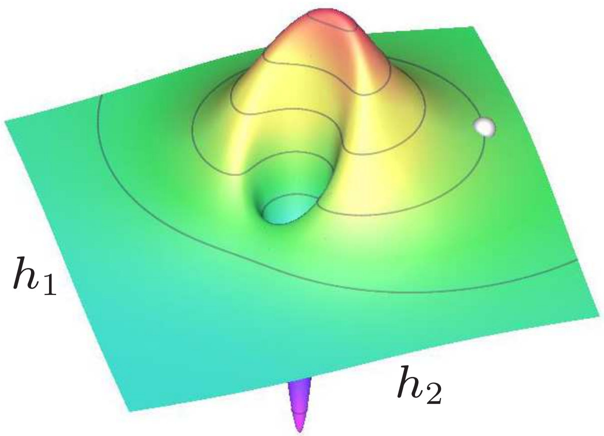
As $L = 0$ cannot correspond to a self-motion of the LP (yields contradiction), it has to arise from the ambiguity in representing a direction of p. As a consequence we can set $L = d_1e_1 + d_2e_2 + d_3e_3$.

# 4. On the line-symmetry of Type 1 & 2 self-motions

$$\Rightarrow \quad \Delta: \ \lambda LF - G = 0$$

The resulting set of four equations arising from the coefficients of $e_1^3e_2$, $e_1^3e_3$, $e_1e_3^3$ and $e_2e_3^3$ of $\Delta$ has the unique solution: $\lambda = 2(h_0^2+h_1^2+h_2^2)$,

$$n_1 = a_cd_2, \ n_2 = -a_cd_1, \ n_3 = (a_r-a_4)d_3. \quad (\circ)$$




$$\Rightarrow \quad \Delta: \ (e_1^2+e_2^2+e_3^2)^2(h_0^2+h_1^2+h_2^2)H = 0,$$

where $H(h_0, h_1, h_2) = 0$ is planar quartic curve.

$H = 0$ can be solved linearly for $p_5$. The corresponding graph is illustrated in dependency of $h_1, h_2$ for $h_0 = 1$.

**Special Case** $(a_4 = a_r)$**:** Then $(\star)$ implies $p_5 = a_4 = a_r$. Now $G$ is fulfilled identically and the self-motion is given by $\Lambda_1 = 0$, which is of degree 4 in $e_1, e_2, e_3$. Moreover for this special case $F = 0$ already holds for $\mathbf{n}$ given in $(\circ)$. □

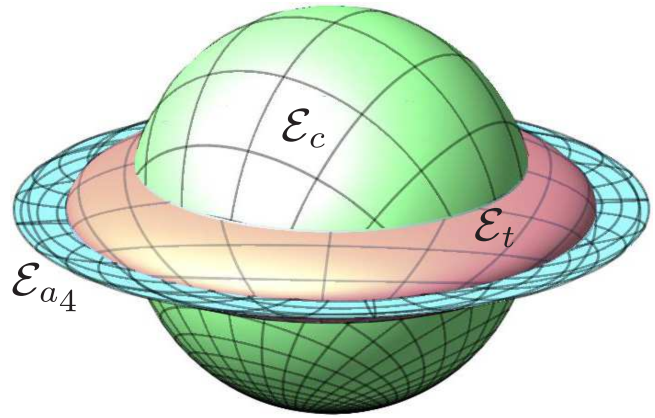
# 5. On the reality of Type 1 & 2 self-motions

We denote real points of p by $\mathrm{p}_t$ with $t \in \mathbb{R}$ and coordinate vector $\mathbf{p}_t = \mathbf{n}+(t-a_r)\mathbf{d}$.

As $L = 0$ corresponds with one configuration of the self-motion, we can compute the locus $\mathcal{E}_t$ of $\mathrm{p}_t$ under the one-parametric set of self-motions by the variation of $(h_0 : h_1 : h_2)$ within $L = 0$. Due to the mentioned ambiguity we can select any solution $(e_0 : e_1 : e_2)$ for $L = 0$ fulfilling $e_1^2 + e_2^2 + e_3^2 = 1$; e.g.:

$$e_1 = \frac{h_2}{\sqrt{h_1^2 + h_2^2}}, \quad e_2 = -\frac{h_1}{\sqrt{h_1^2 + h_2^2}}, \quad e_3 = 0.$$

**Remark:** This implies a rational quadratic parametrization of $\mathcal{E}_t$ in dependency of $(h_0 : h_1 : h_2)$. ◇

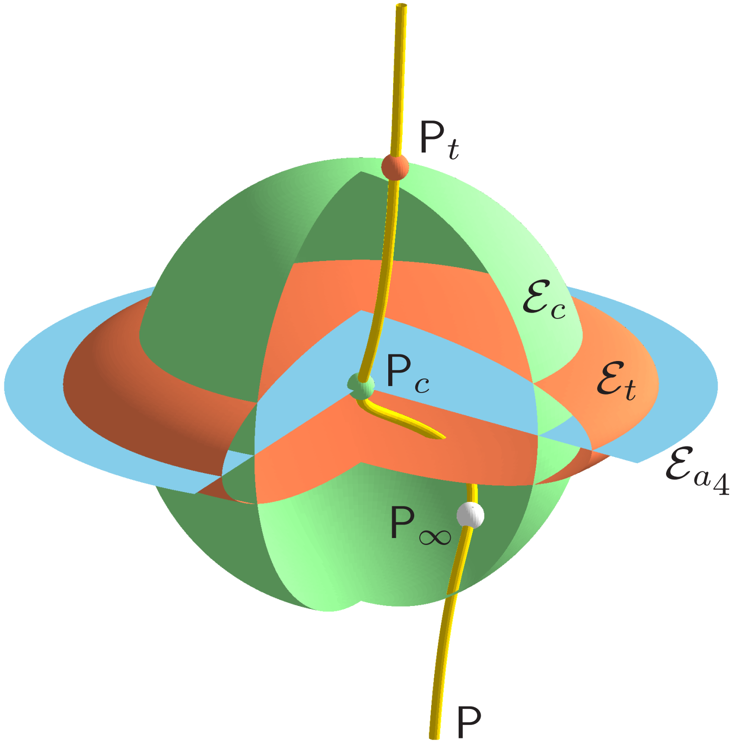



For $h_0 = 1$ the $h_1$- and $h_2$-parameter lines are displayed.

# 5. On the reality of Type 1 & 2 self-motions

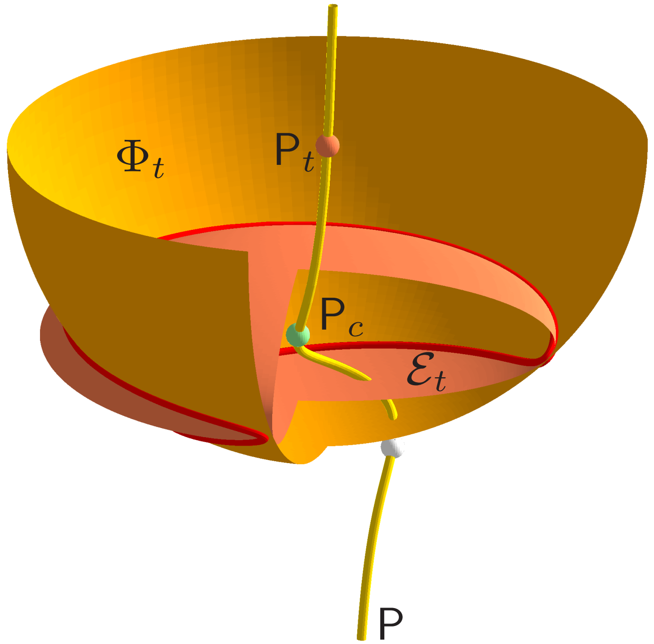
**Remark:** This approach is also valid for the special case $(a_4 = a_r)$ as there always exists a value for $R_1^2$ in dependency of $(h_0 : h_1 : h_2)$ in a way that $\Lambda_1 = 0$ holds. $\diamond$

- For $t \neq a_4$ all $\mathcal{E}_t$ are ellipsoids of rotation, which have the same center point $\mathsf{P}_c \in \mathsf{P}$ and axis c of rotation through W $(= \mathsf{P}_{a_4})$.
  - ⋆ For $a_4 \neq a_r$ the only sphere within the set of ellipsoids is $\mathcal{E}_c$.
  - ⋆ For $a_4 = a_r$ no such sphere exists as $c = \infty$ holds $(\Rightarrow \mathsf{P}_c = \mathsf{M}_5)$.
- $\mathcal{E}_{a_4}$ is a circular disc in the Darboux plane $z = p_4$ centered in $\mathsf{P}_c$.

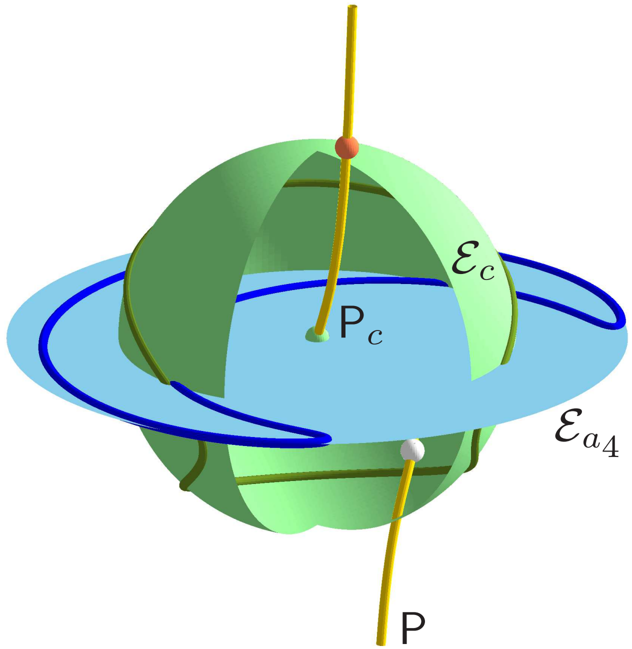


# 5. On the reality of Type 1 & 2 self-motions

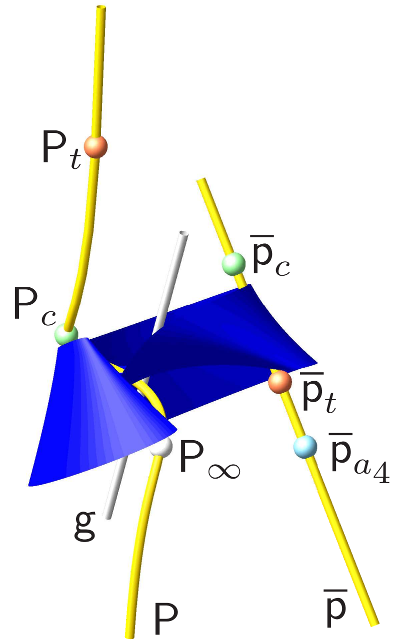
**Remark:** The existence of these ellipsoids was already known to DUPORCQ [7], who used them to show that the spherical trajectories are quartic space curves. ◇




Trajectory of $\mathsf{p}_t$ = intersection curve of $\mathcal{E}_t$ and sphere $\Phi_t$ around $\mathsf{P}_t$.




This intersection procedure fails for the trajectories of $\mathsf{p}_c$ and $\mathsf{p}_{a_4}$.




Quintic basic surface (cf. NAWRATIL [13]).

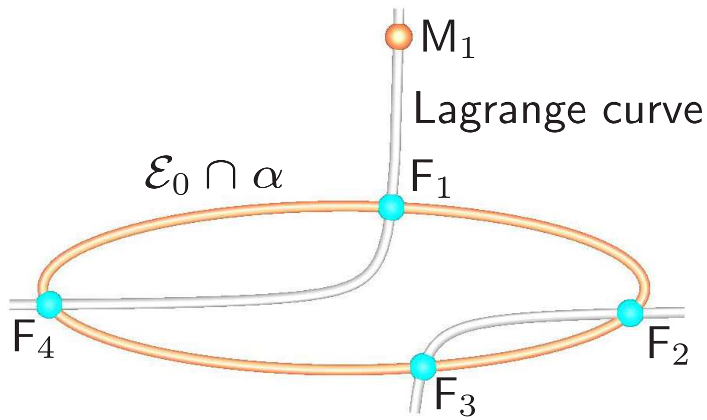
# 5. On the reality of Type 1 & 2 self-motions

Based on this geometric property, recovered by line-symmetric motions, we can formulate the condition for the self-motion to be real as follows:

- $\mathsf{M}_1 \neq \mathsf{P}_c$: We can reduce the problem to a planar one by intersecting the plane $\alpha$ spanned by $\mathsf{M}_1$ $(= \mathsf{P}_0)$ and c with $\mathcal{E}_0$ and the sphere with radius $R_1$ centered in $\mathsf{M}_1$.

  There exists an interval $I_0 = ]I_-, I_+[$ such that for $R_1 \in I_0$ the two resulting conics have at least two distinct real intersection points.
  $\Rightarrow$ real self-motion $\Leftrightarrow R_1 \in I_0$.

- $\mathsf{M}_1 = \mathsf{P}_c$: The interval collapses to the single value $R_1 = |a_4|$.




The limits $I_-$ and $I_+$ can be computed explicitly. $\mathsf{F}_1, \ldots, \mathsf{F}_4$ are the pedal points of the ellipse w.r.t. $\mathsf{M}_1$.

# 5. On the reality of Type 1 & 2 self-motions

⇒ Any LP of Type 1 & 2 has real self-motions if leg-parameters are chosen properly.

**Result of [1].**
LPs with self-motions have at least a quartically solvable direct kinematics.

It is possible to use this advantage (closed form solution) of LPs with self-motions without any risk, by designing LPs of Type 1 & 2, which are guaranteed free of self-motions within their workspace.

A sufficient condition for that is that (at least) for one of the five legs $\mathsf{p}_t\mathsf{P}_t$ of the LP the corresponding reality interval $I_t$ is disjoint with the interval of the maximal and minimal leg length implied by the mechanical realization.
This condition for a self-motion free workspace gets very simple if $\mathsf{p}_c\mathsf{P}_c$ is this leg.

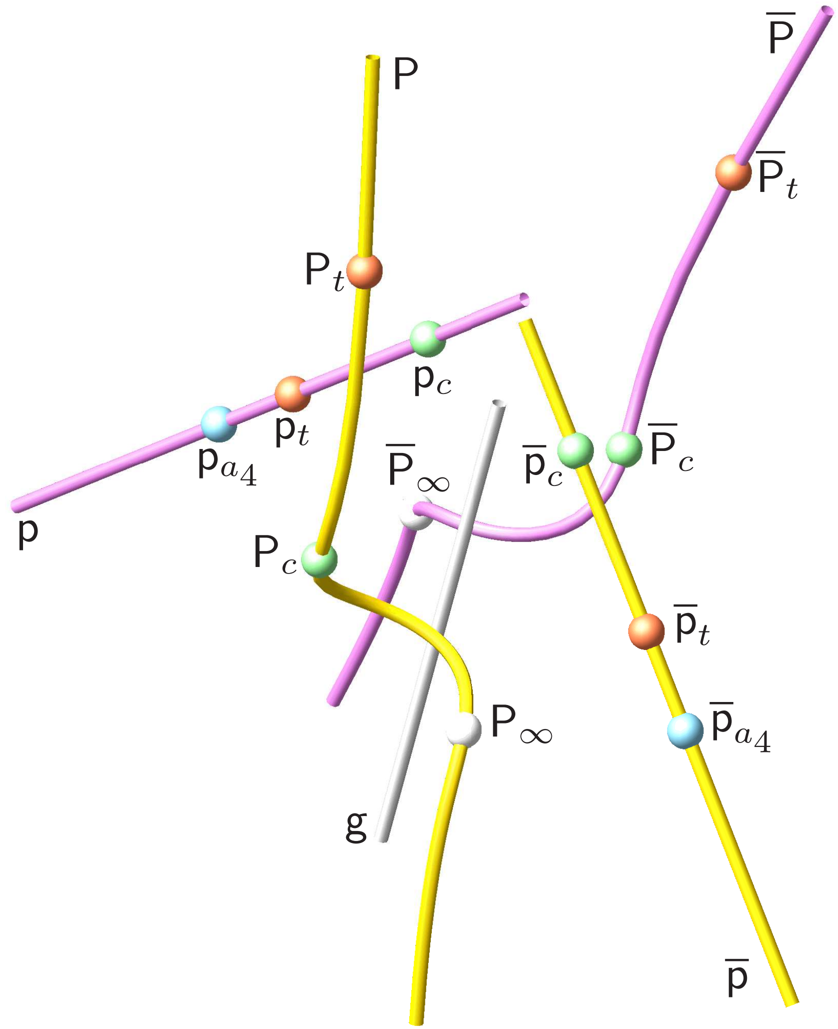
# 6. Krames's construction and open problem

Assume that p is in an arbitrary pose of the self-motion $\mu$, where g denotes the corresponding generator of the basic surface.

Moreover $\overline{\mathsf{p}}$ and $\overline{\mathsf{P}}$ are obtained by the reflexion of p and P with respect to g.

During $\mu$ the points of $\overline{\mathsf{P}}$ are located on spheres with centers on the line $\overline{\mathsf{p}}$ (cf. KRAMES [4]).

**Remark:** A general point of the moving system (as well as one of the cubic $\overline{\mathsf{P}}$) has a trajectory of degree 6 (cf. NAWRATIL [13]). ◇



# 6. Krames's construction and open problem

Krames's construction yields for each line-symmetric motion of the Main Theorem, a new solutions for the Borel Bricard problem, with the exception of one special case where $W \in \overline{p}$ holds, which was already given by Borel [2].

**Remark:** For this special case Borel noted that beside p and $\overline{P}$ only two imaginary planar cubic curves ($\in$ isotropic planes through p) run on spheres. This also holds true for a general example (cf. Nawratil [13]). ◇

**Open problem:** Determine all line-symmetric motions of the Main Theorem where additional real points (beside those of p and $\overline{P}$) run on spheres.
Until now the only known examples with this property are the Borel-Bricard II motions (cf. Hartmann [9], Krames [10]).

**Remark:** References refer to the list of publications given in the presented paper. ◇