Flexible octahedra, their generalization and application

Georg Nawratil

Institute of Discrete Mathematics and Geometry

Research was supported by FWF (I 408-N13)

Bricard octahedra

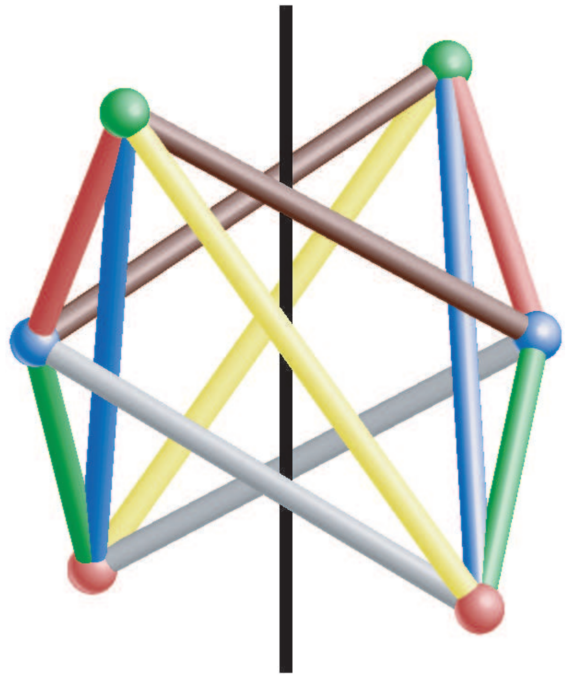
An octahedron is called flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e. every face remains congruent to itself during the flex.

All flexible octahedra in E^3, where no two faces coincide permanently during the flex, were firstly determined by BRICARD [1].

There are 3 types of these so-called Bricard octahedra:

Bricard octahedra of type I

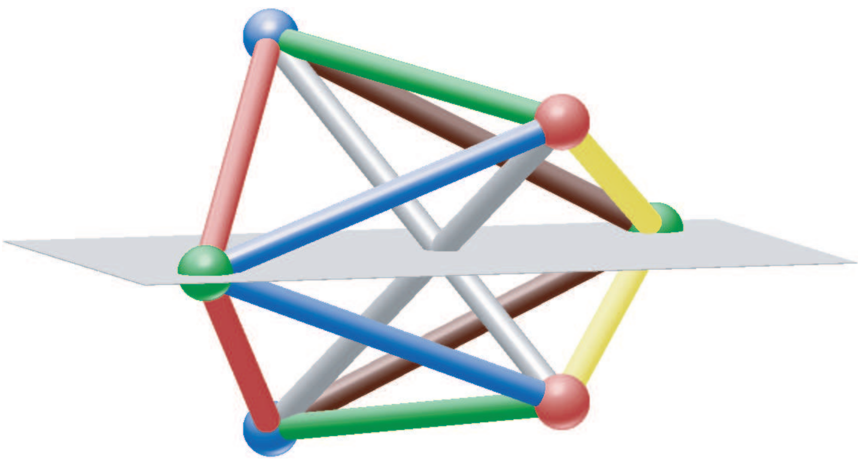
All three pairs of opposite vertices are symmetric with respect to a line.



Bricard octahedra

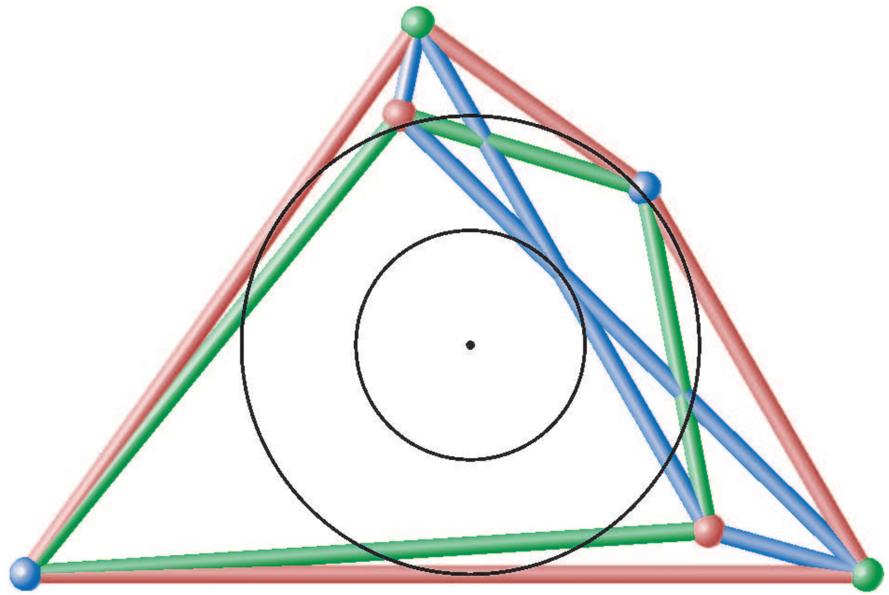
Bricard octahedra of type II

Two pairs of opposite vertices are symmetric with respect to a plane through the remaining two vertices.



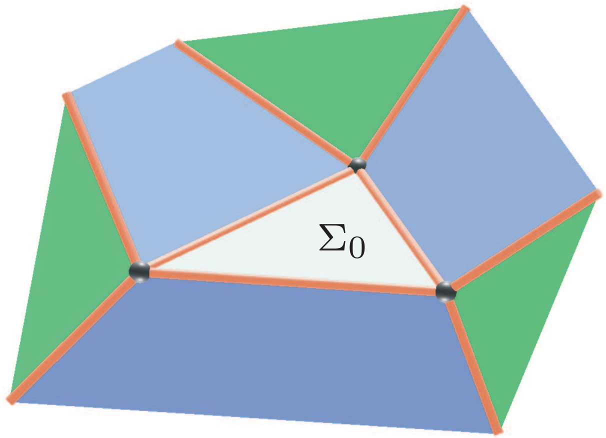
Bricard octahedra of type III

These octahedra possess two flat poses and can be constructed as follows:



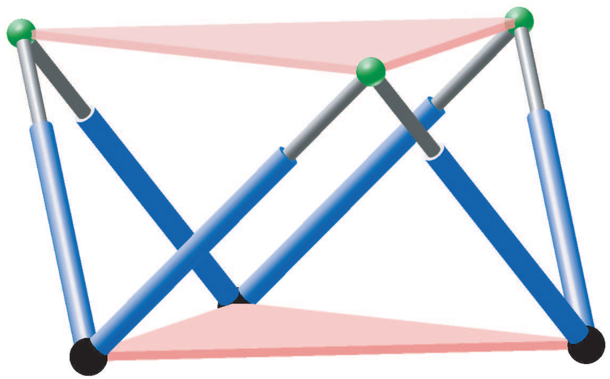
Different points of view

Kokotsakis mesh



A Kokotsakis mesh is a polyhedral structure consisting of a n-sided central polygon Σ_0 surrounded by a belt of polygons.

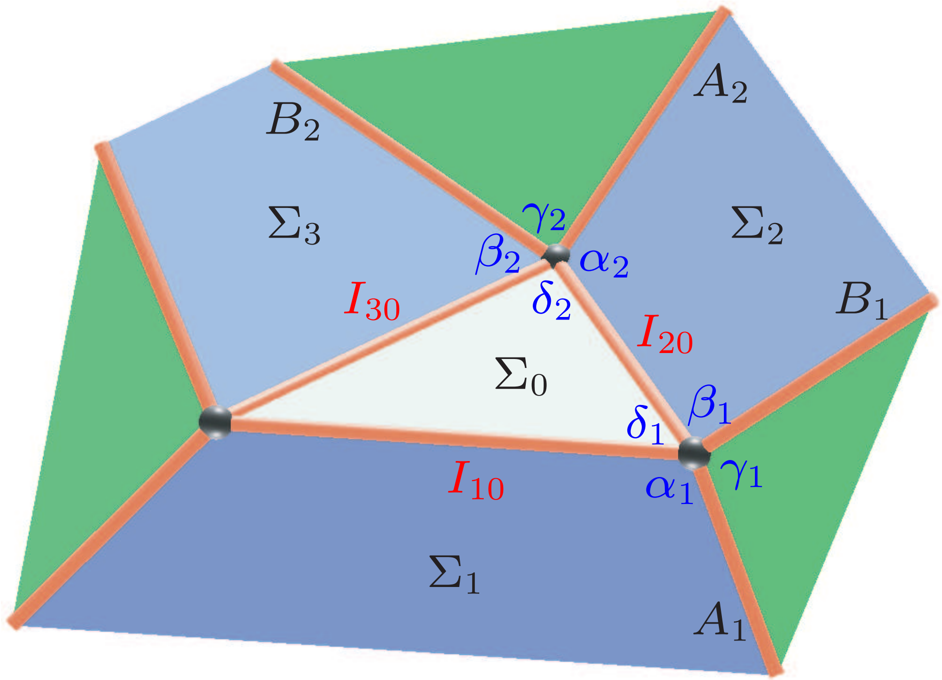
Stewart Gough platform

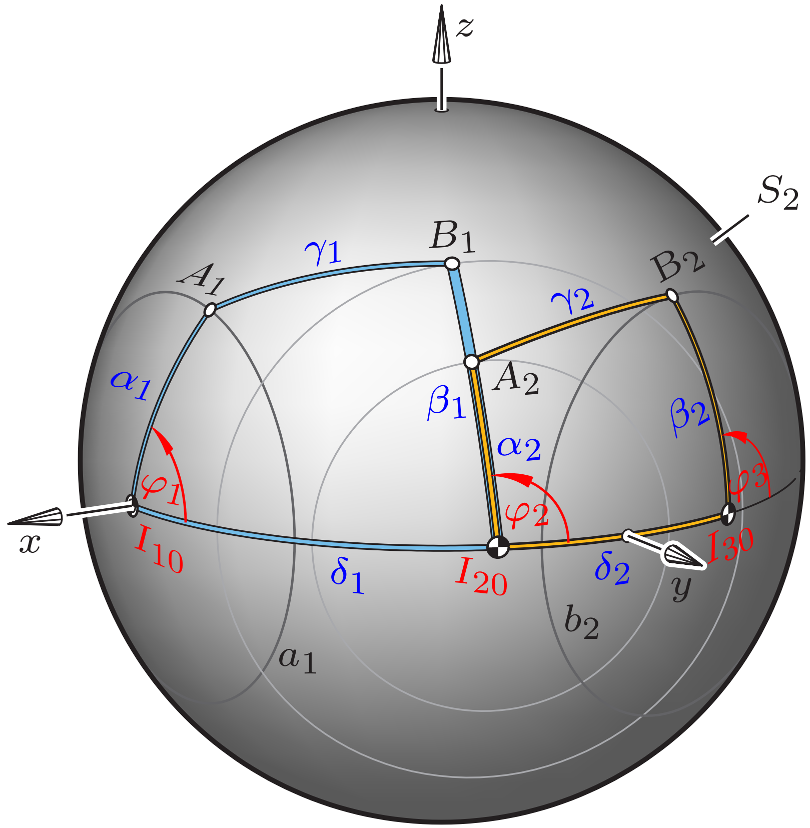


A SGP is a parallel manipulator where the platform is connected via three Spherical-Prismatical-Spherical (S<u>P</u>S) legs with the base.

Table of contents

$[1a]$ Motivation





The relative motions Σ_{i+1}/Σ_i between consecutive systems are spherical four-bars.

$[1a]$ Transmission by a spherical four-bar mechanism

Under consideration of $t_i = \tan(\varphi_i/2)$ the transmission $\varphi_1 \mapsto \varphi_2$ can be written according to STACHEL [2] as follows:

$$\mathsf{C}: \quad \mathsf{c}_{22}\mathsf{t}_1^2\mathsf{t}_2^2+\mathsf{c}_{20}\mathsf{t}_1^2+\mathsf{c}_{02}\mathsf{t}_2^2+\mathsf{c}_{11}\mathsf{t}_1\mathsf{t}_2+\mathsf{c}_{00}=0$$

$$c_{11} = 4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1 \neq 0,$$

$$c_{00} = N_1 - K_1 + L_1 + M_1,$$

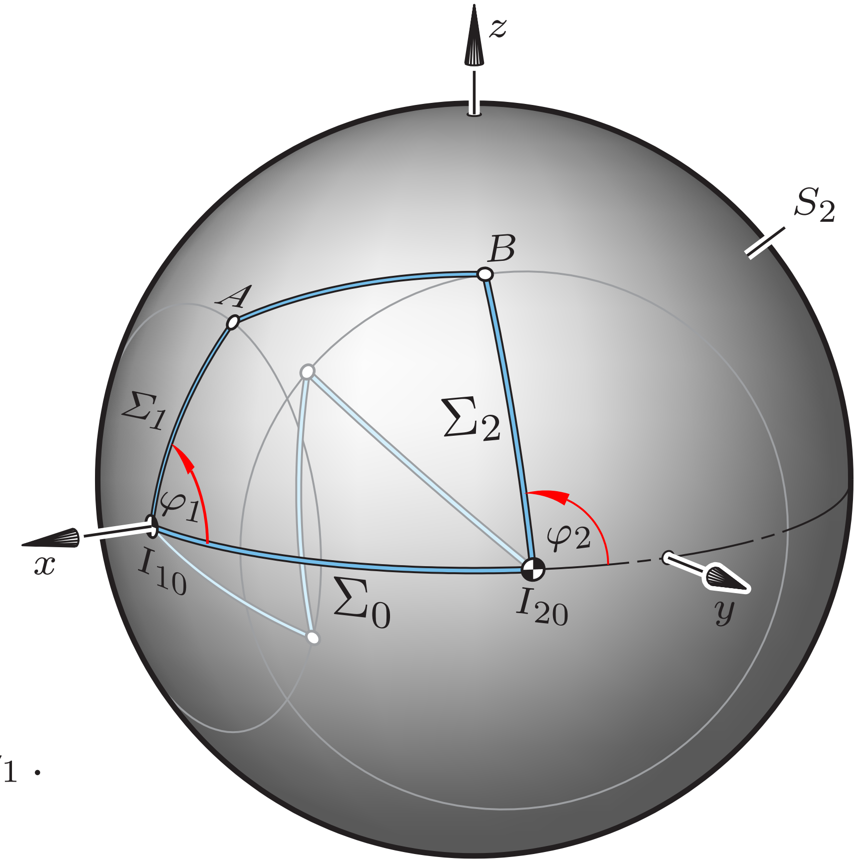
$$c_{02} = N_1 + K_1 + L_1 - M_1,$$

$$c_{20} = N_1 - K_1 - L_1 - M_1,$$

$$c_{22} = N_1 + K_1 - L_1 + M_1,$$

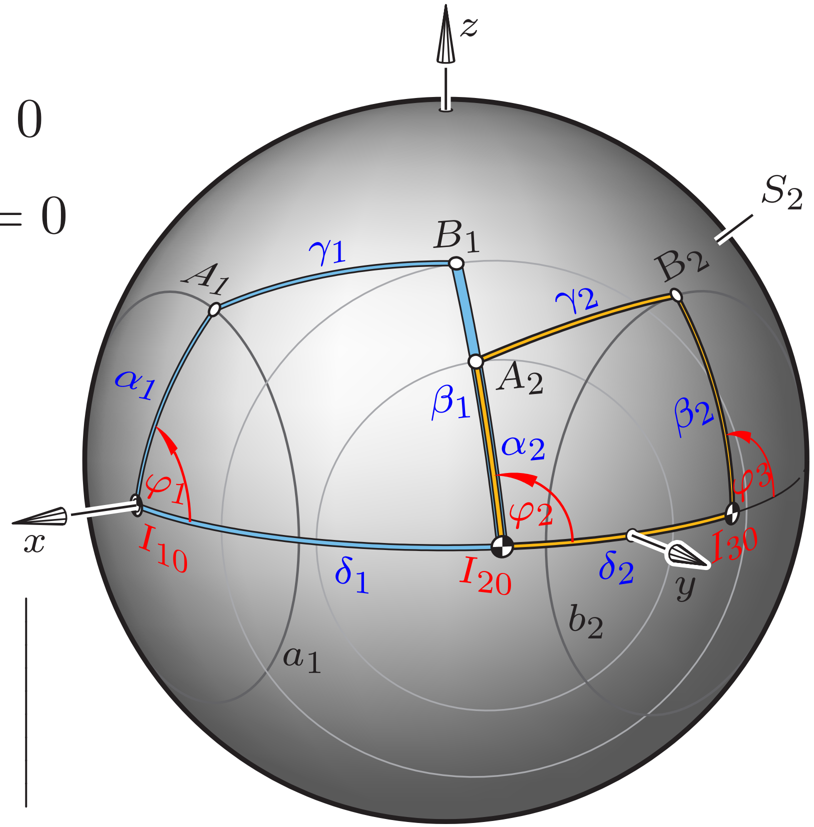
$$K_1 = \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1\,, \qquad M_1 = \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1\,,$$

$$L_1 = \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1\,, \qquad N_1 = \mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\gamma_1\,.$$



$[1a]$ Composition of two spherical four-bar linkages

The transmission between the angles φ_1, φ_2 and φ_3 can be expressed by the two biquadratic equations:



$$\text{C}: \quad c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

$$\text{D}: \quad d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0$$

We eliminate t_2 by computing the resultant of C and D with respect to t_2. This yields a biquartic equation $X = 0$ where X equals:

$$\begin{vmatrix} c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} & 0 \\ 0 & c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} \\ d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00} & 0 \\ 0 & d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00} \end{vmatrix}$$

$[1a]$ Reducible compositions

In the following we are interested in the conditions the c_{ij}'s and d_{ij}'s have to fulfill such that X splits up into the product FG.

If at least one of the factors F or G corresponds to the transmission function of a spherical coupler, i.e. for example

$$\mathsf{F}: \quad \mathsf{f}_{22}\mathsf{t}_1^2\mathsf{t}_3^2 + \mathsf{f}_{20}\mathsf{t}_1^2 + \mathsf{f}_{02}\mathsf{t}_3^2 + \mathsf{f}_{11}\mathsf{t}_1\mathsf{t}_3 + \mathsf{f}_{00} = 0 \quad \text{with} \quad \mathsf{f}_{11} \neq 0,$$

we get a reducible composition with a spherical coupler component.

We denote the coefficients of $t_1^i t_3^j$ of $Y :=$ FG and X by Y_{ij} and X_{ij}. By the comparison of these coefficients we get the 13 equations $Y_{ij} - X_{ij} = 0$ with

$$(i,j) \in \{(4,4),(4,2),(4,0),(3,3),(3,1),(2,4),(2,2),(2,0),(1,3),(1,1),(0,4),(0,2),(0,0)\}.$$

This non-linear system of equations was solved explicitly by the resultant method.

Theorem 1 NAWRATIL [3]
If a reducible composition of two spherical four-bar linkages with a spherical coupler component is given, then it is one of the following cases:

(a) One of the following four cases hold:

$$c_{00} = c_{22} = 0, \quad d_{00} = d_{22} = 0, \quad c_{20} = c_{02} = 0, \quad d_{20} = d_{02} = 0,$$

(b) The following algebraic conditions hold for $\lambda \in \mathbb{R} \setminus \{0\}$:

$$c_{00}c_{20} = \lambda d_{00}d_{02}, \quad c_{22}c_{02} = \lambda d_{22}d_{20},$$

$$c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = \lambda[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})],$$

(c) One of the following two cases hold:

$$c_{22} = c_{02} = d_{00} = d_{02} = 0, \quad d_{22} = d_{20} = c_{00} = c_{20} = 0,$$

(d) One of the following two cases hold for $A \in \mathbb{R} \setminus \{0\}$ and $B \in \mathbb{R}$:

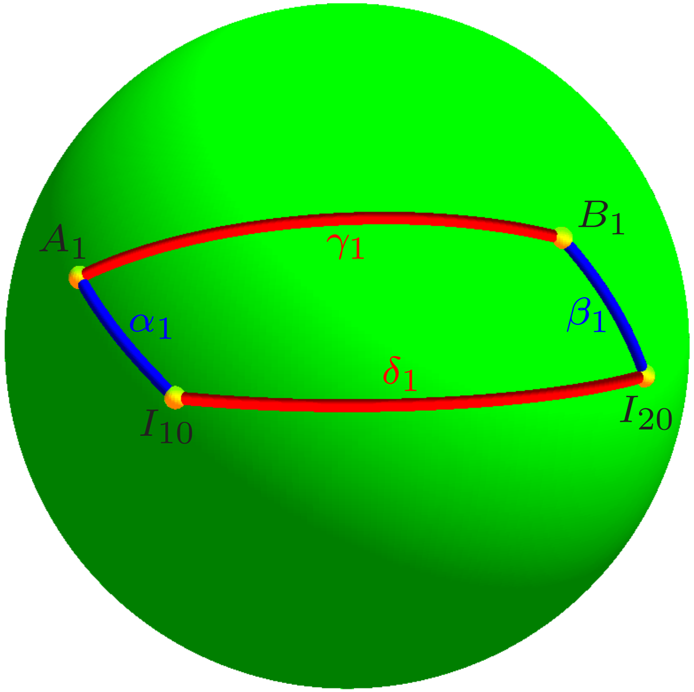
- $c_{20} = Ad_{02},\; c_{22} = Ad_{22},\; c_{02} = Bd_{22},\; c_{00} = Bd_{02},\; d_{00} = d_{20} = 0,$
- $d_{02} = Ac_{20},\; d_{22} = Ac_{22},\; d_{20} = Bc_{22},\; d_{00} = Bc_{20},\; c_{00} = c_{02} = 0.$

$[1a]$ Geometric aspects of Theorem 1(a)

Spherical isogram

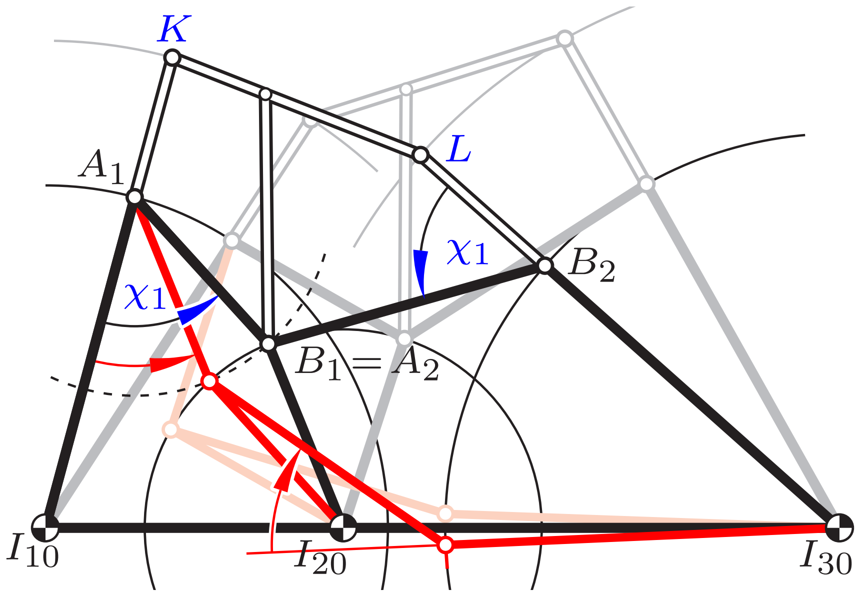
(i) $c_{00} = c_{22} = 0 \iff \beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$, which determine a spherical isogram.

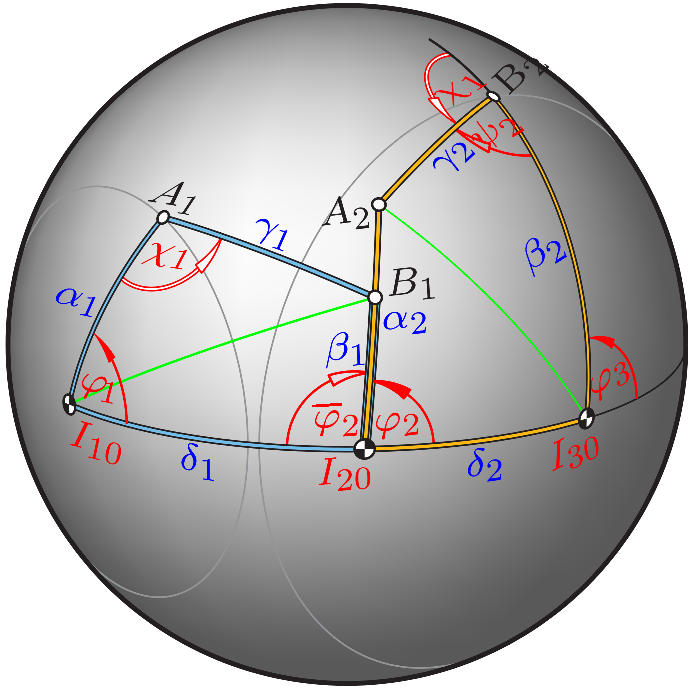
(ii) $c_{20} = c_{02} = 0 \iff \beta_1 = \pi - \alpha_1$ and $\delta_1 = \pi - \gamma_1$. Note that the couplers of both isograms have the same movement because we get item (ii) by replacing I_{20} of item (i) by its antipode.



Remark 1
The cosines of opposite angles in the spherical isograms (of both types) are equal.

$[1a]$ Geometric aspects of Theorem 1(b)





Burmester's focal mechanism:

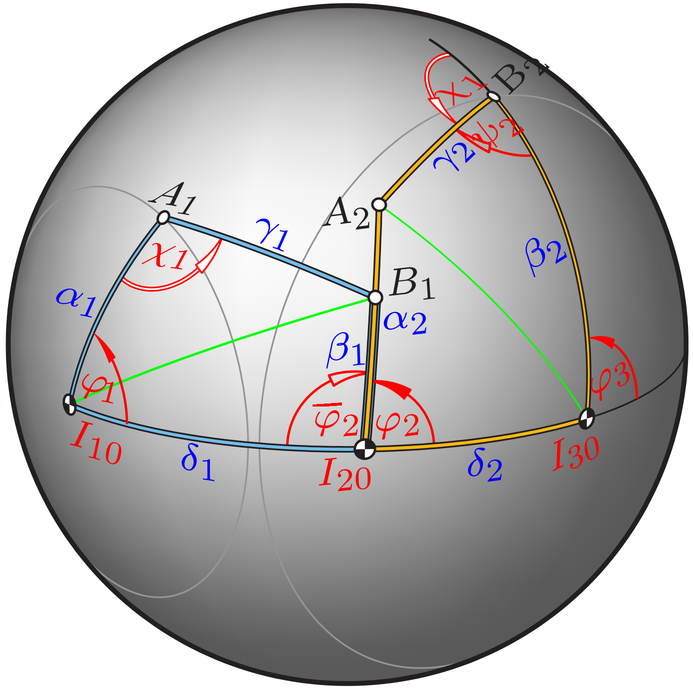
A planar composition of two four-bars with a planar coupler component. Due to WUNDERLICH [4] this composition is characterized by Dixon's angle condition.

$[1a]$ Geometric aspects of Theorem 1(b)

Spherical mechanism of Dixon type

(i) In NAWRATIL & STACHEL [5] it was shown that the algebraic characterization of item (b) is equivalent with Dixon's angle condition. Therefore $c\chi_1 = -c\psi_2$ holds with $\chi_1 = \sphericalangle I_{10}A_1B_1$ and $\psi_2 = \sphericalangle I_{30}B_2A_2$.

(ii) We get the case $c\chi_1 = c\psi_2$ from item (i) by replacing either I_{30} or I_{10} by its antipode.



$[1a]$ Geometric aspects of Theorem 1(c)

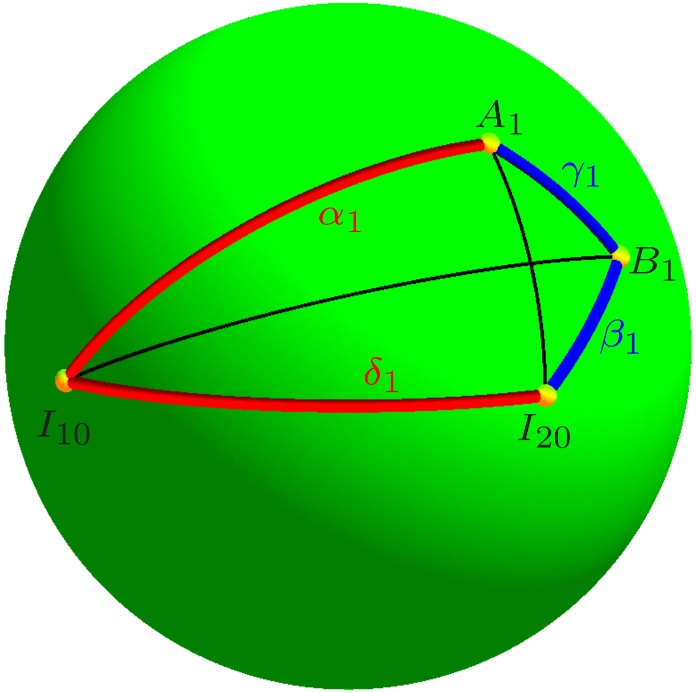
Spherical deltoid

- $\star$ $c_{00} = c_{02} = 0 \iff \alpha_1 = \delta_1$ and $\beta_1 = \gamma_1$

  By replacing I_{20} by its antipode, we get the corresponding mechanism with:
  $\delta_1 = \pi - \alpha_1,\ \beta_1 = \pi - \gamma_1 \iff c_{22} = c_{20} = 0$

- $\star$ $c_{22} = c_{02} = 0 \iff \alpha_1 = \gamma_1$ and $\beta_1 = \delta_1$

  By replacing I_{10} by its antipode, we get the corresponding mechanism with:
  $\alpha_1 = \pi - \gamma_1,\ \delta_1 = \pi - \beta_1 \iff c_{00} = c_{20} = 0$



Remark 2
The cosines of one pair of opposite angles in spherical deltoids are equal.

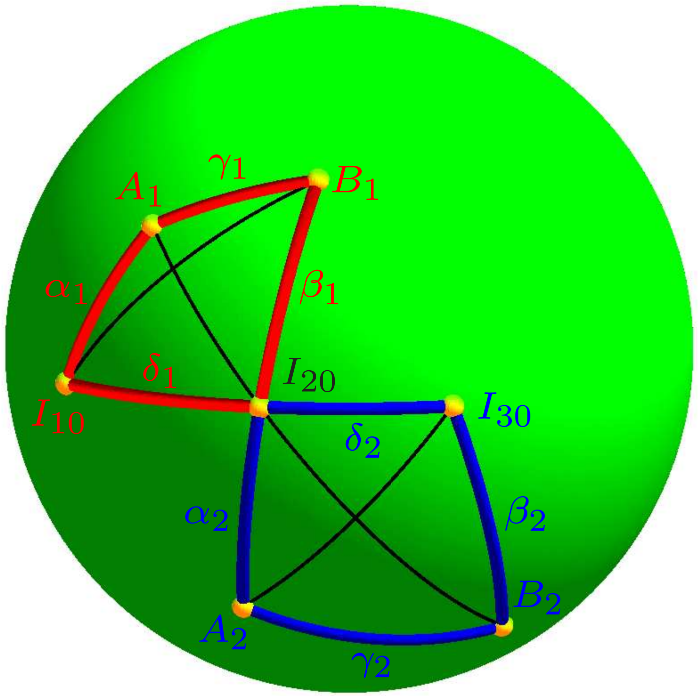
$[1a]$ Geometric aspects of Theorem 1(d)

Orthogonal spherical four-bars

Both couplers are so-called orthogonal spherical four-bar mechanisms (cf. STACHEL [2]), as the diagonals of the spherical quadrangles are orthogonal.

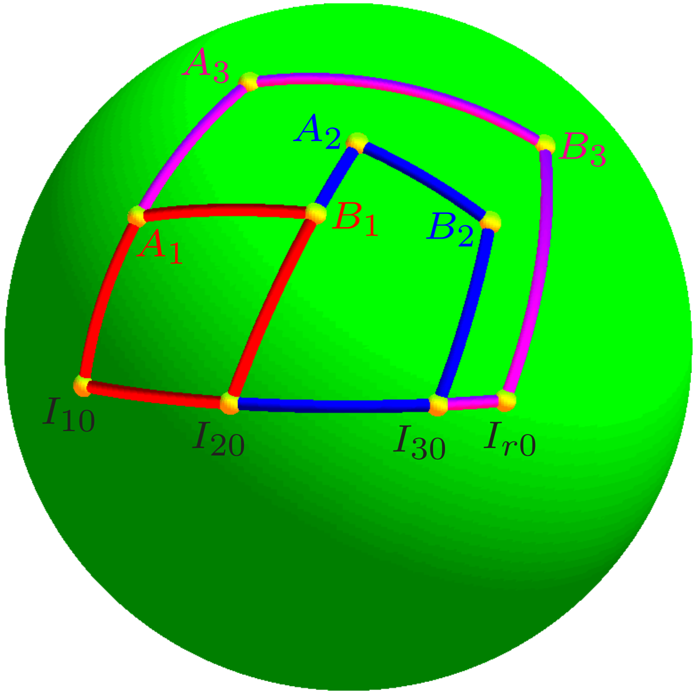
Moreover, the diagonals $A_1 I_{20}$ and $I_{20} B_2$ coincide.

Remark 3
Especially, all spherical deltoid are orthogonal.



$[1b]$ The closure condition

The Kokotsakis mesh for $n = 3$ is flexible if and only if the transmission of the composition of the two spherical four-bar linkages $\mathcal{C}$ and $\mathcal{D}$ equals the one of the single spherical four-bar linkage $\mathcal{R}$ $(= I_{10}I_{r0}B_3A_3)$ which meets the closure condition $I_{r0} = I_{30}$.



Octahedra where no pair of opposite vertices are ideal points possess at least one finite face. We can assume w.l.o.g. that this face is the central polygon of the Kokotsakis mesh.

$[1b]$ No opposite vertices as ideal points

The closure condition $I_{r0} = I_{30}$ can only be fulfilled by spherical mechanisms of Dixon type (ii) and by spherical isograms.

Theorem 2 Nawratil [6]
If an octahedron in the projective extension of E^3 is flexible, where no pair of opposite vertices are ideal points, then its spherical image is a composition of spherical four-bar linkages $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{R}$ of the following type:

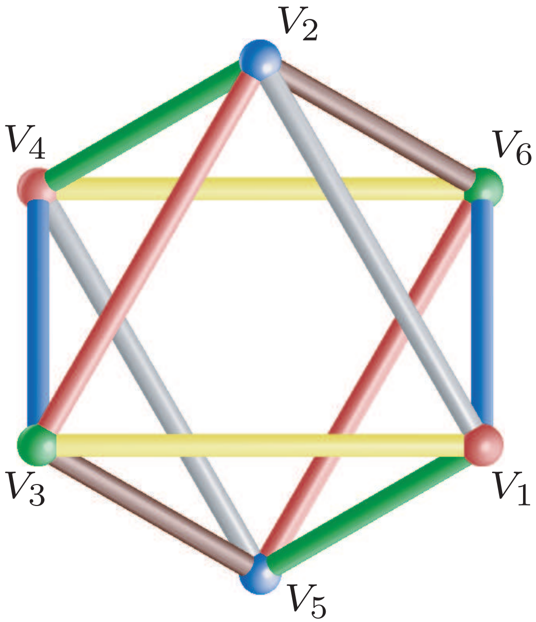
A. $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{R}$ as well as $\mathcal{D}$ and $\mathcal{R}$ form a spherical mechanism of Dixon type (ii),

B. $\mathcal{C}$ and $\mathcal{D}$ form a spherical mechanism of Dixon type (ii) and $\mathcal{R}$ is a spherical isogram,

C. $\mathcal{C}$, $\mathcal{D}$ ($\Rightarrow$ and $\mathcal{R}$) are spherical isograms.

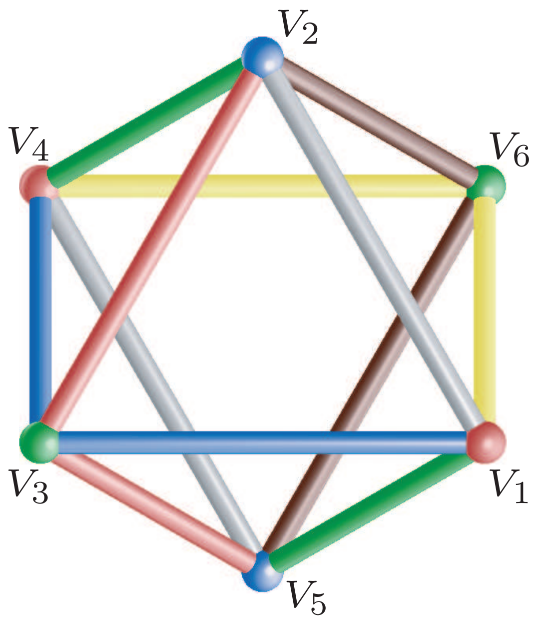
$[1b]$ All vertices are finite

Type A



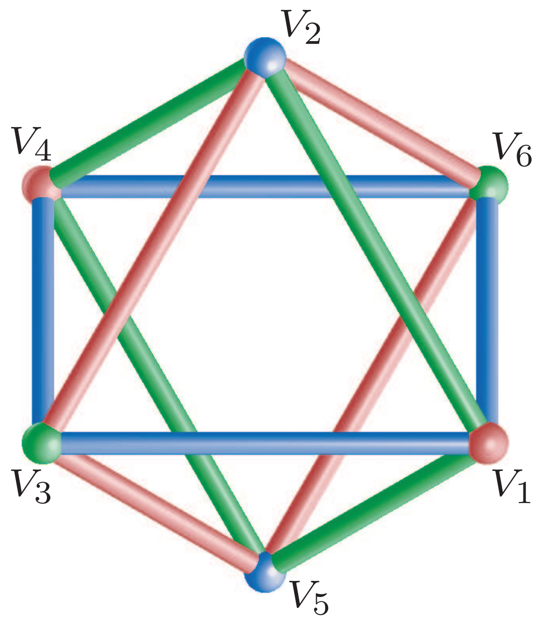
Bricard octahedron I
(cf. KOKOTSAKIS [7])

Type B



Bricard octahedron II
(cf. KOKOTSAKIS [7])

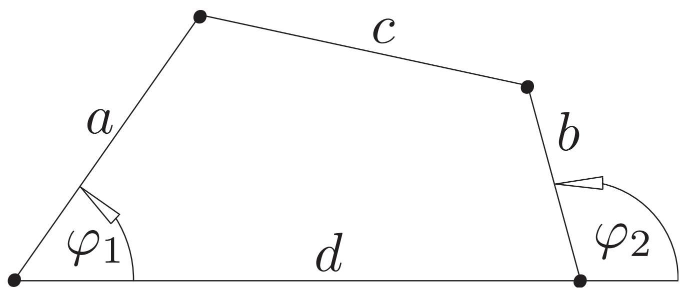
Type C



Bricard octahedron III
(cf. KOKOTSAKIS [7])

$[1b]$ Central Triangles with one ideal point

The four faces of the octahedra through the ideal point form a 4-sides prism. Under consideration of $t_i = \tan(\varphi_i/2)$, the input angle φ_1 and the output angle φ_2 of a planar four-bar linkage (= orthogonal cross section of prism) are related by:

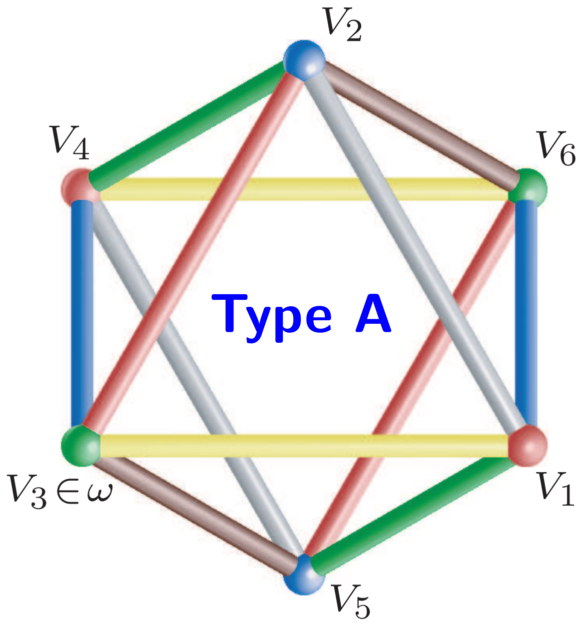


$$p_{22}t_1^2t_2^2+p_{20}t_1^2+p_{02}t_2^2+p_{11}t_1t_2+p_{00} = 0$$

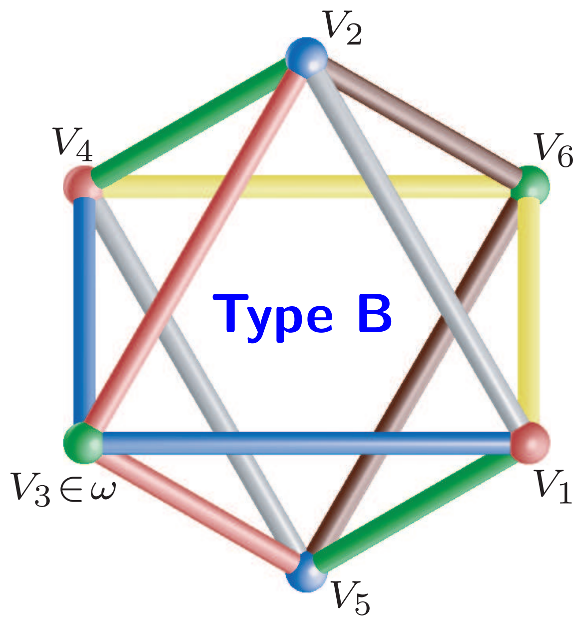
$$p_{11} = -8ab \neq 0,$$
$$p_{22} = (a-b+c+d)(a-b-c+d),$$
$$p_{20} = (a+b+c+d)(a+b-c+d),$$
$$p_{02} = (a+b+c-d)(a+b-c-d),$$
$$p_{00} = (a-b+c-d)(a-b-c-d).$$

Lemma 1 Nawratil [6]
If a reducible composition of a planar and a spherical four-bar linkage with a spherical coupler component is given, then the same conditions as in Theorem 1 are fulfilled.

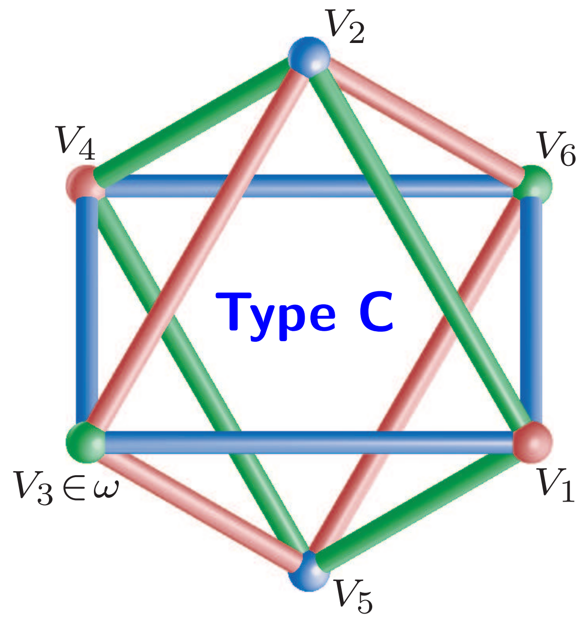
$[1b]$ One vertex is an ideal point



Thm 3 NAWRATIL [6]
There do not exist flexible octahedra of type A where only one vertex is an ideal point.



Thm 4 NAWRATIL [6]
Bricard octahedron II where one vertex located in the plane of symmetry is an ideal point.



Thm 5 NAWRATIL [6]
Bricard octahedron III where one vertex is an ideal point (see also STACHEL [8]).

$[1b]$ Flexible octahedra with edge or face at infinity

Do there exist flexible octahedra with a finite face Σ_0, where one edge or one face is at infinity?

Theorem 6 NAWRATIL [9]
There do not exist flexible octahedron of type C with a finite face Σ_0 and one edge or face at infinity.

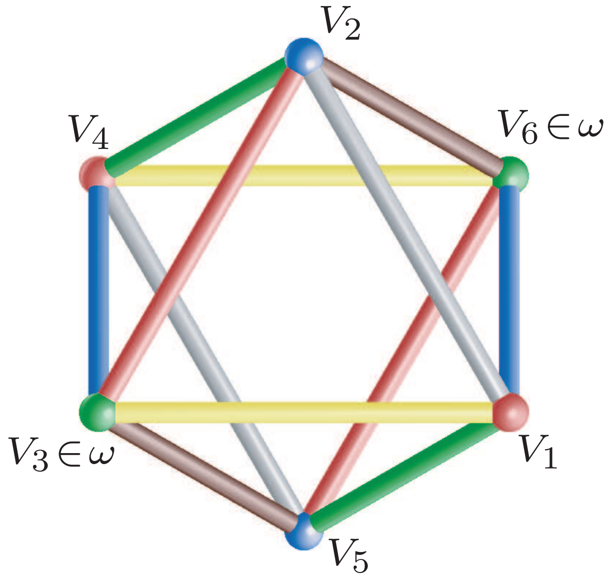
Based on Theorem 3 we can even generalize this result as follows:

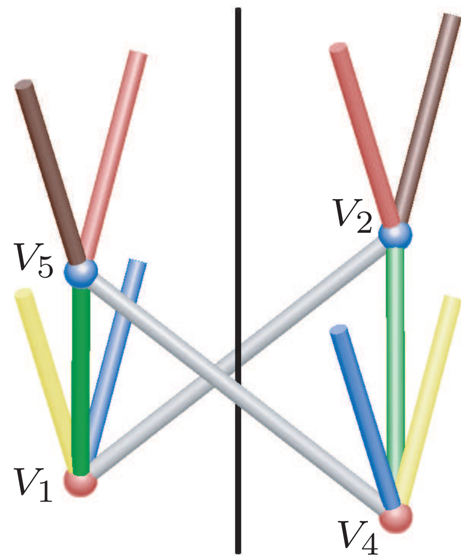
Theorem 7 NAWRATIL [9]
In the projective extension of E^3 there do not exist flexible octahedra with a finite face Σ_0 and one edge or face at infinity.

$[1c]$ One pair of opposite vertices are ideal points

Theorem 8 NAWRATIL [9]
The two pairs of opposite vertices (V_1, V_4) and (V_2, V_5) are symmetric with respect to a common line as well as the edges of the prisms through the ideal points V_3 and V_6, respectively.

Type A

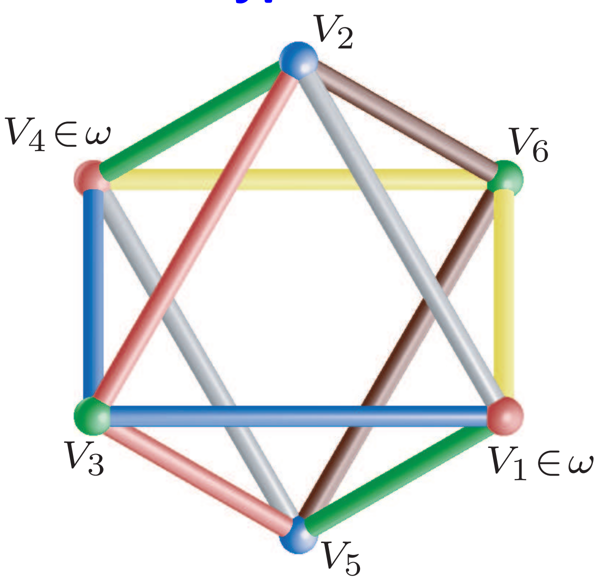


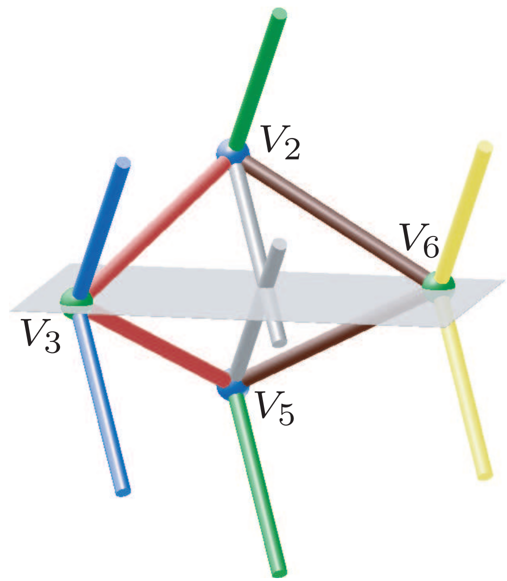


$[1c]$ One pair of opposite vertices are ideal points

Theorem 9 Nawratil [9]
One pair of opposite vertices (V_2, V_5) is symmetric with respect to a plane, which contains the vertices (V_3, V_6). Moreover, also the edges of the prisms through the ideal points V_1 and V_4 are symmetric with respect to this plane.

Type B

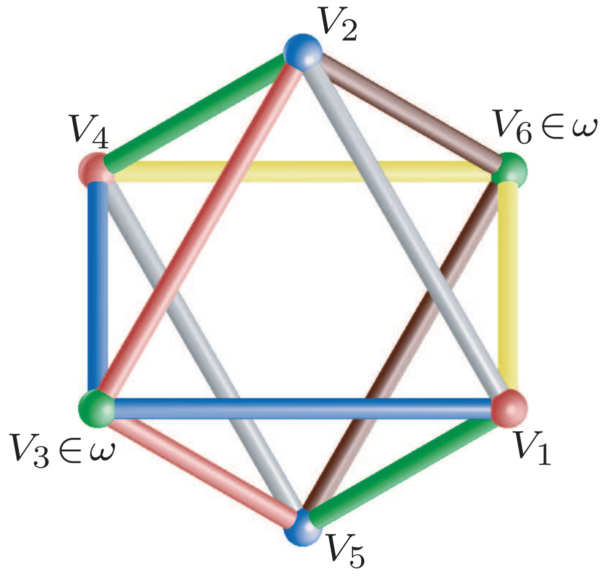


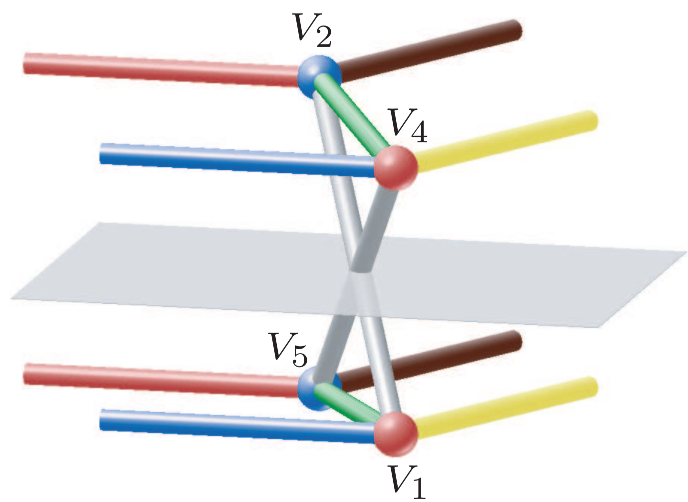


$[1c]$ One pair of opposite vertices are ideal points

Theorem 10 Nawratil [9]
The vertices V_1, V_2, V_4, V_5 are coplanar and form an antiparallelogram and its plane of symmetry is parallel to the edges of the prisms through the ideal points V_3 and V_6, respectively.

Type B

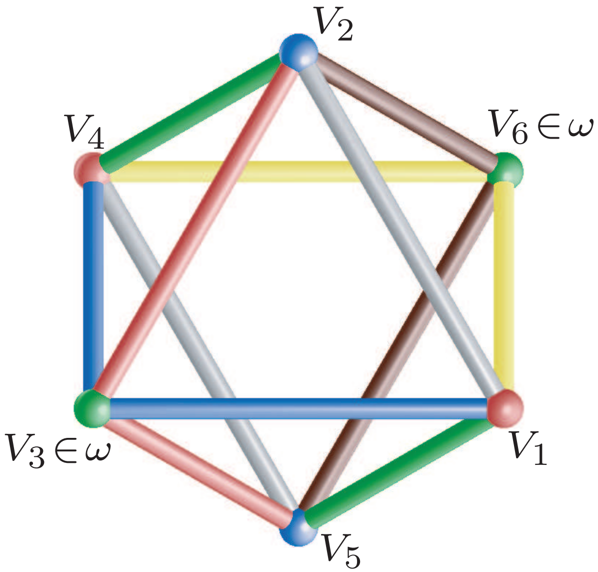


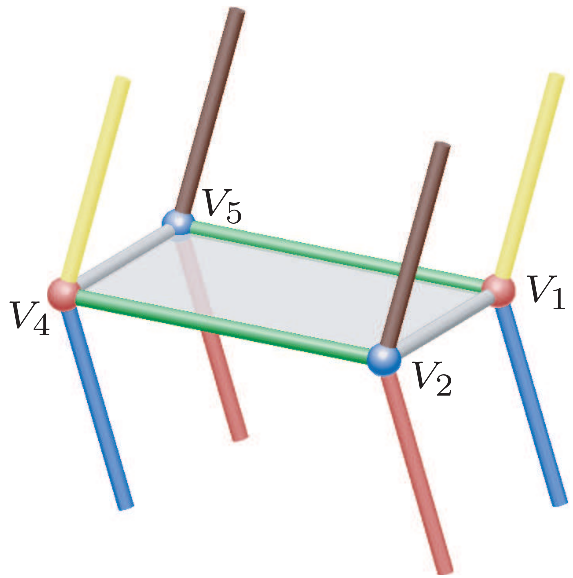


$[1c]$ One pair of opposite vertices are ideal points

Theorem 11 NAWRATIL [9]
The vertices V_1, V_2, V_4, V_5 are coplanar and form a parallelogram. The ideal points V_3 and V_6 can be chosen arbitrarily.

Type B

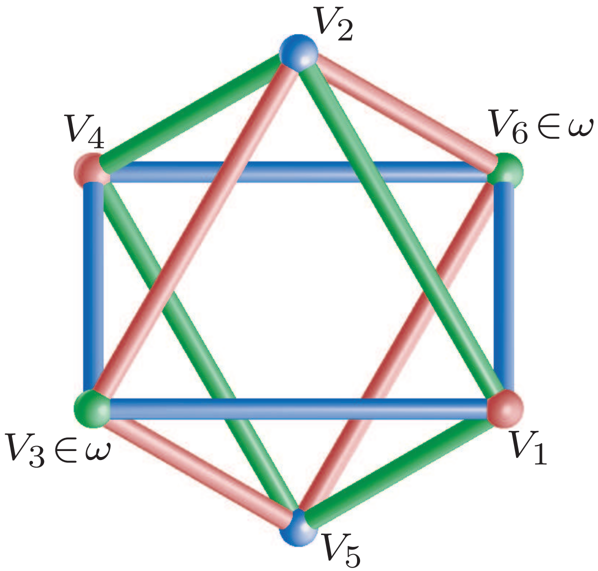


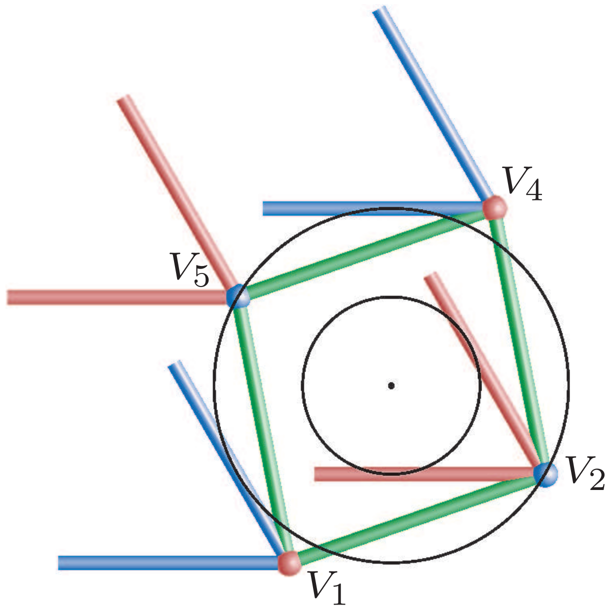


$[1c]$ One pair of opposite vertices are ideal points

Theorem 12 NAWRATIL [9]
This type is characterized by the existence of two flat poses and consists of two prisms through the ideal points V_3 and V_6, where the orthogonal cross sections are congruent antiparallelograms. The construction can be done as follows:

Type C

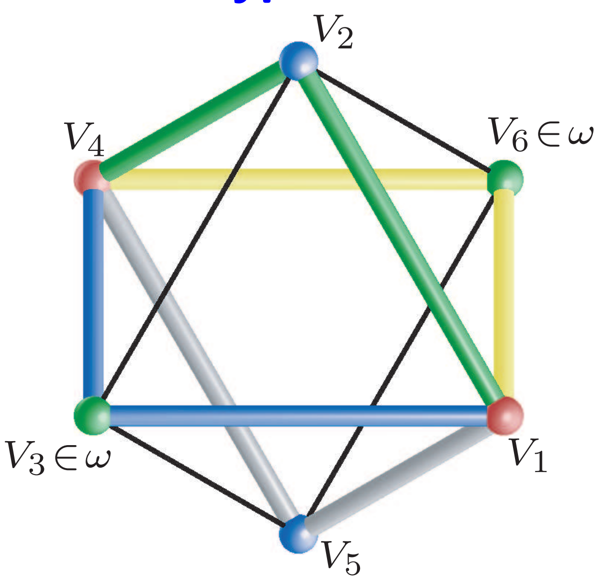


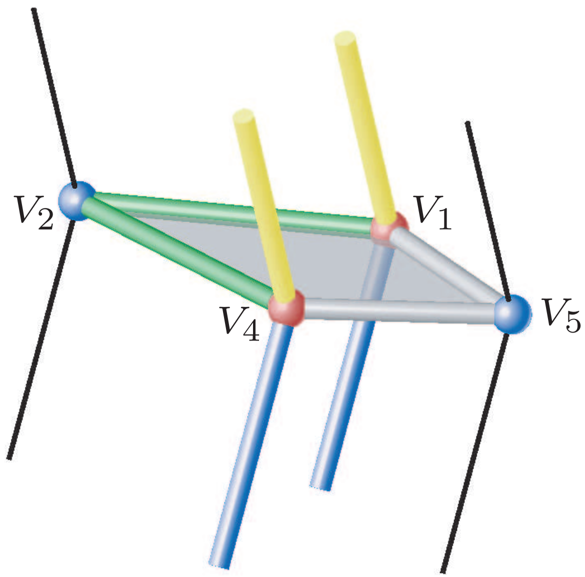


$[1c]$ One pair of opposite vertices are ideal points

Theorem 13 NAWRATIL [9]
The vertices V_1, V_2, V_4, V_5 are coplanar and form a deltoid and the edges of the prisms through the ideal points V_3 and V_6 are orthogonal to the deltoid's line of symmetry.

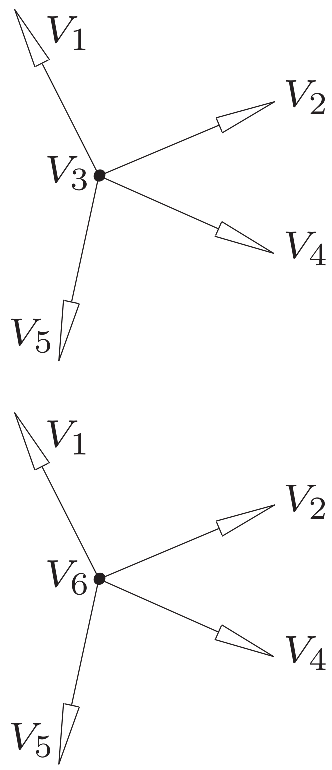
Type D

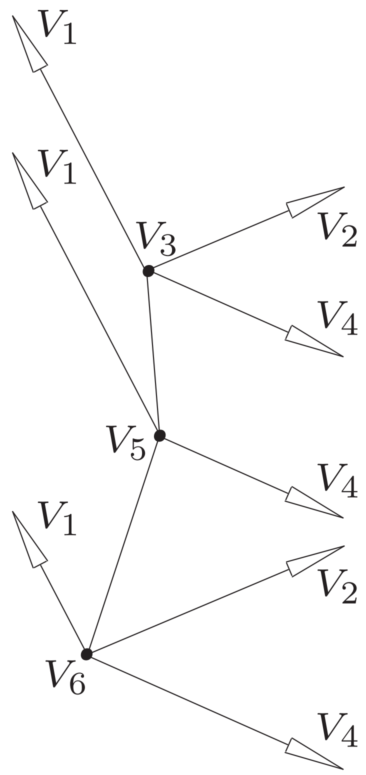




$[1d]$ Special cases

Theorem 14 NAWRATIL [9]
In the projective extension of E^3 any octahedron is flexible where at least two edges are ideal lines but no face coincides with the plane at infinity.





There are only two types of octahedra fulfilling the requirements of theorem 14.

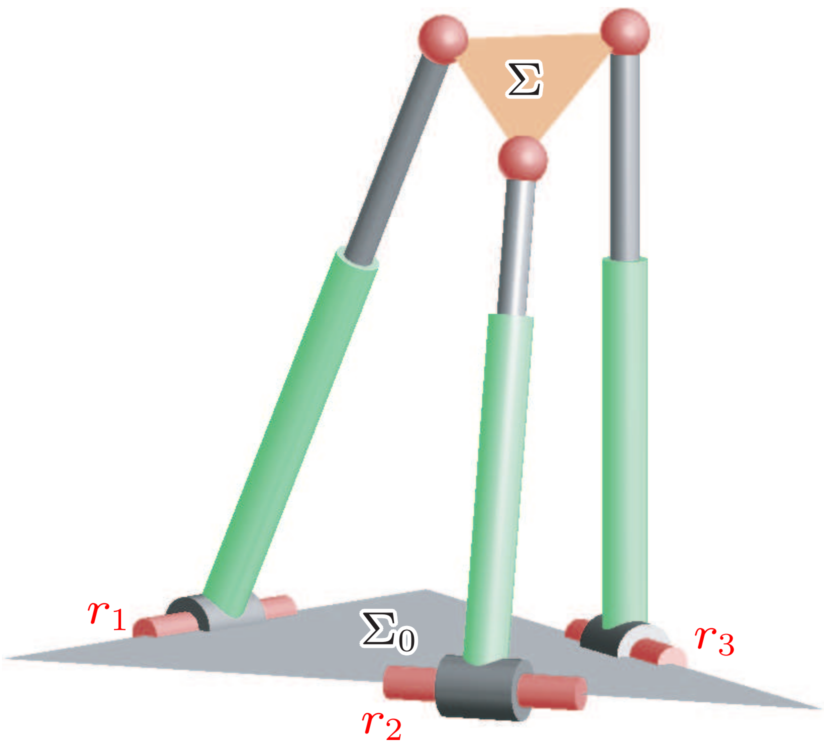
$[1d]$ Application in robotics

A TSSM consists of a platform Σ, which is connected via three S<u>P</u>R legs with the base Σ_0, where the axes r_i of the R-joints are coplanar.

Following was shown in NAWRATIL [10]:

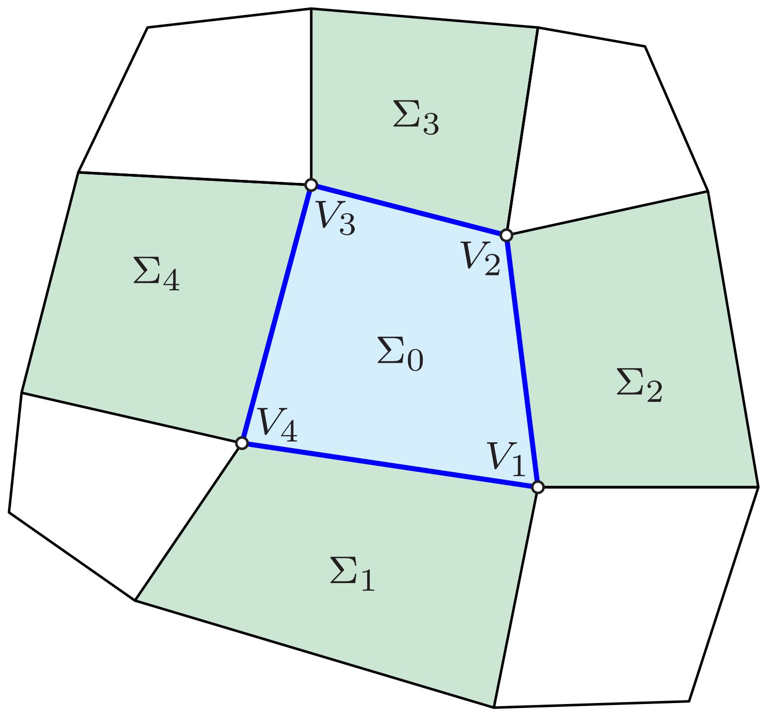
Self-motions of TSSMs can only be:

- ⋆ circular translations,
- ⋆ pure rotations,
- ⋆ planar four-bar motions,
- ⋆ spherical four-bar motions,
- ⋆ self-motions of Bricard octahedra,
- ⋆ self-motions of flexible octahedra with one vertex at infinity.



[2] **Flexible 3×3 complexes**

A 3×3 complex, which is also known as *Neunflach*, is a Kokotsakis mesh with a quadrilateral as central polygon.



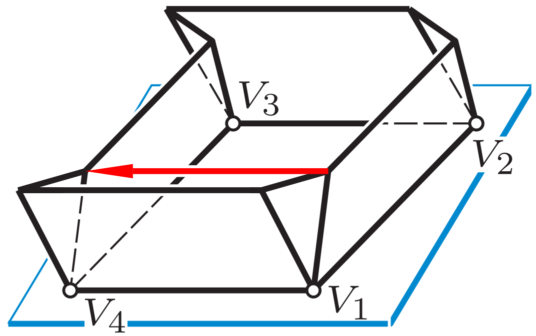
$\mathcal{M}$ denotes as polyhedral mesh with valence 4 composed of planar quadrilaterals.

BOBENKO, HOFFMANN, SCHIEF [11]:
$\mathcal{M}$ in general position is flexible $\Longleftrightarrow$ all its 3×3 complexes are flexible

Application: architectural design of flexible claddings composed of planar quads.

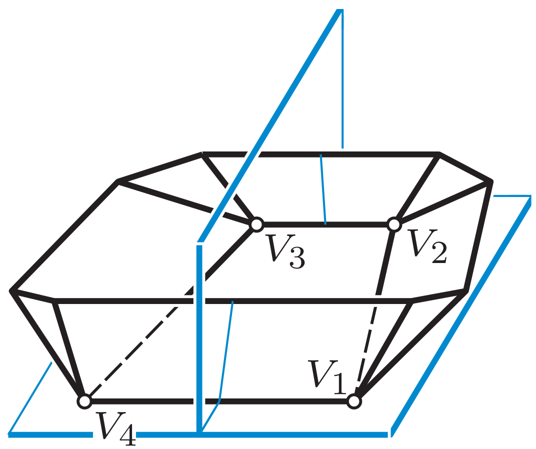
$[2a]$ Stachel's conjecture

A 3×3 complex is non-trivially flexible if and only if the transmission $\varphi_1 \mapsto \varphi_3$ can be decomposed in at least two different ways into two spherical four-bars.



Stachel's conjecture

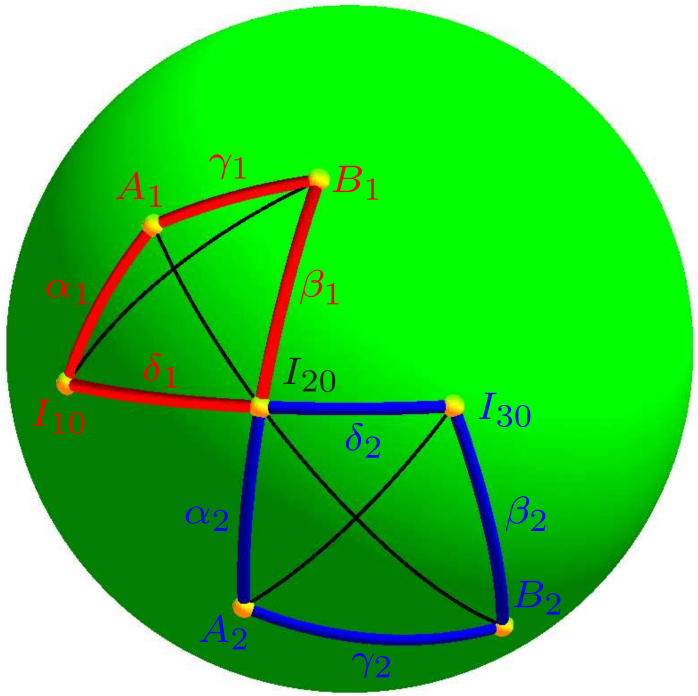
All multiply decomposable compounds of two spherical four-bars are reducible with exception of the translatory type and planar-symmetric type.



$[2b]$ Classification of reducible compositions

Theorem 15 NAWRATIL [12]
If a reducible composition of two spherical four-bar mechanisms $\mathcal{C}$ and $\mathcal{D}$ is given, then it is one of the following cases:

I. One of the quadrangles $\mathcal{C}$ or $\mathcal{D}$ is an isogram.
II. $\mathcal{C}$ and $\mathcal{D}$ form a spherical mechanism of Dixon type.
III. $\mathcal{C}$ and $\mathcal{D}$ are orthogonal four-bars and the diagonals A_1I_{20} and $I_{20}B_2$ coincide.
IV. One of the quadrangles $\mathcal{C}$ or $\mathcal{D}$ is a deltoid.



The work on a complete classification of 3×3 complexes is in progress.

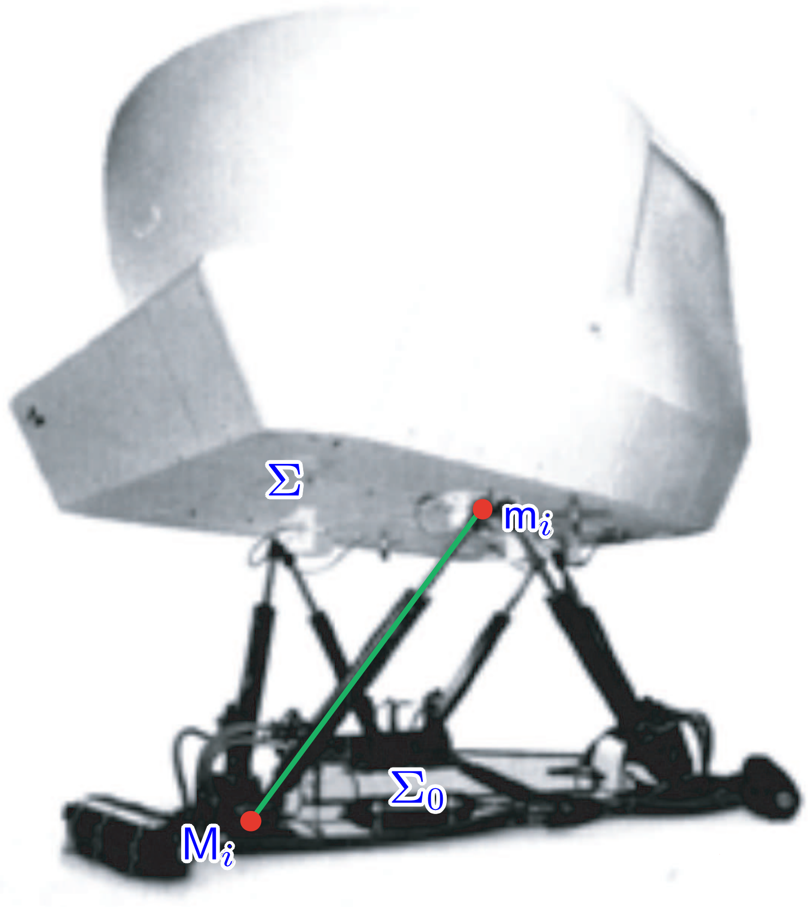
[3] Stewart Gough Platform

The geometry of a SGP is given by the six base anchor points $M_i \in \Sigma_0$ and by the six platform points $m_i \in \Sigma$.

A SGP is called planar, if $M_1, \ldots, M_6$ are coplanar and $m_{,1}, \ldots, m_6$ are coplanar.

M_i and m_i are connected with a S<u>P</u>S leg.

MERLET [13]
A SGP is singular (infinitesimal flexible, shaky) if and only if the carrier lines of the six S<u>P</u>S legs belong to a linear line complex.

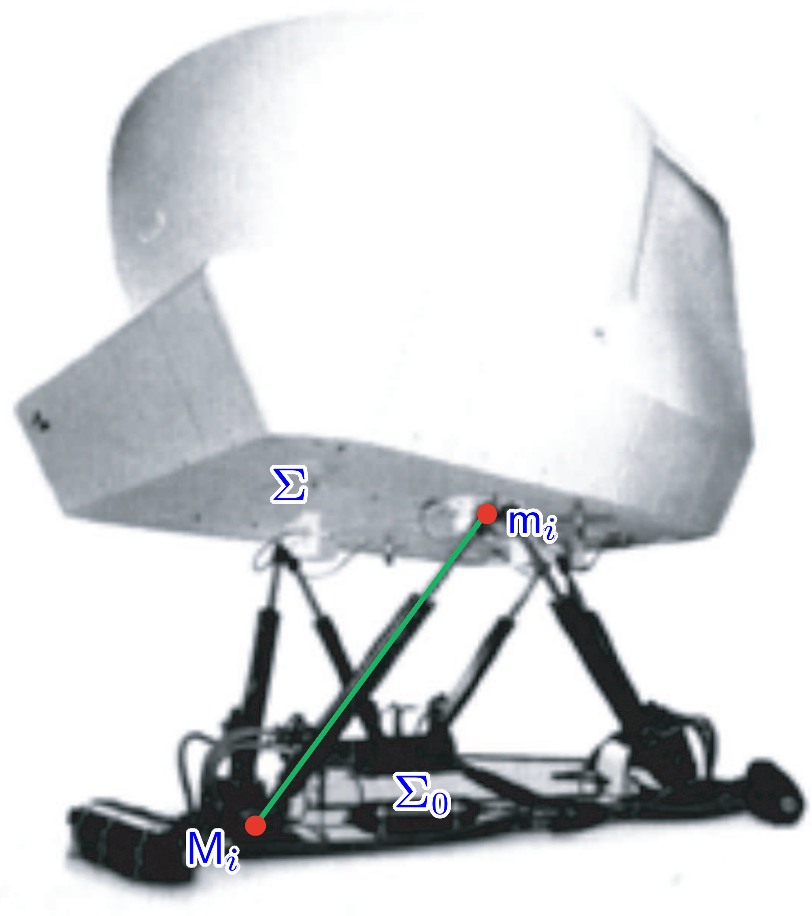


[3a] Self-motions and the Borel Bricard problem

If all P-joints are locked, a SGP is in general rigid. But, in some special cases the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion.

Note that in each pose of the self-motion, the SGP has to be singular. Moreover, all self-motions of SGPs are solutions to the famous Borel Bricard problem.

Borel Bricard problem (still unsolved)
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.



$[3a]$ Architecturally singular SGPs

SGPs which are singular in every configuration, are called architecturally singular.

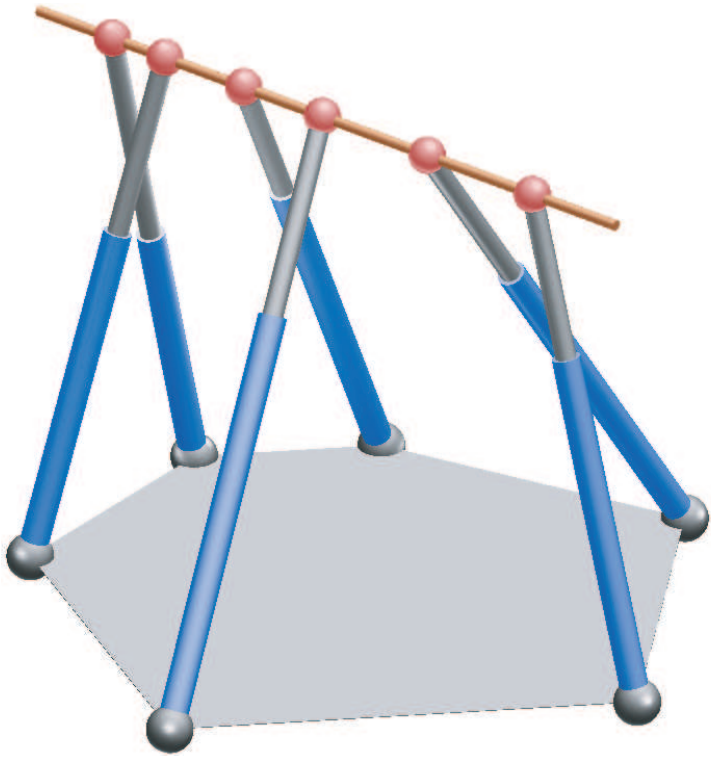
Architecturally singular SGPs are well studied:

⋆ For the planar case see Röschel & Mick [14], Karger [15], Nawratil [16], Wohlhart [17].

⋆ For the non-planar case see Karger [18] and Nawratil [19].

It is well known, that architecturally singular SGPs possess self-motions in each pose.

Therefore we are only interested in self-motions of non-architecturally singular SGPs. Only a few such motions are known.



$[3b]$ Redundant SGPs

According to HUSTY [20], the "sphere constraint" that m_i is located on a sphere with center M_i and radius R_i can be expressed by a homogeneous quadratic equation Λ_i in the Study parameters.

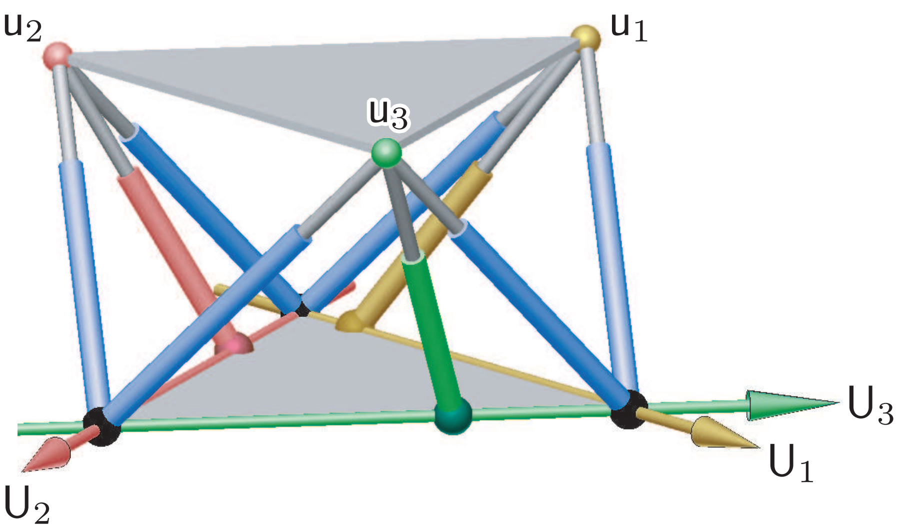
Therefore the direct kinematic problem corresponds to the solution of the system $\Lambda_1, \ldots, \Lambda_6, \Psi$ where Ψ denotes the equation of the Study quadric.

If a planar SGP is not architecturally singular, then at least a one-parametric set of legs Λ_+ can be added without changing the direct kinematics (cf. HUSTY ET AL [21]) and singularity surface (cf. BORRAS ET AL [22]):

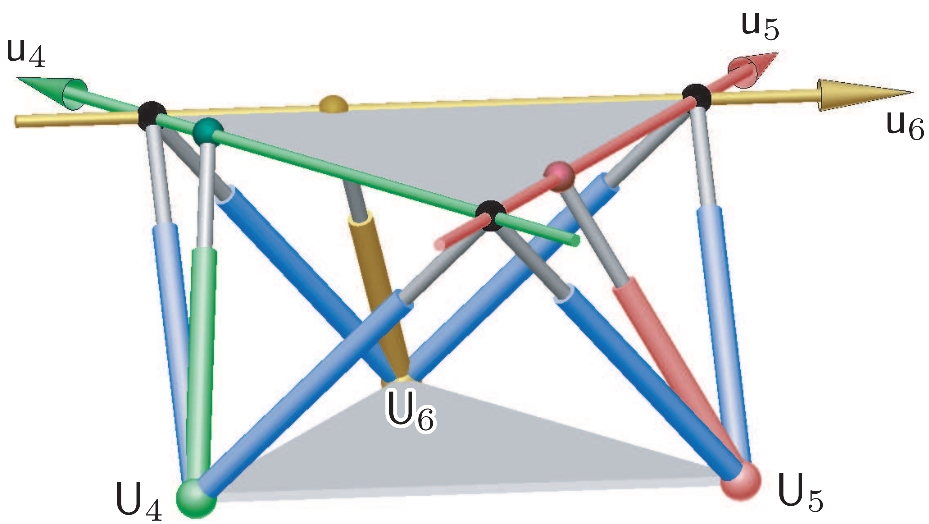
$$\Lambda_+ = \lambda_1 \Lambda_1 + \ldots + \lambda_6 \Lambda_6$$

$[3b]$ Redundant SGPs

Moreover, it was shown in HUSTY ET AL [21] that in general the base anchor points M_i as well as the corresponding platform anchor points m_i are located on planar cubic curves C and c, which can also split up.



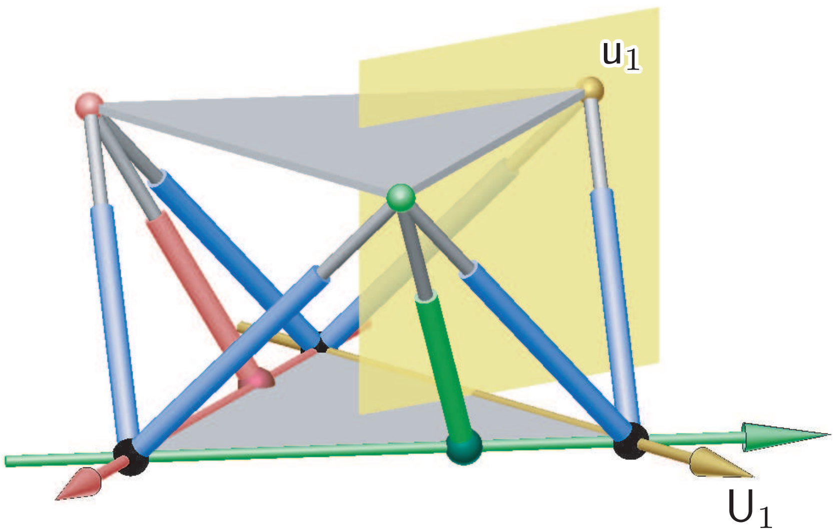
Cubic C of the octahedral SGP



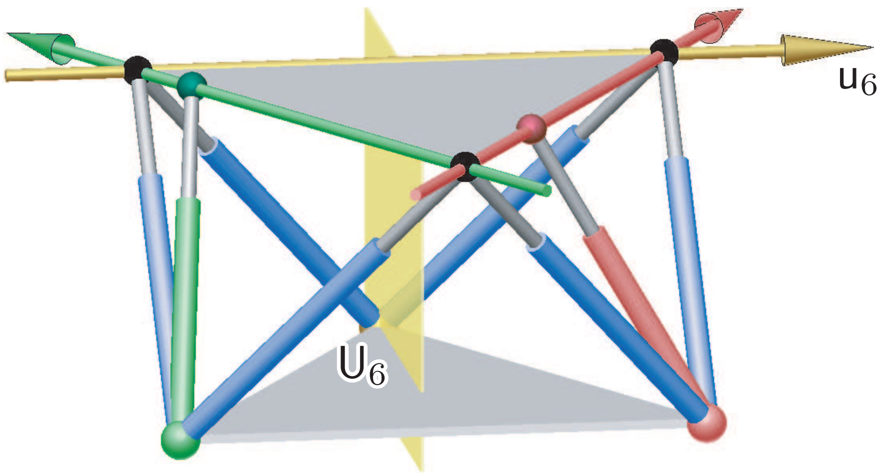
Cubic c of the octahedral SGP

$[3b]$ Darboux and Mannheim motion

The Darboux constraint that u_i moves in a plane $\in \Sigma_0$ orthogonal to the direction of the ideal point U_i is a homogeneous quadratic equation Ω_i in the Study parameters $(i = 1, 2, 3)$.



The Mannheim constraint that a plane of Σ orthogonal to u_j slides through the point $U_j \in \Sigma_0$ is a homogeneous quadratic equation Π_j in the Study parameters $(j = 4, 5, 6)$.



$[3b]$ Self-motions implied by Bricard octahedra I

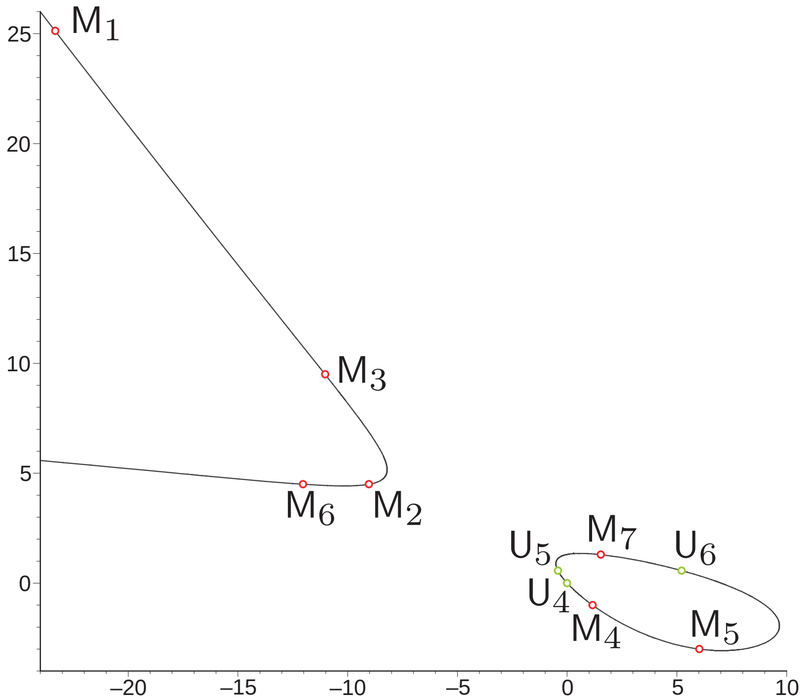
It was shown in NAWRATIL [23], that the system $\Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5, \Pi_6$ is redundant $\Longrightarrow$ manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is architecturally singular.

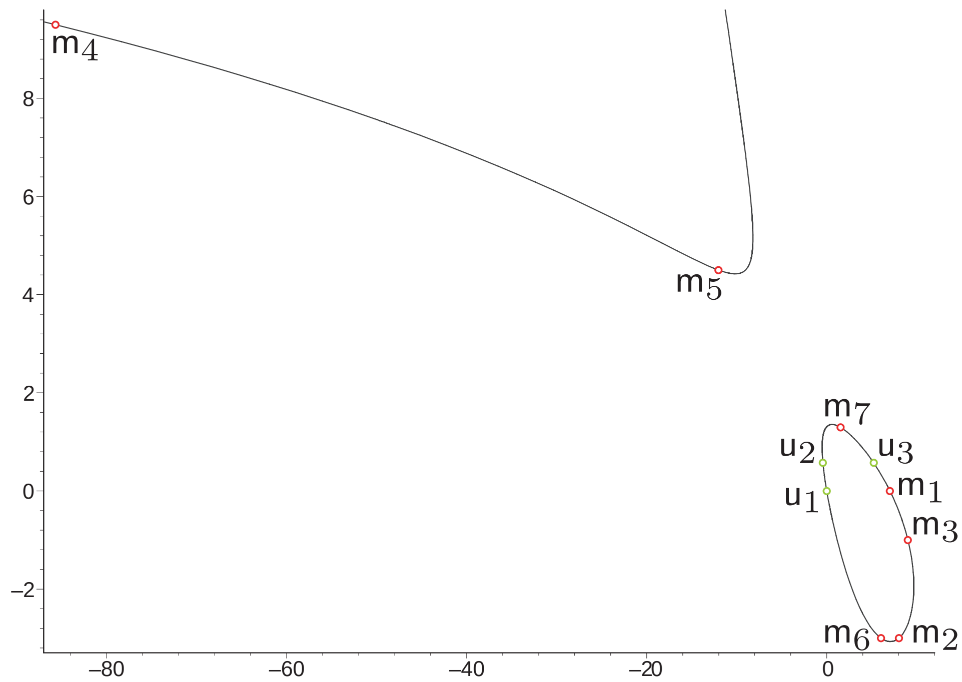
Moreover, if the underlying SGP is a Bricard octahedron of type I, then $\mathsf{u}_1, \ldots, \mathsf{U}_6$ has even a two-parametric self-motion (cf. NAWRATIL [23]).

By adding an arbitrary leg Λ to $\Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5$ we get an one-parametric self-motion. Further legs Λ_+ are determined by:

$$\Lambda_+ = \lambda\Lambda + \sum_{i=1}^{3} \nu_i \Omega_i + \sum_{j=4}^{5} \mu_j \Pi_j.$$

$[3b]$ Example





Remark 4

All self-motions implied by Bricard octahedra of type I are line-symmetric motions.

$[3b]$ Future research

This approach can also be applied to general planar SGPs:

Definition 1 NAWRATIL [23]
Assume $\mathcal{M}$ is a 1-parametric self-motion of a non-architecturally singular SGP $\mathrm{m}_1, \ldots, \mathrm{M}_6$. Then $\mathcal{M}$ is of type n DM (Darboux Mannheim) if the corresponding architecturally singular manipulator $\mathrm{u}_1, \ldots, \mathrm{U}_6$ has an n-parametric self-motion $\mathcal{U}$ (which includes $\mathcal{M}$).

Moreover, it was shown in NAWRATIL [23], that all 1-parametric self-motions of general planar SGPs (non-architecturally singular) are type I or type II DM self-motions.

Based on the Darboux and Mannheim constraints we were able to present a set of 24 equations yielding a type II DM self-motion.

$[3b]$ Future research

$$\Gamma_{080} = F_1[8]F_2[18]^2, \qquad \Gamma_{800} = (b_2 - b_3)^2(L_1 - g_4)^2F_3[8],$$

$$\Gamma_{170} = F_2[18]F_4[283], \qquad \Gamma_{710} = (b_2 - b_3)(L_1 - g_4)F_5[170],$$

$$\begin{array}{llll}
\Gamma_{620}[2054], & \Gamma_{602}[1646], & \Gamma_{260}[6126], & \Gamma_{062}[4916], \\
\Gamma_{026}[5950], & \Gamma_{116}[3066], & \Gamma_{530}[4538], & \Gamma_{512}[4512], \\
\Gamma_{152}[6514], & \Gamma_{440}[7134], & \Gamma_{422}[6314], & \Gamma_{242}[7622], \\
\Gamma_{044}[6356], & \Gamma_{314}[6934], & \Gamma_{224}[7096], & \Gamma_{134}[6656], \\
\Gamma_{206}[5950], & \Gamma_{350}[7166], & \Gamma_{404}[5766], & \Gamma_{332}[6982].
\end{array}$$

Based on these 24 equations $\Gamma_{ijk} = 0$ (in 14 unknowns), we were already able to compute first results for type II DM self-motions in NAWRATIL [24], which raise the hope of giving a complete classification of these self-motions in the future.

References

[1] Bricard R (1897) Mémoire sur la théorie de l'octaèdre articulé. Journal de Mathématiques pures et appliquées, Liouville 3:113–148

[2] Stachel H (2010) A kinematic approach to Kokotsakis meshes. Comput Aided Geom Des 27:428-437

[3] Nawratil G (2011) Reducible compositions of spherical four-bar linkages with a spherical coupler component. Mech Mach Theory 46(5):725-742

[4] Wunderlich W (1968) On Burmester's focal mechanism and Hart's straight-line motion. J Mechanism 3:79–86

[5] Nawratil G, Stachel H (2010) Composition of spherical four-bar-mechanisms. In: Pisla et al (eds) New Trends in Mechanisms Science, Springer, 99–106

[6] Nawratil G (in press) Self-motions of TSSM manipulators with two parallel rotary axes. ASME J Mech Rob

[7] Kokotsakis A (1932) Über bewegliche Polyeder. Math Ann 107:627–647

[8] Stachel H (2002) Remarks on Bricard's flexible octahedra of type 3. Proc. 10th International Conference on Geometry and Graphics, Kiev, Ukraine, 1:8–12

[9] Nawratil G (2010) Flexible octahedra in the projective extension of the Euclidean 3-space. J Geom Graphics 14(2):147-169

[10] Nawratil G (2011) Self-motions of parallel manipulators associated with flexible octahedra. Austrian Robotics Workshop, Hall in Tyrol, Austria (2011)

[11] Bobenko AI, Hoffmann T, Schief WK (2008) On the integrability of infinitesimal and finite deformations of polyhedral surfaces. In: Bobenko et al (eds) Discrete Differential Geometry, Series: Oberwolfach Seminars 38:67–93

[12] Nawratil G (2011) Reducible compositions of spherical four-bar linkages without a spherical coupler component. Technical Report No. 216, Geometry Preprint Series, TU Vienna

[13] Merlet J-P (1989) Singular configurations of parallel manipulators and Grassmann geometry. Int J Rob Res 8(5) 45–56

[14] Röschel O, Mick S (1998) Characterisation of architecturally shaky platforms. In: Lenarcic J, Husty ML (eds) Advances in Robot Kinematics: Analysis and Control, Kluwer, pp 465–474

[15] Karger A (2003) Architecture singular planar parallel manipulators. Mech Mach Theory 38(11):1149–1164

[16] Nawratil G (2008) On the degenerated cases of architecturally singular planar parallel manipulators. J Geom Graphics 12(2):141–149

[17] Wohlhart K (2010) From higher degrees of shakiness to mobility. Mech Mach Theory 45(3):467–476

[18] Karger A (2008) Architecturally singular non-planar parallel manipulators. Mech Mach Theory 43(3):335–346

[19] Nawratil G (2009) A new approach to the classification of architecturally singular parallel manipulators. In: Kecskemethy A, Müller A (eds) Computational Kinematics, Springer, pp 349–358

[20] Husty ML (1996) An algorithm for solving the direct kinematics of general Stewart-Gough platforms. Mech Mach Theory 31(4):365–380

[21] Husty M, Mielczarek S, Hiller M (2002) A redundant spatial Stewart-Gough platform with a maximal forward kinematics solution set. In: Lenarcic J, Thomas F (eds) Advances in Robot Kinematics: Theory and Applications, Springer, pp 147–154

[22] Borras J, Thomas F, Torras C (2010) Singularity-invariant leg rearrangements in doubly-planar Stewart-Gough platforms. In: Proceedings of Robotics Science and Systems, Zaragoza, Spain (2010)

[23] Nawratil G (under review) Types of self-motions of planar Stewart Gough platforms.

[24] Nawratil G (2011) Basic result on type II DM self-motions of planar Stewart Gough platforms. In Proc. of 1st Workshop on Mechanisms, Transmissions and Applications, Timisoara, Romania, Springer (2011) to appear