CONGRUENT STEWART GOUGH PLATFORMS WITH NON-TRANSLATIONAL SELF-MOTIONS

Georg Nawratil
Vienna University of Technology, Austria

ABSTRACT: It is well known that each Stewart Gough (SG) manipulator, where the platform is congruent with the base (= congruent SG manipulator), has a 2-dimensional translational self-motion, if all legs have equal (non-zero) length. As congruent SG platforms with planar platform and planar base are only special cases of so-called planar affine/projective SG platforms, which were already studied by the author in foregoing publications, we focus on the non-planar case. In this paper we give a geometric characterization of all non-planar congruent SG platforms, which have further self-motions beside the above mentioned translational one. The main result is obtained by means of bond theory.

Keywords: Stewart Gough platform, Self-motion, Bond theory, Wren platform, Schönflies motion, Cylinder of revolution

1. INTRODUCTION

The geometry of a Stewart Gough (SG) platform is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ with respect to the fixed system and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$ with respect to the moving system (for $i = 1, \dots, 6$). Each pair $(\mathsf{M}_i, \mathsf{m}_i)$ of corresponding anchor points is connected by a SPS-leg, where only the prismatic joint (P) is active and the spherical joints (S) are passive (cf. Fig. 1).

If the geometry of the manipulator is given as well as the leg lengths, the SG platform is generically rigid. But, under particular conditions, the manipulator can perform a n-dimensional motion ($n > 0$), which is called self-motion.

Note that self-motions are also solutions to the still unsolved problem posed by the French Academy of Science for the "Prix Vaillant" of the year 1904, which is also known as Borel Bricard problem (cf. [1], [2], [9]) and reads as follows: *"Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."*

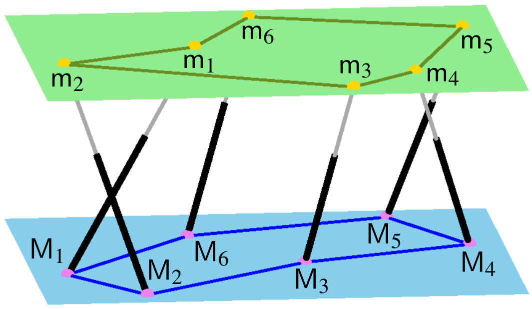

Figure 1: SG manipulator with planar platform and planar base (= planar SG manipulator).

1.1 Bond Theory

In this section we give a short introduction into the theory of bonds for SG manipulators presented in [18], which was motivated by publication [7]. We start with the direct kinematic problem of parallel manipulators of SG type and further with the definition of bonds.

Due to the result of Husty [8], it is advantageous to work with Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ for solving the forward kinematics. Note that the first four homogeneous coordinates $(e_0 : e_1 : e_2 : e_3)$ are the so-called Euler parameters. Now all real points of the 7-dimensional Study parameter space P^7, which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the 3-dimensional subspace E of Ψ given by $e_0 = e_1 = e_2 = e_3 = 0$, as its points cannot fulfill the condition $N \neq 0$ with $N = e_0^2 + e_1^2 + e_2^2 + e_3^2$. The translation vector $\mathbf{v} := (v_1, v_2, v_3)^T$ and the rotation matrix $\mathbf{R} := (r_{ij})$ of the corresponding Euclidean displacement $\mathbf{Rx} + \mathbf{v}$ are given by:

$$
\begin{aligned}
v_1 &= 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2),\\
v_2 &= 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3),\\
v_3 &= 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1),\\
r_{11} &= e_0^2 + e_1^2 - e_2^2 - e_3^2,\\
r_{22} &= e_0^2 - e_1^2 + e_2^2 - e_3^2,\\
r_{33} &= e_0^2 - e_1^2 - e_2^2 + e_3^2,\\
r_{12} &= 2(e_1 e_2 - e_0 e_3), \quad r_{21} = 2(e_1 e_2 + e_0 e_3),\\
r_{13} &= 2(e_1 e_3 + e_0 e_2), \quad r_{31} = 2(e_1 e_3 - e_0 e_2),\\
r_{23} &= 2(e_2 e_3 - e_0 e_1), \quad r_{32} = 2(e_2 e_3 + e_0 e_1),
\end{aligned}
$$

if $N = 1$ is fulfilled. All points of the complex extension of P^7, which cannot fulfill this normalizing condition, are located on the so-called exceptional cone $N = 0$ with vertex E.

By using the Study parametrization of Euclidean displacements the condition that the point m_i is located on a sphere centered in M_i with radius R_i, is a quadratic homogeneous equation according to Husty [8]. This so-called sphere condition Λ_i has the following form:

$$
\begin{aligned}
\Lambda_i : \quad & (a_i^2 + b_i^2 + c_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)N +\\
& 2[(a_iA_i + b_iB_i - c_iC_i)e_3^2 - (a_iA_i + b_iB_i + c_iC_i)e_0^2\\
& - (a_iA_i - b_iB_i - c_iC_i)e_1^2 + (a_iA_i - b_iB_i + c_iC_i)e_2^2\\
& + 2(c_iB_i - b_iC_i)e_0e_1 + 2(a_i - A_i)(e_0f_1 - e_1f_0)\\
& - 2(c_iA_i - a_iC_i)e_0e_2 + 2(b_i - B_i)(e_0f_2 - e_2f_0)\\
& + 2(b_iA_i - a_iB_i)e_0e_3 + 2(c_i - C_i)(e_0f_3 - e_3f_0)\\
& - 2(b_iA_i + a_iB_i)e_1e_2 + 2(a_i + A_i)(e_3f_2 - e_2f_3)\\
& - 2(c_iA_i + a_iC_i)e_1e_3 + 2(b_i + B_i)(e_1f_3 - e_3f_1)\\
& - 2(c_iB_i + b_iC_i)e_2e_3 + 2(c_i + C_i)(e_2f_1 - e_1f_2)\\
& + 2(f_0^2 + f_1^2 + f_2^2 + f_3^2)] = 0.
\end{aligned} \tag{1}
$$

Now the solution of the direct kinematics over $\mathbb{C}$ can be written as the algebraic variety V of the ideal $\mathscr{I}$ spanned by $\Psi, \Lambda_1, \ldots, \Lambda_6, N = 1$. In general V consists of a discrete set of points with a maximum of 40 elements.[1]

We consider the algebraic motion of the mechanism, which are the points on the Study quadric that the constraints define; i.e. the common points of the seven quadrics $\Psi, \Lambda_1, \ldots, \Lambda_6$. If the manipulator has a n-dimensional self-motion then the algebraic motion also has to be of this dimension. Now the points of the algebraic motion with $N \neq 0$ equal the kinematic image of V. But we can also consider the points of the algebraic motion, which belong to the exceptional cone $N = 0$. An exact mathematical definition of these so-called bonds can be given as follows (cf. Remark 5 of [18]):

Definition 1 *For a SG manipulator the set $\mathscr{B}$ of bonds is defined as:*

$$
\mathscr{B} := ZarClo(V^\star) \cap \{(e_0 : \ldots : f_3) \in P^7 \mid \Psi, \Lambda_1, \ldots, \Lambda_6, N = 0\},
$$

where $V^\star$ denotes the variety V after the removal of all components, which correspond to pure translational motions. Moreover $ZarClo(V^\star)$ is the Zariski closure of $V^\star$, i.e. the zero locus of all algebraic equations that also vanish on $V^\star$.

We have to restrict to non-translational motions for the following reason: A component of V, which corresponds to a pure translational motion, is projected to a single point O (with $N \neq 0$) of the Euler parameter space P^3 by the elimination of $f_0, \ldots, f_3$. Therefore the intersection of O and $N = 0$ equals $\varnothing$. Clearly, the kernel of this projection equals the group of translational motions. Moreover it is important to note that the set of bonds only depends on the geometry of the manipulator and not on the leg lengths (cf. Theorem 1 of [18]). For more details please see [18].

[1] Note that for non-planar congruent SG manipulators this number drops to 24 according to [12] and [15].

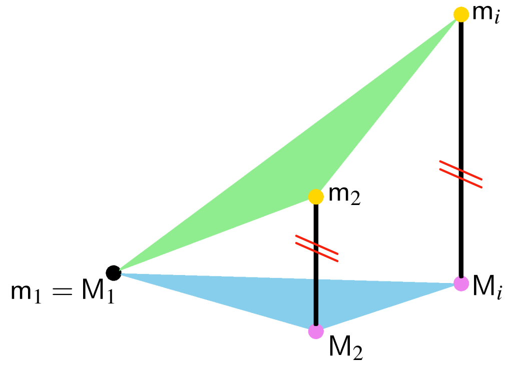

Figure 2: Illustration of Eq. (2) with $\mathsf{m}_1 = \mathsf{M}_1$.

Due to Theorem 2 of [18] a SG platform possesses a pure translational self-motion if and only if the platform can be rotated about the center $\mathsf{m}_1 = \mathsf{M}_1$ into a pose, where the vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i = 2,\ldots,6$ fulfill the condition (cf. Fig. 2):

$$rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2},\ldots,\overrightarrow{\mathsf{M}_6\mathsf{m}_6}) \leq 1. \tag{2}$$

Moreover all 1-dimensional self-motions are circular translations, which can easily be seen by considering a normal projection of the SG manipulator in direction of the parallel vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i = 2,\ldots,6$. If all these five vectors are zero-vectors, the platform and the base are congruent and therefore we get a so-called congruent SG manipulator. This type of SG manipulator has a well known 2-dimensional translational self-motion $\mathscr{T}$, if all legs have equal (non-zero) length.

Note that $\mathscr{T}$ is the only 2-dimensional translational self-motion and that higher-dimensional translational self-motions do not exist (cf. [20]).

2. PRELIMINARY CONSIDERATIONS ON CONGRUENT SG PLATFORM

In this article we are interested in designs of congruent SG platforms, which can perform additional self-motions beside $\mathscr{T}$. First of all we clarify whether congruent SG platforms can have further translational self-motions beside $\mathscr{T}$. Clearly, due to the result given in the last paragraph of Section 1.1, these translational self-motions can only be 1-dimensional ones.

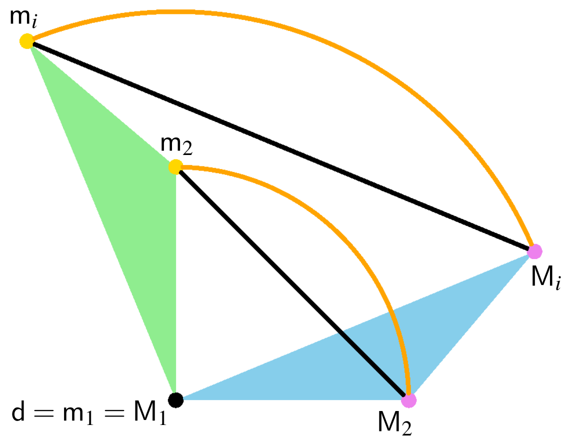

Figure 3: Projection in direction of the axis d: The chords of the rotation around d are parallel if and only if d equals the line of intersection of the planar platform and the planar base.

Lemma 1 *A non-planar congruent SG platform cannot have a* 1*-dimensional translational self-motion.*

PROOF: If a congruent SG manipulator has a 1-dimensional translational self-motion there has to exist an orientation of the platform with $\mathsf{m}_1 = \mathsf{M}_1$ and $rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2},\ldots,\overrightarrow{\mathsf{M}_6\mathsf{m}_6}) = 1$. We assume that the manipulator is in this configuration.

As the platform and the base are congruent there exists a rotation ρ around the axis d through $\mathsf{m}_1 = \mathsf{M}_1$ with $\mathsf{M}_i\rho = \mathsf{m}_i$ for $i = 2,\ldots,6$. Therefore the vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ are chords of ρ. A projection in direction of the axis d shows immediately the validity of this lemma (cf. Fig. 3). □

For our study we can focus on the non-planar case, as planar congruent SG platforms were already discussed in detail by Karger [14] and by the author [17] as special cases of so-called planar projective SG platforms. Therefore it remains to determine all non-planar congruent SG platforms, which possess non-translational self-motions. This is done in the remainder of this article, which is structured as follows:

In Section 2.1 we discuss non-planar congruent SG platforms, which are architecturally singular. In Section 2.2 we demonstrate on the basis of a modified Wren platform that non-architecturally singular congruent SG platforms can have non-translational self-motions. After a short review on cylinders of revolution in Section 3, we present a remarkable geometric characterization of all non-planar congruent SG platforms with non-translational self-motions by means of bond theory in Section 4. We close the paper by discussing some non-planar congruent SG platforms with remarkable self-motions in Section 5.

2.1 Architecture singularity

A SG platform is called architecturally singular if it is singular in every possible configuration. All manipulators which have this property are well studied and classified (for a review on this topic see Section 3.1 of [19]). Therefore the following lemma can easily be proven:

Lemma 2 *A non-planar congruent SG platform is architecturally singular if and only if four anchor points are collinear. These manipulators possess self-motions in each pose over $\mathbb{C}$.*

PROOF: The first sentence follows directly from the list of non-planar architecturally SG platforms given by Karger in Theorem 3 of [13].

If four anchor points are collinear, the corresponding four legs belong to a regulus and one can remove any of the four legs without changing the direct kinematics ($\Rightarrow$ redundant SG platform). This already proves the second statement of Lemma 2. □

Until now only a few non-architecturally singular SG platforms with self-motions are known. A detailed review of this topic was given in [16], which is as complete as possible to the best knowledge of the author. Based on the body of literature cited within this review, it is not difficult to come up with the following example of a non-planar congruent SG platform with no four anchor points collinear, which possesses a non-translational self-motion.

2.2 Example: Wren platform

We start with considering a special planar congruent SG manipulator, which is also known as Wren platform (cf. [23]). As in this case the anchor points are located on a circle (see Fig. 4, left), the Wren platform is an architecturally singular manipulator (cf. [2], [3], [14]). If all legs have equal length, there also exists a 1-dimensional Schönflies self-motion (see Fig. 5, left), beside $\mathscr{T}$ (see Fig. 6, left). In the left picture of Fig. 4, the *branching singularity* (cf. [6]) of these two self-motions is displayed.

Remark 1 *Due to Wohlhart [23], the Wren platform is called kinematotropic, as it can change the dimension of mobility.* ◇

Now we consider the Wren platform in its branching configuration (cf. Fig. 4, left). If we translate each leg (including their anchor points) arbitrarily in direction of its carrier line, we end up with a configuration of the modified Wren platform, which is still a congruent SG manipulator, but not longer planar and therefore not architecturally singular (cf. Fig. 4, right).

Clearly the modified Wren platform also has the self-motion $\mathscr{T}$ (cf. Fig. 6, right). Moreover due to Husty and Karger [10] this modification has no influence on the Schönflies self-motion (cf. Fig. 5, right). Therefore this is an example of a non-translational self-motion of a non-planar congruent SG platform, which is not architecturally singular. Note that this existence is not self-evident, as planar congruent SG platforms can only have translational self-motions if they are not architecturally singular (cf. [17]).

Remark 2 *The modified Wren platform also demonstrates that the property of kinematotropy is not restricted to architecturally singular manipulators.* ◇

Motivated by this example, we are interested in all congruent SG manipulators with non-translational self-motions. Before they are determined in Section 4, we review some known results about cylinders of revolution.

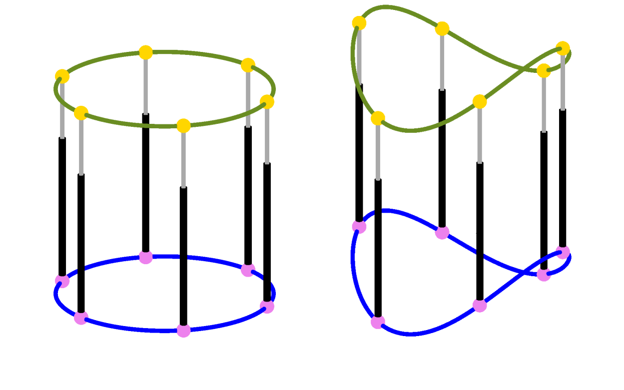

Figure 4: Branching singularity of the Wren platform (left) and the modified one (right).

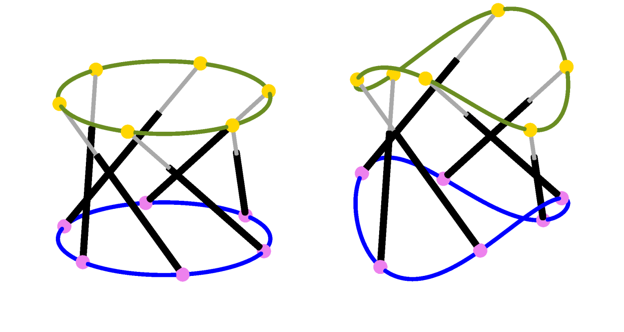

Figure 5: Schönflies self-motion of the Wren platform (left) and the modified one (right).

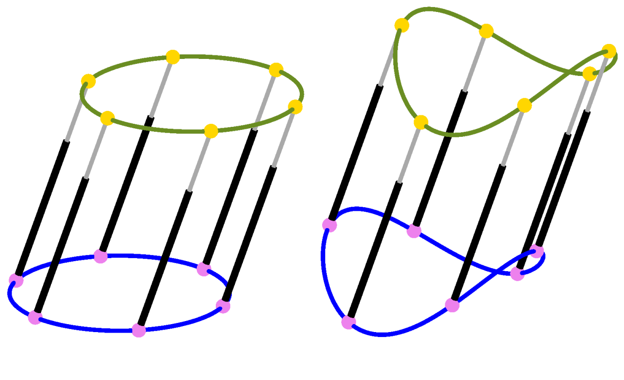

Figure 6: Self-motion $\mathscr{T}$ of the Wren platform (left) and the modified one (right).

3. CYLINDERS OF REVOLUTION

A cylinder of revolution Φ equals the set of all points, which have equal distance to its rotation axis s (finite line). Under the assumption that Φ has at least one real point, we can distinguish the following four cases:

1. s is real and Φ is not reducible: Φ is a cylinder of revolution over $\mathbb{R}$.
2. s is real and Φ is reducible: Φ equals a pair of isotropic planes[2] γ_1 and γ_2, which are conjugate complex. Trivially s carries the only real points of Φ.
3. s is imaginary and Φ is not reducible: Φ is a cylinder of revolution over $\mathbb{C}$. The real points of Φ are located on the 4th order intersection curve of Φ and its conjugate $\overline{\Phi}$.
4. s is imaginary and Φ is reducible: In this case Φ equals a pair of isotropic planes γ_1 and γ_2, which are not conjugate complex. Moreover Φ contains two real lines g_i ($i = 1,2$), which are the intersections of γ_i and its isotropic conjugate $\overline{\gamma}_i$.

Remark 3 *It is a well known fact from projective geometry that the axis* s *is the line, where the tangent planes* γ_1 *and* γ_2 *through* s *onto* Φ *are isotropic planes.* $\diamond$

3.1 Computation of cylinders of revolution

In this section we focus on the determination of all cylinders of revolution through a given set of real points $\mathsf{X}_1,\ldots,\mathsf{X}_n$. There exist many papers on this well studied problem (see e.g. [5], [22], [24] and the references therein).

In the following we want to use the computational approach of Schaal [22], which was furthered by Zsombor-Murray and El Fashny in [24]. They pointed out that this problem is equivalent with the solution of the following system of equations if X_1 equals the origin U of the reference frame:

$$\mathbf{s}^2 = 1, \tag{3}$$
$$\Upsilon: \quad \mathbf{s}\cdot\mathbf{t} = 0, \tag{4}$$
$$\Omega_i: \quad (\mathbf{x}_i\times\mathbf{s})^2 - 2\mathbf{s}^2(\mathbf{x}_i\cdot\mathbf{t}) = 0, \tag{5}$$

for $i = 2,\ldots,n$, where $\mathbf{x}_i$ is the coordinate vector of the point X_i, $\mathbf{s} := (s_1,s_2,s_3)^T$ the direction

[2] A plane is called isotropic if its ideal line is tangent to the absolute quadric.

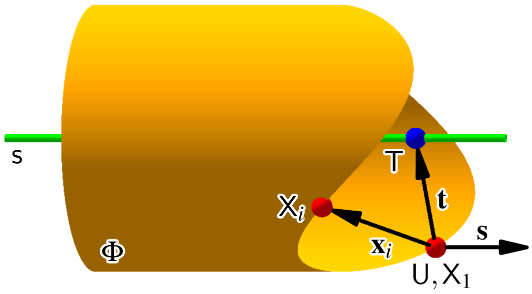

Figure 7: Notation used for computation.

vector of the rotation axis s, and $\mathbf{t} := (t_1,t_2,t_3)^T$ the coordinate vector of the footpoint T on s with respect to $U = X_1$ (cf. Fig. 7).

The rough procedure for solving this system of equations is as follows: In the first step, one solves the equations $\Upsilon, \Omega_2, \ldots, \Omega_n$, which already gives the solutions up to a common factor; i.e. we get $s_1 : s_2 : s_3 : t_1 : t_2 : t_3$. In the second step, we normalize these 6-tuples with respect to the normalizing condition given in Eq. (3). This normalization is always possible as the axis cannot be isotropic[3], because it is the intersection of two isotropic planes (cf. Remark 3).

Remark 4 *For $n = 5$ there exist in general six cylinders of revolution over $\mathbb{C}$ (e.g. [24]). There even exist examples, where all six cylinders are real (e.g. [5]). For $n > 5$ no solution exists, if $X_1, \ldots, X_n$ are in general configuration.* $\diamond$

4. MAIN THEOREM

Based on these considerations regarding cylinders of revolution, we can formulate the following main theorem:

Theorem 1 *A non-planar congruent SG manipulator can have a real non-translational self-motion only if the six base (resp. platform) anchor points have equal distance to a finite line* s*, i.e. they are located on a cylinder of revolution of type 1, 3 or 4 listed in Section 3. Moreover this condition is also sufficient for the existence of self-motions over $\mathbb{C}$.*

[3]The line is called isotropic if its ideal point is located on the absolute quadric.

PROOF: The proof of this theorem is based on the following fact: If a non-translational self-motion exists, then the bond-set has to be non-empty. Therefore we have to determine the conditions for which the set of bonds consists of at least one element. The computation of these conditions is outlined next.

Without loss of generality (w.l.o.g.) we can choose Cartesian coordinate systems in the platform and base with $a_i = A_i$, $b_i = B_i$, $c_i = C_i$ for $i = 1, \ldots 6$ and $a_1 = b_1 = b_2 = c_1 = c_2 = c_3 = 0$. In order to eliminate the factor of similarity we can set $a_2 = 1$. Clearly we assume for the remainder of this article that all anchor points are distinct, as otherwise two legs coincide due to the congruence of the platform and the base. Moreover we can assume (after a possible necessary reindexing of anchor points) that the first four points are not coplanar; i.e. $b_3c_4 \neq 0$. We distinguish three cases.

4.1 No three anchor points are collinear

According to [18] the set of bonds can be computed as follows: We compute $\Delta_{j,i} := \Lambda_j - \Lambda_i$, which is only linear in the Study parameters $f_0, \ldots, f_3$. We distinguish three cases.

General case $e_3 \neq 0$. Assuming a real motion we can solve the linear system of equations $\Psi, \Delta_{2,1}, \Delta_{3,1}$ for f_1, f_2, f_3 without loss of generality. We plug the obtained expressions for f_1, f_2, f_3 into $\Delta_{4,1}, \Delta_{5,1}, \Delta_{6,1}$ and consider their numerators, which are homogeneous polynomials P_4, P_5, P_6 of degree three in the Euler parameters. Note that these polynomials do not depend on f_0. Therefore f_0 can be calculated from Λ_1, but this is not of interest for the further computation.

We eliminate e_0 from P_i and $N = 0$ by computing the resultant Q_i of these two expressions for $i = 4, 5, 6$. Q_i factors into $16F_i^2$ with

$$F_i = \sum_{j+k+l=3} g_{jkl} e_1^j e_2^k e_3^l \quad \text{and} \quad j,k,l \in \{0, \ldots, 3\}.$$

Moreover the coefficients g_{jkl} are given by:

$$g_{210}=c_ib_3^2,\quad g_{201}=b_3(b_i^2-b_3b_i+c_i^2),$$
$$g_{300}=0,\quad g_{012}=c_i(b_3^2+a_3^2-a_3-2b_3b_i),$$
$$g_{120}=b_3c_i(1-2a_3),\quad g_{102}=b_3c_i(1-2a_i),$$
$$g_{111}=2b_3b_i(a_3-a_i),\quad g_{030}=a_3c_i(a_3-1),$$
$$g_{021}=a_i^2b_3-a_ib_3+a_3b_i-a_3^2b_i+c_i^2b_3,$$
$$g_{003}=a_3b_i-a_3^2b_i+a_i^2b_3+b_i^2b_3-a_ib_3-b_ib_3^2.$$

We proceed with the computation of the resultant U_k of F_i and F_j with respect to e_1 for pairwise distinct $i,j,k\in\{4,5,6\}$. From each U_k we can factor out $b_3^2e_3^2$ and remain with an expression V_k with 3827 terms. Moreover V_k is a homogeneous polynomial of degree 6 in the remaining Euler parameters e_2,e_3.

Now the necessary condition for the existence of a bond is that V_4,V_5,V_6 have a common solution; i.e. the resultant W_k of V_i and V_j with respect to e_2 with pairwise distinct $i,j,k\in\{4,5,6\}$, has to be fulfilled identically for all e_3. Due to the large number of terms and the high degree, W_k cannot be computed in general, and therefore it seems that we cannot prove the theorem.

But due to the example of the modified Wren platform, we conjecture that bonds can only exist if the six anchor points are located on a cylinder of revolution. Therefore we consider the system of equations $\Upsilon,\Omega_2,\dots,\Omega_6$ given in Eqs. (4) and (5) with respect to the six anchor points.

Under the assumption $s_3\neq 0$, we can solve $\Upsilon,\Omega_2,\Omega_3$, which are linear in t_1,t_2,t_3, for these unknowns. We plug the obtained expressions into $\Omega_4,\Omega_5,\Omega_6$ and consider their numerators, which are homogeneous polynomials $F_4^\star,F_5^\star,F_6^\star$. For the substitution $s_i\leftrightarrow e_i$ for $i=1,2,3$ the polynomial $F_j^\star$ equals F_j for $j=4,5,6$.

Therefore the existence of a cylinder of revolution through the six anchor points implies the existence of a bond and vice versa. This already closes the general case.

Special case $e_3=0$, $e_2\neq 0$. The procedure for this case only differs slightly from the general one. Assuming a real motion we can solve $\Psi,\Delta_{2,1},\Delta_{4,1}$ for f_1,f_2,f_3 without loss of generality. Then we plug the obtained expressions into $\Delta_{3,1},\Delta_{5,1},\Delta_{6,1}$ and consider their numerators, which are homogeneous polynomials P_3,P_5,P_6 of degree two in the Euler parameters. We eliminate e_0 from P_i and $N=0$ by computing the resultant Q_i of these two expressions for $i=3,5,6$. Q_i factors into $16e_2^2F_i^2$ with $F_i=g_{20}e_1^2+g_{11}e_1e_2+g_{02}e_2^2$ and

$$g_{20}=c_ic_4^2+c_ib_4^2-b_i^2c_4-c_i^2c_4,$$
$$g_{11}=2a_ib_ic_4-b_ic_4+c_ib_4-2c_ia_4b_4,$$
$$g_{02}=c_ia_4^2-c_ia_4-c_i^2c_4+a_ic_4+c_ic_4^2-a_i^2c_4.$$

Now we proceed with the computation of the cylinders of revolution with $s_3=0$ and $s_2\neq 0$. W.l.o.g. we can solve $\Upsilon,\Omega_2,\Omega_4$ for t_1,t_2,t_3. We plug the obtained expressions into $\Omega_3,\Omega_5,\Omega_6$ and consider their numerators, which are homogeneous polynomials $F_3^\star,F_5^\star,F_6^\star$. Again the substitution $s_i\leftrightarrow e_i$ for $i=1,2$ shows that $F_j^\star$ equals F_j for $j=3,5,6$. Therefore we can draw the same conclusion as in the general case.

Very special case $e_2=e_3=0$. If $e_1=0$ holds, the platform has the same orientation during the whole self-motion. As a consequence we can only end up with a translational self-motion, which has to be 2-dimensional due to Lemma 1.

Assuming a real motion with $e_1\neq 0$ we can solve $\Psi,\Delta_{3,1},\Delta_{4,1}$ for f_1,f_2,f_3 without loss of generality. Then we plug the obtained expressions into $\Delta_{2,1},\Delta_{5,1},\Delta_{6,1}$ and consider their numerators, which are homogeneous polynomials P_2,P_5,P_6 of degree two in the Euler parameters. Note that P_2 factors into $N(R_1-R_2)$. Therefore we only have to compute the resultant Q_i of $N=0$ and P_i with respect to e_0 for $i=5,6$. Q_i factors into $16b_3^2F_i^2$ with $F_i=g_2e_1^2$ and

$$g_2=b_3c_ib_4+b_i^2c_4+c_i^2c_4-c_ic_4^2-c_3c_ic_4-c_ib_4^2.$$

Now we proceed with the computation of the cylinders of revolution with $s_2=s_3=0$ and $s_1\neq 0$. W.l.o.g. we can solve $\Upsilon,\Omega_3,\Omega_4$ for

t_1, t_2, t_3. If we plug the obtained expression into Ω_2, we see that it is fulfilled identically. Therefore we consider the numerators of Ω_5, Ω_6, which are homogeneous polynomials $F_5^\star, F_6^\star$. Again the substitution $s_1 \leftrightarrow e_1$ shows that $F_j^\star$ equals F_j for $j = 5, 6$. Therefore we can draw the same conclusion as in the general case.

Remark 5 *One also has to check in the general, special and very special case that Q_i can always be computed by means of resultant. This is true if the coefficient H_i of e_0^2 in P_i does not vanish.*[4] *As the bonds are independent of the leg lengths, H_i has to vanish independently from $R_1, \ldots, R_6$. It can easily be seen that this cannot be the case without contradicting our assumptions.* ⋄

This closes the study of cases, where no three anchor points are collinear. Clearly in this case we can only obtain cylinders of revolution of type 1 and 3 of the list given in Section 3.

4.2 Three anchor points are collinear but no four anchor points are collinear

We assume that the first, second and fifth anchor points are collinear, which implies $b_5 = c_5 = 0$. We distinguish again the following three cases:

General case $e_3 \neq 0$. Everything can be done similarly to the general case of Section 4.1.

Special case $e_3 = 0$, $e_2 \neq 0$. Everything can be done similarly to the special case of Section 4.1.

Very special case $e_2 = e_3 = 0$. Everything can be done similarly to the very special case of Section 4.1. It should only be noted that in this case also P_5 factors into $N(R_1 - R_5)$. As a consequence not only Ω_2 is fulfilled identically in the corresponding computation of the cylinders of revolution, but also Ω_5.

Due to the collinearity of three anchor points and the exclusion of architecturally singular designs, we can only obtain cylinders of revolution

[4] There are no terms with e_0^3 and e_0 in P_i.

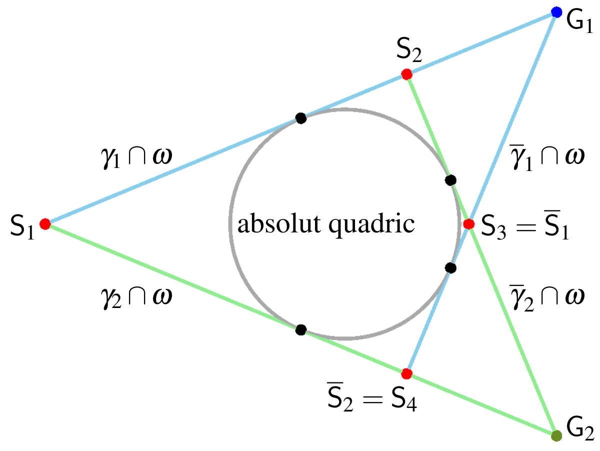

Figure 8: Sketch of the situation in the ideal plane ω: G_i denotes the ideal point of the line g_i for $i = 1, 2$. Moreover $\mathsf{S}_1, \ldots, \mathsf{S}_4$ denote the ideal points of the axes $\mathsf{s}_1, \ldots, \mathsf{s}_4$ of the four different cylinders of revolution of type 4. Note that the first and third solution as well as the second and fourth one are conjugate complex.

of type 4 of the list given in Section 3 if the carrier line of the three collinear points is no generator of Φ (general case and special one). Then the six points have to be located on two skew lines g_1 and g_2, where each line carries three pairwise distinct anchor points. For this non-planar congruent SG manipulator there always exists four cylinders of revolution of type 4 (cf. Section 5.1), which can easily be seen on basis of Fig. 8.

In the very special case, the carrier line of the three collinear points is a generator of Φ, thus we can only obtain a cylinder of revolution of type 1 of the list given in Section 3.

4.3 Four anchor points are collinear

As the anchor points are located on two skew lines, there also exist the above mentioned four cylinders of revolution of type 4 (cf. Fig. 8). Moreover there is one cylinder of revolution Φ of type 1, where the carrier line of the four collinear anchor points is a generator of Φ. The cylinder Φ is of multiplicity two, thus in sum this redundant manipulator (cf. Proof of Lemma 2) has the expected six solutions (cf. Remark 4).

The existence of bonds follows trivially from the second part of Lemma 2.

Remark 6 *With exception of this architecturally singular case, there are no examples known to the author, where the anchor points are located on more than four cylinder of revolution (cf. Sections 5.1 and 5.2).* $\diamond$

4.4 Sufficiency

In this section we want to point out that the condition that the six points are located on a cylinder of revolution of type 1, 3 or 4 is already sufficient for the existence of a self-motion over $\mathbb{C}$. Geometrically this is clear, as for each cylinder of revolution there exists the corresponding Schönflies self-motion, which was recognized for the modified Wren platform (cf. Section 2.2). This circumstance can also easily be verified algebraically as follows:

As for this Schönflies self-motion all legs have to have equal length, we set $R_1 = \ldots = R_6$. Then we do not have to eliminate e_0 by applying the resultant with $N = 0$, but P_i already equals F_i up to a non-zero factor. Therefore e_0 remains free and parametrizes the self-motion. This finishes the proof of Theorem 1. $\square$

5. SELF-MOTIONS

From the algebraic proof of the sufficiency given in Section 4.4, it also follows that the Schönflies self-motion is real if and only if the points are located on a real cylinder of revolution, as the direction $(s_1 : s_2 : s_3)$ of the axis s equals the direction $(e_1 : e_2 : e_3)$ of the rotation axis of the Schönflies self-motion.

But we cannot conclude from Theorem 1 that the Schönflies self-motions are the only non-translational self-motions, which can be performed by the manipulators characterized in Theorem 1. A trivial counter example is the architecture singularity, as the self-motions are the motions of the 5-legged manipulator, which results from the removal of one of the four legs, whose anchor points are collinear (cf. Lemma 2).

Further counter examples read as follows:

5.1 Butterfly self-motions

If $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear and $\mathsf{M}_4, \mathsf{M}_5, \mathsf{M}_6$ are collinear, then there also exist the following self-motion: If the platform is placed in a way that $\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6$ (resp. $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3$) are located on the carrier line g of $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ (resp. $\mathsf{M}_4, \mathsf{M}_5, \mathsf{M}_6$), then we get a pure rotational self-motion about g, which is called butterfly motion.

It is already known (cf. Examples 1 and 2 of [18]) that a Schönflies self-motion possesses only one bond (up to conjugation of coordinates), which is singular[5], and that a butterfly self-motion is given by two 1-parametric bonds (up to conjugation of coordinates).

As there are two butterfly self-motions we get four 1-parametric bonds (up to conjugation of coordinates). They possess four singular bonds (up to conjugation of coordinates), which correspond with the four cylinders of revolution of type 4 (cf. Fig. 8) implying four complex Schönflies self-motions.

5.2 New self-motions

We study non-planar congruent SG platforms which are plane-symmetric, i.e. the fourth, fifth and sixth anchor point are obtained by reflecting the first, second and third one on a plane ε. Therefore there always exists a cylinder of revolution Φ of type 1 with generators orthogonal to ε.

W.l.o.g. we can assume that ε is the xy-plane and that the rotation axis of Φ is the z-axis. Moreover we can eliminate the factor of similarity by setting the radius of Φ equal to 1. Finally we can rotate the coordinate system about the z-axis in a way that the i-th and j-th anchor point have the same y-coordinate, which results in the following coordinatization:

$$\begin{aligned} a_i &= a_{i+3} = \sin(\mu), & b_i &= b_{i+3} = \cos(\mu), \\ a_j &= a_{j+3} = \sin(-\mu), & b_j &= b_{j+3} = \cos(\mu), \\ a_k &= a_{k+3} = \sin(\lambda), & b_k &= b_{k+3} = \cos(\lambda), \end{aligned}$$

$$c_i = -c_{i+3} \neq 0,\ c_j = -c_{j+3} \neq 0,\ c_k = -c_{k+3} \neq 0$$

[5] A bond is called singular if it is located in the vertex space E of $N = 0$.

with pairwise distinct $i,j,k \in \{1,2,3\}$ and the angles $\mu \in (0,\pi)$ and $\lambda \in [0,2\pi)$.

Due to the plane-symmetry these six anchor points are located on further three cylinders of revolution[6] beside Φ, whose three axes of rotation are located within ε. Therefore the bond-set of this plane-symmetric congruent SG platform consists (up to conjugation of coordinates) of the four singular bonds of the Schönflies self-motions implied by the four cylinders of revolution. But beside these self-motions there exist the following ones characterized by $e_3 = 0$, which are new to the best knowledge of the author:

As we can assume $e_2 \neq 0$ w.l.o.g.[7], the unknowns f_1, f_2, f_3 can be computed from $\Psi, \Delta_{j,i}, \Delta_{i+3,i}$. If we plug the obtained expressions into $\Delta_{j+3,i}$, it can easily be seen that it vanishes for

$$R_{j+3}^2 = \frac{c_j}{c_i}(R_{i+3}^2 - R_i^2) + R_j^2.$$

Moreover, if additionally

$$R_{k+3}^2 = \frac{c_k}{c_i}(R_{i+3}^2 - R_i^2) + R_k^2$$

is fulfilled, $\Delta_{k,i} = \Delta_{k+3,i}$ holds. The numerator of this condition is a homogeneous polynomial P_k of degree 3 in the Euler parameters e_0, e_1, e_2. Note that P_k does not depend on f_0, which is determined by the only remaining equation $\Lambda_i = 0$. Hence, for given five design parameters $c_i, c_j, c_k, \mu, \lambda$, the cubic P_k implies a 4-parametric set $\mathscr{S}$ of self-motions, as it depends on the four leg lengths R_i, R_j, R_k, R_{i+3}.

Subset $\mathscr{X}$ of $\mathscr{S}$. For the following special choice of R_j and R_k:

$$R_j^2 = \frac{c_j}{2c_i}(R_i^2 - R_{i+3}^2) + \frac{1}{2}(R_i^2 + R_{i+3}^2),$$
$$R_k^2 = \frac{c_k}{2c_i}(R_i^2 - R_{i+3}^2) + \frac{1}{2}(R_i^2 + R_{i+3}^2),$$

P_k does not depend on the Euler parameter e_0 and the remaining leg lengths R_i and R_{i+3}. For reasons given later on, the three solutions of the cubic equation $P_k = 0$ with respect to e_1 and e_2 have to equal the axial directions of the three cylinder axes located in ε. Therefore each of these three directions determines a 2-parametric set $\mathscr{X}$ of Schönflies self-motions, as the leg lengths R_i and R_{i+3} can still be chosen arbitrarily. Note that the choice of R_i and R_{i+3} only influences the translational component of the Schönflies self-motion.

For the extra condition $R_i = R_{i+3}$ we obtain the 1-parametric set $\mathscr{X}_-$ of Schönflies self-motions with equal leg lengths $R_1 = \ldots = R_6$, which were recognized for the modified Wren platform (cf. Section 2.2). This proves the validity of the above given solutions of $P_k = 0$, as this cubic equation (and therefore its solutions) does not depend on R_i and R_{i+3}.

Remark 7 *Due to the limitation of length, a discussion of exemplary self-motions generated by a plane-symmetric congruent SG platform can be downloaded from the author's homepage. Furthermore animations of these self-motions are provided as supplementary data.* $\diamond$

5.3 Addendum

After finishing this paper the author became aware of Example 3.3.8 given by Husty et al. [11], where a full discussion of the self-motions of the following special congruent SG platform is given:

$$a_1 = -a_2 \neq 0, \; b_3 = -b_4 \neq 0, \; c_5 = -c_6 \neq 0,$$

and all remaining coordinates are zero.

Beside the 2-dimensional translational self-motion $\mathscr{T}$, there exist Schönflies self-motions[8] and more complicated self-motions of 4th order, which possess exact six points with spherical trajectories. The latter self-motions were firstly reported by Dietmaier in [4] and have the property $R_1 = R_2$, $R_3 = R_4$ and $R_5 = R_6$.

[6] Either all three are real (type 1) or one is real and the remaining two are conjugate complex (type 3).

[7] If $e_2 = 0$ holds, the self-motion is a Schönflies motion, where the rotation axis is parallel to the x-axis. But in this case i,j,k can be permuted ($\Rightarrow$ change of coordinate system) in a way that this parallelism vanishes ($\Rightarrow e_2 \neq 0$).

[8] Due to the symmetry of the manipulator there always exist cylinders of revolution through the six anchor points, which imply the Schönflies self-motions.

6. CONCLUSIONS AND OUTLOOK

The first research on cylinders of revolution through a given non-planar set of points by Schaal [22] was motivated kinematically. After a pure geometric and algebraic study done by several researchers (see e.g. [5], [24] and the references therein), it was shown within the paper at hand that this problem is strongly connected with the following one, which is again kinematic: The determination of all non-planar congruent SG platforms with a real non-translational self-motion. It turns out that the six base (resp. platform) anchor points have to be located on a cylinder of revolution (cf. Theorem 1). Note that this geometric characterization is also sufficient for the existence of self-motions over $\mathbb{C}$. Based on the modified Wren platform given in Section 2.2, this main theorem was proven by means of bond theory.

Although this result is known, a complete list of all possible non-translational self-motions of congruent SG platforms is still missing. All known examples are given in Section 5, where also a family of new self-motions is presented (cf. Section 5.2). Moreover a restriction of the sufficiency condition with respect to $\mathbb{R}$ remains also open.

In the future we are interested in the generalization of the presented result with respect to the linear coupling κ of the non-planar platform and the base. Based on the paper at hand, this was already done in [21] for the case that κ is a similarity. The problem for the case, where κ is an affinity or even a projectivity, remains open.

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ABOUT THE AUTHOR

Georg Nawratil is a senior researcher in the research group "Differential Geometry and Geometric Structures" at the Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Austria. His research interests include the geometry of mechanisms, kinematics, robotics and line geometry. He can be reached by email: nawratil@geometrie.tuwien.ac.at, by phone: +43-1-58801-104362, by fax: +43-1-58801-9104362, or through the postal address: Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8-10/104, A-1040 Vienna, Austria, Europe. More informations about the author can be found on his homepage: http://www.geometrie.tuwien.ac.at/nawratil/.