Singularity Distance for Parallel Manipulators of Stewart Gough Type

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Abstract. The number of applications of parallel robots, ranging from medical surgery to astronomy, has increased enormously during the last decades due to their advantages of high speed, stiffness, accuracy, load/weight ratio, etc. One of the drawbacks of these parallel robots are their singular configurations, where the manipulator has at least one uncontrollable instantaneous degree of freedom. Furthermore, the actuator forces can become very large, which may result in a breakdown of the mechanism. Therefore singularities have to be avoided. As a consequence the kinematic/robotic community is highly interested in evaluating the singularity closeness, but a geometric meaningful distance measure between a given manipulator configuration and the next singular configuration is still missing. We close this gap for parallel manipulators of Stewart Gough type by introducing such measures. Moreover the favored metric has a clear physical meaning, which is very important for the acceptance of this index by mechanical/constructional engineers.

Keywords: parallel robot, singularity, distance, metric

1 Introduction

Under the term "parallel manipulators of Stewart Gough (SG) type" we subsume the following three robot architectures (cf. Fig. 1) within this paper:

(A) Hexapod: The moving platform is connected via six spherical-prismatic-spherical (SPS) legs with the base. A hexapod is in a singular configuration[1] if and only if the six lines $\mathsf{l}_1, \ldots, \mathsf{l}_6$ spanned by the centers of corresponding spherical joints belong to a linear line complex [10].

(B) Linear pentapod: In this case the platform degenerates to a line, which is connected via five SPS-legs to the fixed base. The linear pentapod is shaky if and only if the five lines $\mathsf{l}_1, \ldots, \mathsf{l}_5$ belong to a congruence of lines.

(C) 3-RPR manipulator: The moving platform is connected via three rotational-prismatic-rotational (RPR) legs with the base. It is well known that this planar analogue of the hexapod is infinitesimal movable if and only if the three lines $\mathsf{l}_1, \mathsf{l}_2, \mathsf{l}_3$ belong to a pencil of lines.

[1] Also known as *shaky* configuration or *infinitesimal movable* configuration.

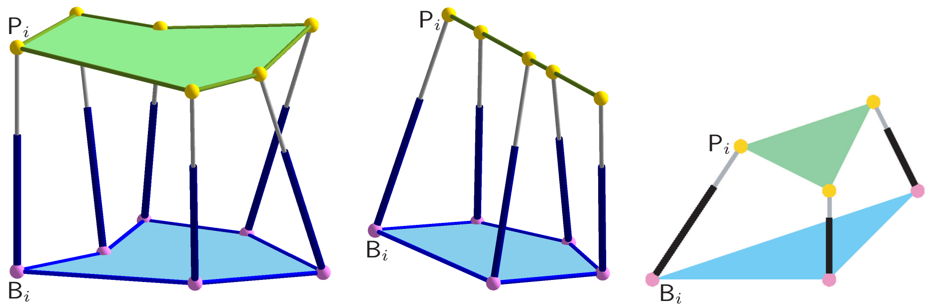

Fig. 1. Sketch of a hexapod (left), linear pentapod (center) and 3-RPR manipulator (right). For the planar mechanism as well as the spatial mechanical devices the anchor points of the legs are denoted by B_i (at the base) and P_i (at the platform). For all three parallel manipulators only the prismatic joints are active.

From the given geometric characterizations of shakiness an algebraic one (i.e. the equation of the singularity variety) can be obtained over the linear dependence of the Plücker coordinates of the involved n lines[2], which form also the rows of the manipulator's Jacobian matrix $\mathbf{J}$. Note that beside this line-geometric criterion one can also characterize singular poses as multiple solutions of the direct kinematic problem.

1.1 Review on the closeness to singular configuration

In the following we give a literature review on works dealing with the determination of the closest singular configuration to a given non-singular one:

- For 3-RPR manipulators the following two approaches has to be mentioned:
  - ⋆ Li et al [8] determined singularity-free zone around a non-singular configuration as follows: They parametrized the 3-dimensional configuration space by x, y, ζ, where x, y are the two position variables and ζ the orientation angle. Then point (x, y, ζ) of the singularity variety which minimizes the function
$$d := (x - x_0)^2 + (y - y_0)^2 \tag{1}$$
where (x_0, y_0) corresponds with the position of the given non-singular configuration. Note that the orientation of the given configuration is not taken into account thus $\sqrt{d}$ is the radius of the circular directrix centered in (x_0, y_0) of the "singularity-free cylinder". This concept was also used in [1].
  - ⋆ Zein et al [20] presented a procedure for the determination of a maximal singularity-free cube in the joint space centered in (ρ_1, ρ_2, ρ_3), where ρ_i is the length of the i-th leg in the given non-singular configuration. But the edge length e of this cube is not very well suited as a closeness index due to

[2] Note that in the context of hexapods $n = 6$ holds and that we have $n = 5$ and $n = 3$ for linear pentapods and 3-RPR manipulators, respectively.

the fact that the mapping from the configuration space to the joint space is 6 to 1 (cf. [4]). As in general not all six configurations, which correspond to a point on the singularity variety in the joint space, are singular ones, it can be the case that even in a non-singular configuration e equals zero.

- For hexapods Li et al [9] computed the "maximally singularity-free hypersphere" around a non-singular configuration as follows: They parametrized the 6-dimensional configuration space by $x, y, z, \theta, \varphi, \psi$, where x, y, z are the three position variables and θ, φ, ψ the Euler angles representing the orientation. Then the authors of [9] are looking for the point $(x, y, z, \theta, \varphi, \psi)$ of the singularity variety which minimizes the function

$$\begin{aligned} D :=& W\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right] + \\ & (1-W)\left[(\tan\tfrac{\theta}{2} - \tan\tfrac{\theta_0}{2})^2 + (\tan\tfrac{\varphi}{2} - \tan\tfrac{\varphi_0}{2})^2 + (\tan\tfrac{\psi}{2} - \tan\tfrac{\psi_0}{2})^2\right] \end{aligned}$$

where $(x_0, y_0, z_0, \theta_0, \varphi_0, \psi_0)$ corresponds with the given non-singular configuration and $W \in [0, 1]$ is a weighting coefficient, which can be used by the designer to "*favour either the position workspace or the orientation workspace*".

Li, Gosselin and Richard were aware of the drawbacks of their objective function [9, page 500]: "*... the above formulation poses the problem of defining a distance in the 6-D workspace in order to find the 'closest' point on the singularity manifold. Clearly, an Euclidean distance cannot be defined in this space since it is composed of mixed dimensions (position coordinates and orientation coordinates). Therefore, the above index D cannot be called a distance in the mathematical sense of the term and the singularity-free region obtained cannot properly speaking be termed a hyper-sphere.*"

Computing the distance to the next singularity for fixed orientation [9, 5] and position [9, 13], respectively, are further concepts known in kinematics but from these two separated informations no conclusion about the closeness to the next singular configuration within the n-dimensional configuration space can be drawn. Thus the question of a suitable distance function arises.

2 Distance function

It is well known (cf. Park [16] and Murray et al [12, page 427]), that there does not exist a bi-invariant[3] (positive-definite) metric on SE(3). Therefore it is not possible to define a geometric meaningful distance between two poses, which reasons the following statement in [11, page 275]: "*Measuring closeness between a pose and a singular configuration is a difficult problem: there exists no mathematical metric defining the distance between a prescribed pose and a given singular pose. Hence, a certain level of arbitrariness must be accepted in the definition of the distance to a singularity ...*"

[3] A metric is called bi-invariant if it is invariant with respect to changes of the fixed frame (left invariant) and the moving frame (right invariant), respectively.

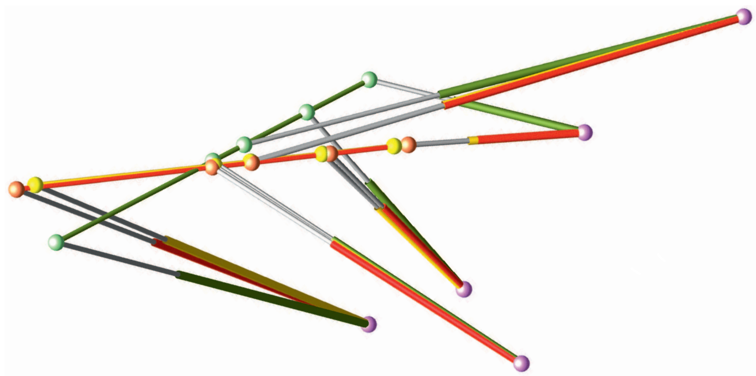

Fig. 2. A linear pentapod in the given (green) configuration and the closest singularity (red). The yellow configuration is the closest singularity under equiform motions.

Due to Park [16], there is an approach to come up with a geometric meaningful distance function, as he mentioned an alternative to distance metrics on SE(3) by changing the point of view as follows: One can consider the distance between two poses of the same rigid body, which yields so-called object depended metrics firstly studied by Kazerounian and Rastegar [6].

As the moving platform has n exceptionally points (i.e. platform anchor points) it suggests itself to measure the distance between two poses of the moving platform (given pose P_i and transformed pose P_i^{α}) by the distance measure

$$d_n := \sqrt{\frac{1}{n}\sum_{i=1}^{n}\langle \mathsf{P}_i^{\alpha} - \mathsf{P}_i, \mathsf{P}_i^{\alpha} - \mathsf{P}_i\rangle} \tag{2}$$

where $\langle\, , \rangle$ denotes the standard scalar product and $\alpha \in SE(3)$. A similar metric was introduced by Pottmann et al [17] within the context of motion design, which was also used by the author [14, Section 2] or Schröcker and Weber [19].

The considerations, done in this section so far, do not only hold for the configuration space SE(3) of hexapods, but also for the configuration space SE(2) of 3-RPR manipulators as well as the set of oriented line elements of $\mathbb{R}^3$, which is the configuration space of linear pentapods (cf. [15]).

2.1 Singularity distance

The distance function of Eq. (2) has been used by Rasoulzadeh and Nawratil [18] to compute the distance of linear pentapods to the next singularity (cf. Fig. 2). It turns out that the determination of the pedal points on the singularity variety with respect to the given configuration is an algebraic problem of degree 80, which can be relaxed by allowing $\alpha \in$ equiform motion group[4]. Then the

[4]The composition of Euclidean displacements and uniform scalings yields the group of equiform transformations.

degree drops to 28 and the corresponding solution is also drawn in Fig. 2. As the obtained distance of the relaxed problem is less or equal the distance of the original problem, it can be used as the radius of a hypersphere, which is guaranteed singularity-free. These results motivate the following systematic procedure for defining distance measures for parallel manipulators of SG type.

(A) For hexapods the set of transformations (α belongs to) can be extended step by step from the Euclidean group to

- ⋆ *equiform transformations*
- ⋆ *affine transformations*
- ⋆ *projective transformations*
- ⋆ *general transformations* which denote the mapping $\mathsf{P}_i \mapsto \mathsf{P}_i^{\alpha}$ for $i = 1, \ldots, n$.

The distance measure given in Eq. (2) has the following drawback: Assume we compute the distance p of a given configuration to the closest singularity in the sense of Eq. (2). Then we change our point of view by considering the platform as fixed and the base as moving part (i.e. platform and base are changing their roll) and compute again the distance to the next singularity according to Eq. (2). We get a second distance b which differs from p in the general case. This circumstance is less satisfactory from the geometric point of view.

Clearly, an ad hoc solution of this point of criticism would be $(b + p)/2$. Another more sophisticated approach is based on the idea to transform base and platform anchor points simultaneously and use the distance function

$$D_n := \sqrt{\tfrac{1}{2n}\sum_{i=1}^{n}\left[\langle \mathsf{P}_i^{\alpha} - \mathsf{P}_i, \mathsf{P}_i^{\alpha} - \mathsf{P}_i\rangle + \langle \mathsf{B}_i^{\beta} - \mathsf{B}_i, \mathsf{B}_i^{\beta} - \mathsf{B}_i\rangle\right]} \tag{3}$$

where B_i^{β} denote the transformed base points by the *base transformation* β.

Remark 1. Alternatively one can consider the shape space (e.g. [7]) of the n oriented line segments $\mathsf{P}_i\mathsf{B}_i$. Then D_n is a metric on this shape space, which is implied by the distance function between oriented line segments $\mathsf{P}_i\mathsf{B}_i$ and $\mathsf{P}_i^{\alpha}\mathsf{B}_i^{\beta}$ given in [15, Section 4.2]. From this point of view one can also use the distance function between oriented line segments $\mathsf{B}_i\mathsf{P}_i$ and $\mathsf{P}_i^{\alpha}\mathsf{B}_i^{\beta}$ introduced by Chen and Pottmann [3][5], which results in the following metric

$$\sqrt{\tfrac{1}{3n}\left[\sum_{i=1}^{n}\langle \mathsf{P}_i^{\alpha} - \mathsf{P}_i, \mathsf{P}_i^{\alpha} - \mathsf{P}_i\rangle + \langle \mathsf{B}_i^{\beta} - \mathsf{B}_i, \mathsf{B}_i^{\beta} - \mathsf{B}_i\rangle + \langle \mathsf{P}_i^{\alpha} - \mathsf{P}_i, \mathsf{B}_i^{\beta} - \mathsf{B}_i\rangle\right]} \tag{4}$$

on the mentioned shape space. But this metric does not fit with the kinematic reasoning of a singularity, as a singular configuration only depends on the pose of the base points and platform points but not on the leg itself; i.e. the connection between the two spherical joints has not to be a straight line segment but can have an arbitrary shape. ⋄

[5]This distance metric equals the square root of the mean of the squared distances of corresponding points over the entire line-segment (see also [15, Section 4.1]).

Then the singularity distance equals the minimizer of D_n under the side condition that the configuration of n lines $[\mathsf{P}_i^{\alpha}, \mathsf{B}_i^{\beta}]$ is singular. Clearly, the obtained singularity distance depends on the set (*Euclidean*, *equiform*, *affine*, *projective* or *general transformation*) both transformations α and β belong to. These singularity distances decrease (or remain unchanged) with respect to every extension step of the transformation set. Therefore all of them can be used as radius of a hypersphere, which is guaranteed singularity-free.

Let G_n denote the minimizer of Eq. (3), where the platform and the base transformations are both *general ones*. Due to the following important physical interpretation we favor this singularity distance G_n over all others possible singularity distances mentioned in this section.

Theorem 1. *If the radial clearance of the* $2n$ *passive joints is smaller than* G_n *then the parallel manipulator is guaranteed to be not in a singular configuration.*

Remark 2. The set of affine/projective/general transformations equipped with the metric d_n is an Euclidean space, thus in these three cases d_n is a geodesic distance. In contrast the embedding of the group of Euclidean/equiform transformations into the group of affine transformations yields a curved space $\mathcal{C}$, thus in these two cases d_n does not give the geodesic distance with respect to $\mathcal{C}$ (it gives the geodesic distance in the ambient space).

The same considerations hold for the metric D_n and the involved transformations α and β (belonging to the same set of transformations). $\diamond$

(B) For linear pentapods the singular distance G_5 can be defined as above for $n = 5$ and Theorem 1 holds too. Note that in this case the *general transformation* of the base is a projectivity if the base is non-planar. Moreover, euqiform and affine transformations affect the linear platform in the same way.

(C) For 3-RPR manipulators the singular distance G_3 can be defined as above for $n = 3$ and Theorem 1 holds too. In this case the *general transformation* of the base/platform is an affinity if the base/platform points are not collinear. Note that in context of Remark 2 planar equiform transformations imply geodesic distances (in contrast to the spatial case).

3 Results

The presented singularity distances are demonstrated on the basis of a 3-RPR manipulator as it is very well-suited for a graphical representation. The coordinates of the base/platform points with respect to the fixed/moving frame are:

$$\mathsf{B}_1 = \mathsf{P}_1 = (0,0)^T, \quad \mathsf{B}_2 = (11,0)^T, \quad \mathsf{B}_3 = (5,7)^T, \quad \mathsf{P}_2 = (3,0)^T, \quad \mathsf{P}_3 = (1,2)^T.$$

We consider the following one-parametric motion with parameter $\varphi \in [0, 2\pi[$:

$$\mathsf{P}_i \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \mathsf{P}_i + \frac{1}{2} \begin{pmatrix} 11 - 6\sin\varphi \\ 3 - 3\cos\varphi \end{pmatrix}. \tag{5}$$

For this 3-RPR manipulator we can extend the planar Euclidean motion group SE(2) only in the following two steps:

- ⋆ *planar equiform motion group*
- ⋆ *group of planar affine transformations*

We denote the singularity distance with respect to SE(2) and d_3 (resp. D_3) by s_3 (resp. S_3). Moreover we denote the singularity distance with respect to the equiform motion group and d_3 (resp. D_3) by e_3 (resp. E_3). Finally the singularity distance with respect to the affine motion group and d_3 (resp. D_3) is denoted by g_3 (resp. G_3). This notation is summarized in the following table:

	Euclidean group	Equiform group	Affine group
d_3	s_3	e_3	g_3
D_3	S_3	E_3	G_3

The constrained optimization problem is solved by the Lagrange approach. For its formulation we use the following notation:

$$\mathsf{P}_i^\alpha = (x_i, y_i)^T \qquad \mathsf{B}_i^\beta = (X_i, Y_i)^T \tag{6}$$

If α and β are affine transformations then we have $i = 1, 2, 3$. For an equiform transformation we set $i = 1, 2$ and

$$\mathsf{P}_3^\alpha = \mathsf{P}_1^\alpha + \left(\overrightarrow{\mathsf{P}_1^\alpha \mathsf{P}_2^\alpha} \quad \overrightarrow{\mathsf{P}_1^\alpha \mathsf{P}_2^\alpha}^\perp\right) \frac{\overrightarrow{\mathsf{P}_1 \mathsf{P}_3}}{\overline{\mathsf{P}_1 \mathsf{P}_2}} \qquad \mathsf{B}_3^\beta = \mathsf{B}_1^\beta + \left(\overrightarrow{\mathsf{B}_1^\beta \mathsf{B}_2^\beta} \quad \overrightarrow{\mathsf{B}_1^\beta \mathsf{B}_2^\beta}^\perp\right) \frac{\overrightarrow{\mathsf{B}_1 \mathsf{B}_3}}{\overline{\mathsf{B}_1 \mathsf{B}_2}} \tag{7}$$

where the $\perp$ sign indicates the rotation of the vector by 90°. Then the Lagrange function L for the computation of e_3, g_3 and E_3, G_3, respectively, can be written as

$$L: \quad d_3^2 - \lambda V_3 = 0 \qquad L: \quad D_3^2 - \lambda V_3 = 0 \tag{8}$$

where V_3 denotes the algebraic condition that the three legs of the transformed 3-RPR manipulator belong to a pencil of lines. If we add the conditions

$$M: \quad \overline{\mathsf{P}_1^\alpha \mathsf{P}_2^\alpha}^2 - \overline{\mathsf{P}_1 \mathsf{P}_2}^2 = 0 \qquad N: \quad \overline{\mathsf{B}_1^\beta \mathsf{B}_2^\beta}^2 - \overline{\mathsf{B}_1 \mathsf{B}_2}^2 = 0 \tag{9}$$

to the ansatz of Eq. (7), we end up with Euclidean displacements. Thus the Lagrange function L for computing s_3 and S_3, respectively, can be formulated as

$$L: \quad d_3^2 - \lambda V_3 - \mu M = 0 \qquad L: \quad D_3^2 - \lambda V_3 - \mu M - \nu N = 0 \tag{10}$$

In the following table the number u of unknowns (incl. the Lagrange multipliers) within the Lagrange function L are given:

singularity distance	s_3	e_3	g_3	S_3	E_3	G_3
u	6	5	7	11	9	13
# local extrema	32	19	22	88	34	50

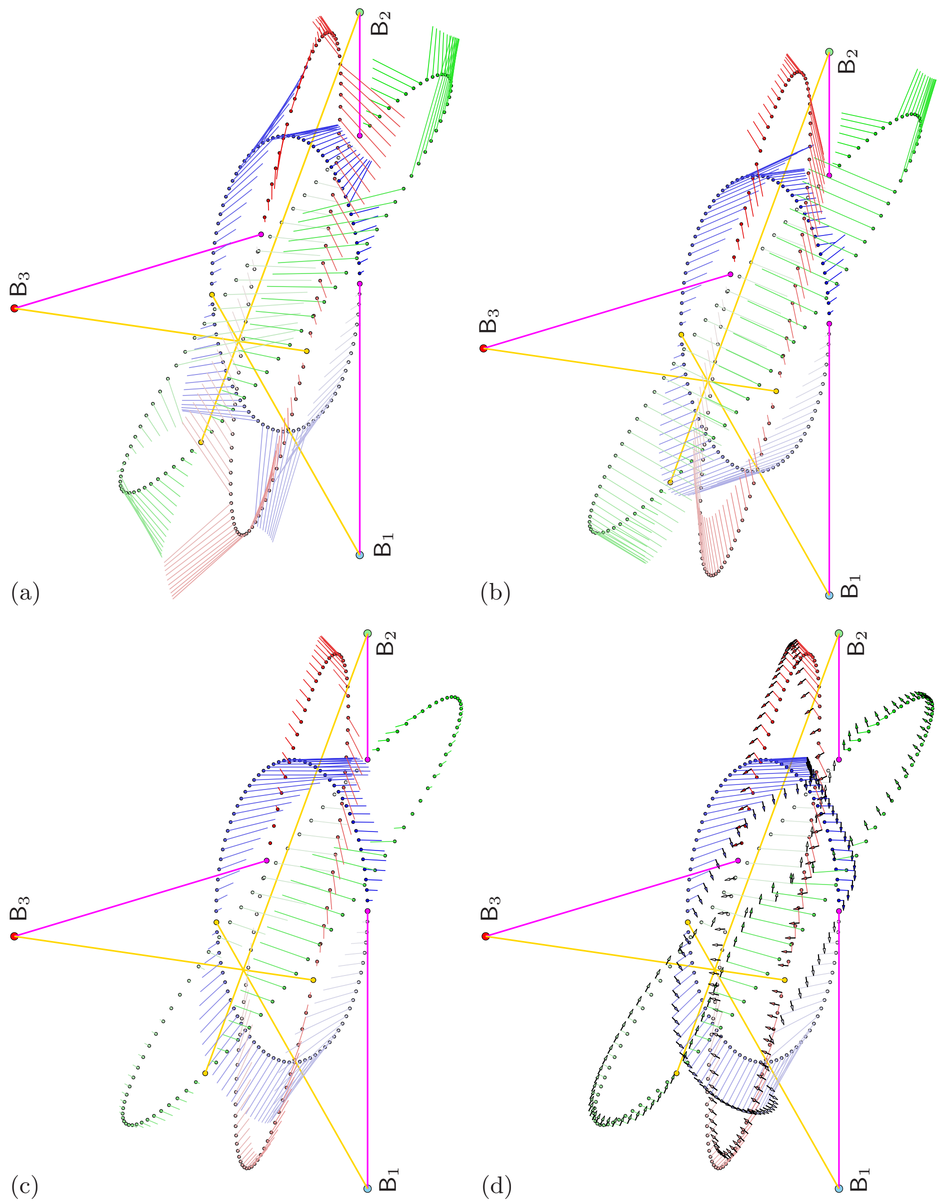

Fig. 3. For reasons of layout we rotated the four figures by 90°. The dots indicate the pose of the platform points and the end points of the attached lines correspond to the platform points of the closest singular configuration in the sense of (a) s_3, (b) e_3 and (c) g_3. In (d) the closest singular configuration with respect to G_3 is visualized. In the end-points of the attached lines we added orthogonal arrows indicating the direction of the leg in the closest singular configuration. The pedal points on these legs with respect to the corresponding base points B_i equal the base points of the closest singular configuration.

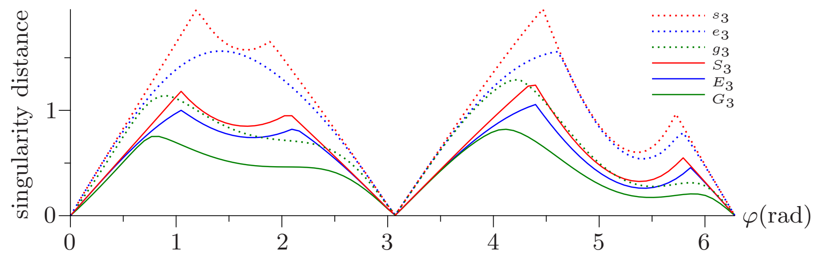

Fig. 4. Comparison of the singularity distances computed for the 3-RPR manipulator.

The system of u partial derivatives L_i $(i = 1, \dots, u)$ of L is solved using the Gröbner base method. For the case of G_3 the pseudo Maple code reads e.g. as:

$$[> \; B := Basis([L_1, \dots, L_{13}], tdeg(\lambda, x_1, y_1, X_1, Y_1, \dots x_3, y_3, X_3, Y_3)):$$
$$[> \; E := Basis([op(B)], plex(\lambda, x_1, y_1, X_1, Y_1, \dots x_3, y_3, X_3, Y_3)):$$

The degree of the univariate polynomial (given by $E[1]$ in the Maple code) equals the number of local extrema over $\mathbb{C}$ listed in the table above. Within this set of local extrema we pick out the one causing the smallest singularity distance.

For the illustration given in Fig. 3 the motion of Eq. (5) is discretized into 90 poses, where two of them are singular ones. In these two poses the legs are displayed in yellow and magenta, respectively. One has to point out the discontinuity of the closest singular pose in Fig. 3 (a,b), which is caused by passing through the cut locus[6] of the singularity variety.

A comparison of all proposed singularity distances is displayed in Fig. 4. In this context it should be noted that the replacement of $\frac{1}{n}$ by $\frac{1}{2n}$ in Eq. (2) for the computation of s_3, e_3, g_3 yields also an upper bound of S_3, E_3, G_3, respectively.

4 Conclusion and future research

We presented measures for evaluating the distance of a parallel manipulator of SG type to the next singularity and demonstrated them based on the 3-RPR manipulator. For the case of hexapods and linear pentapods the computation of the local extrema of the Lagrange function is in general no longer doable by Gröbner base method (due to the degree and number of unknowns). Therefore our future studies will use the homotopy continuation method (e.g. Bertini [2]). Clearly the proposed distance functions can also be adopted for redundant designs or other mechanisms (e.g. spherical 3-RPR manipulator).

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[6] The cut locus consists of all poses with more than one closest singular configuration with respect to the used distance function.

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