Linear Pentapods with a Simple Singularity Variety – Part I: Determination and Redundant Designs

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Abstract. There exists a bijection between the configuration space of a linear pentapod and all points $(u,v,w,p_x,p_y,p_z) \in \mathbb{R}^6$ located on the singular quadric $\Gamma : u^2+v^2+w^2=1$, where (u,v,w) determines the orientation of the linear platform and (p_x,p_y,p_z) its position. Then the set of all singular robot configurations is obtained by intersecting Γ with a cubic hypersurface Σ in $\mathbb{R}^6$, which is only quadratic in the orientation variables and position variables, respectively. This article investigates the restrictions to be imposed on the design of this mechanism in order to obtain a reduction in degree. In detail we study the cases where Σ is (1) linear in position variables, (2) linear in orientation variables and (3) quadratic in total. Finally we propose three kinematically redundant designs of linear pentapods with a simple singularity surface.

Keywords: Linear pentapods, Singularity variety, Design, Kinematic redundancy

1 Introduction and Review

A *linear pentapod* is defined as a five degree-of-freedom (DOF) *line-body component* of a Gough-Stewart platform consisting of a linear motion platform ℓ with five identical spherical-prismatic-spherical (S$\underline{\text{P}}$S) legs, where the prismatic joints are active and the rest are passive [1]. The pose of ℓ is uniquely determined by a position vector $\mathbf{p} \in \mathbb{R}^3$ and an orientation given by a unit-vector $\mathbf{i} \in \mathbb{R}^3$. The coordinate vector $\mathbf{m}_j$ of the platform anchor point m_j of the j-th leg is defined by the equation $\mathbf{m}_j = \mathbf{p} + r_j\mathbf{i}$ with $r_j \in \mathbb{R}$ and the base anchor points M_j of the j-th leg has coordinates $\mathbf{M}_j = (x_j, y_j, z_j)^T$ for $j = 1,\ldots,5$. In the following we list the results relevan It turns out that this kind of manipulator is an interesting alternative to serial robots handling axis-symmetric tools. The singularity analysis of linear pentapods, which are interesting alternatives to serial robots handling axis-symmetric tools, has undergone an acceptable level of investigations over the past few years. In the following sum up the relevant results for the paper at hand:

There exists a bijection between the configuration space of a linear pentapod and all points $(u,v,w,p_x,p_y,p_z) \in \mathbb{R}^6$ located on the singular quadric $\Gamma : u^2+v^2+w^2=1$, where (u,v,w) determines the orientation of the linear platform ℓ and (p_x,p_y,p_z) its position. Then the set of all singular robot configurations is obtained as the intersection of Γ with a cubic hypersurface Σ of $\mathbb{R}^6$, which can be written as $\Sigma : \det(\mathbf{S}) = 0$ (according to [2]), which from now on will be called *singularity polynomial*, with

$$\mathbf{S} = \begin{pmatrix} 1 & u & v & w & p_x & p_y & p_z \\ 0 & p_x & p_y & p_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u & v & w \\ r_2 & x_2 & y_2 & z_2 & r_2x_2 & r_2y_2 & r_2z_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_5 & x_5 & y_5 & z_5 & r_5x_5 & r_5y_5 & r_5z_5 \end{pmatrix}. \tag{1}$$

under the assumption that $x_1 = y_1 = z_1 = r_1 = 0$. Note that this assumption can always be made without loss of generality as the fixed/moving frame can always be chosen in a way that the first base/platform anchor point is its origin.

Remark 1. Until now the relation between the 7×7 matrix $\mathbf{S}$ and the five 6-tuples of homogenous Plücker coordinates implied by the legs of the linear pentapod is not well explained in the literature (cf. [3], [4] and [5]). For the interested reader this gap is closed in the Appendix. $\diamond$

A further well-studied field within the singularity analysis of linear pentapods are designs, which are singular in any configuration. These so-called architecture singular designs are completely classified in [6, Section 1.3]. Finally it should be noted, that Borràs and Thomas have studied how to move the leg attachments in the base and the platform of 5-S$\underline{\text{P}}$S linear pentapod without altering the robot's singularity locus (for a planar base see [5] and for a non-planar one see [2]).

1.1 Motivation and outline

Using a parallel manipulator with a simple singularity variety (with respect to the position variables) was first proposed by Karger [7] for the case of Stewart-Gough platforms[1]. This work was furthered in [8] and [9], where the necessary conditions for the design of Stewart-Gough platforms with linear or quadratic singularity surface with respect to positioning variables are determined.

It can easily be seen that the equation of the cubic hypersurface Σ is only quadratic in position as well as in orientation variables. Therefore the intention here is to find necessary conditions for the linear pentapods such that $\det(\mathbf{S}) = 0$ is:

- linear in position variables (cf. Section 2.1),
- linear in orientation variables (cf. Section 2.2),
- quadratic in total (cf. Section 2.3).

Clearly, due to the degree reduction it becomes easier to obtain closed form information about singular poses. But the main motivation for our research is the computational simplification of singularity-free zones (cf. [10]). The designs computed in Section 2 imply a degree reduction of the polynomials associated with the problem of determining singularity-free zones and even lead to singularity distances computable in closed form

[1] For Stewart-Gough platforms the singularity loci is in general cubic in the position variables.

[10], which offers interesting new concepts and strategies concerning path optimization [11] and singularity avoidance. In the latter context we propose three kinematic redundant linear pentapods with a simplified singularity variety (cf. Section 3) as the optimal reconfiguration of the base (regarding the distance from the singularity) can easily be obtained by the closed form solution.

1.2 Notation and preparatory work

The following notations are used in the rest of the paper:

- The compact notations $\mathbf{X} = (x_2, x_3, x_4, x_5)^T$, $\mathbf{Y} = (y_2, y_3, y_4, y_5)^T$, $\mathbf{Z} = (z_2, z_3, z_4, z_5)^T$ are introduced for the coordinates related to base anchor points.
- The compact notation $\mathbf{r} = (r_2, r_3, r_4, r_5)^T$ is used for the coordinates related to platform anchor points.
- The *component-wise* product of two vectors is given as $\mathbf{rX} = (r_2x_2, r_3x_3, r_4x_4, r_5x_5)^T$ and $\mathbf{rY}$ as well as $\mathbf{rZ}$ are defined analogously.
- For the sake of simplicity in notation as well as interpretation, we use the *bracket*:

$$[\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4] = \det(\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_4}) \quad \text{with} \quad \mathbf{A_i} \in \{\mathbf{r}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{rX}, \mathbf{rY}, \mathbf{rZ}\}. \quad (2)$$

  It is noteworthy that in the coming sections, the Roman letters inside the bracket are interpreted as points in projective space while the bold letters denote the corresponding homogeneous coordinates expressed as a vector.

A linear pentapod is an *architecturally singular manipulator* if for every position and orientation, the matrix of $\det(\mathbf{S})$ (cf. [3]) becomes rank deficient. By defining the *architecture matrix* of linear pentapods, namely:

$$\mathbf{A} = \big(\mathbf{r},\, \mathbf{X},\, \mathbf{Y},\, \mathbf{Z},\, \mathbf{rX},\, \mathbf{rY},\, \mathbf{rZ}\big). \quad (3)$$

we can identify such singularities by considering the rank deficiency of this matrix (obtained from the last four rows of the matrix $\mathbf{S}$, cf. [3]).

Lemma 1. *If the "architecture matrix" is rank deficient then the linear pentapod is an "architecturally singular manipulator" (cf.[4]).*

Since in computational kinematics most of the computations are of symbolic type, and naturally expensive in the sense of time consumption, it will be highly favourable if we are able to eliminate some extra symbols. The following lemma shows that it is possible to alleviate the burden of extra symbols in computations to come:

Lemma 2. *If the linear pentapod is not architecturally singular then there exists a triple of base points M_i, M_j and M_k which form a triangle and $m_i \neq m_j$ holds.*

Proof. For the proof please see [12].

Based on this lemma one can assume $M_1 = (0,0,0)$, $M_2 = (x_2,0,0)$ and $M_3 = (x_3, y_3, 0)$ where $x_2y_3 \neq 0$. Moreover due to $m_1 \neq m_2$ we can assume a scaling upon which, $r_2 = 1$ holds. The following Lemma would give us a geometric intuition of the coming algebraic computations in the later sections:

Lemma 3. *The "architecture matrix" is rank deficient iff the points* r, X, Y, Z, rX, rY *and* rZ *are coplanar in* $\mathbb{PR}^3$.

2 Simple singularity variety

2.1 Linear in p_x, p_y and p_z

For the determination of all non-architectural singular designs, where the singularity polynomial $\det(\mathbf{S}) = 0$ is only linear in position variables, we distinguish between linear pentapods with/without coplanar base anchor points (planar/non-planar case).

Planar case Assume that the manipulator is planar ($z_4 = z_5 = 0$). The desired goal is that all terms containing position variables of degree two should vanish. These terms form a polynomial, which we call the *undesired polynomial* through the remainder of the article. Here the *undesired polynomial* is as follows:

$$\det\left(\mathbf{S}\,{}^{4,7}_{1,2}\right) p_z^2 + \det\left(\mathbf{S}\,{}^{4,5}_{1,2}\right) p_x p_z - \det\left(\mathbf{S}\,{}^{4,6}_{1,2}\right) p_y p_z = 0. \tag{4}$$

If Eq. (4) is fulfilled independently of the position variables then all the coefficients have to be zero. Based on the resulting conditions one can prove the following theorem:

Theorem 1. *A non-architecturally singular linear pentapod with a planar base has a singularity polynomial linear in position variables, iff there is a singular affine mapping κ from the base plane to the platform line ℓ with $M_i \mapsto m_i$ for $i = 1, \ldots, 5$.*

Proof. Using *Laplace expansion by minors*, $\det\left(\mathbf{S}\,{}^{4,7}_{1,2}\right)$ is:

$$[\mathrm{r}, \mathrm{X}, \mathrm{Y}, \mathrm{rX}]v - [\mathrm{r}, \mathrm{X}, \mathrm{Y}, \mathrm{rY}]u = 0. \tag{5}$$

For all possible orientations, Eq. (5) holds whenever both bracket coefficients vanish. Again by considering the *Laplace expansion by minors* for $\det\left(\mathbf{S}\,{}^{4,5}_{1,2}\right)$ and $\det\left(\mathbf{S}\,{}^{4,6}_{1,2}\right)$ respectively, one obtains:

$$[\mathrm{r}, \mathrm{X}, \mathrm{Y}, \mathrm{rY}]w = [\mathrm{r}, \mathrm{X}, \mathrm{Y}, \mathrm{rX}]u = 0. \tag{6}$$

As it is also desired to have these equations vanished for all possible orientations, the bracket coefficients should be equal to zero simultaneously. Hence, independently of all possible orientations, the following statement holds:

$$\det\left(\mathbf{S}\,{}^{4,7}_{1,2}\right) \text{ vanishes} \Longleftrightarrow \det\left(\mathbf{S}\,{}^{4,5}_{1,2}\right) \text{ and } \det\left(\mathbf{S}\,{}^{4,6}_{1,2}\right) \text{ vanish.} \tag{7}$$

Finally, based on Eq. (7) the necessary and sufficient condition for having a singularity polynomial linear in position variables will be:

$$[\mathrm{r}, \mathrm{X}, \mathrm{Y}, \mathrm{rY}] = [\mathrm{r}, \mathrm{X}, \mathrm{Y}, \mathrm{rX}] = 0. \tag{8}$$

Using the literature of bracket algebra available at [13], [14] these brackets vanish whenever the four points characterizing them are coplanar. We denote the planes associated with the two brackets of Eq. (8)-left and Eq. (8)-right by $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. Then the following two cases have to be distinguished:

1. If the points r, X and Y are not collinear (or in other words if the vectors $\mathbf{r}$, $\mathbf{X}$ and $\mathbf{Y}$ are linearly independent) then the linear pentapod would be an *architecturally singular manipulator* since geometrically, by Lemma 3 this is equivalent to having the planes $\mathscr{P}_1$ and $\mathscr{P}_2$ coincident.
2. If the points r, X and Y are collinear (or in other words if the vectors $\mathbf{r}$, $\mathbf{X}$ and $\mathbf{Y}$ are linearly dependent) then $\mathbf{r} \in \operatorname{span}\{\mathbf{X}, \mathbf{Y}\}$; i.e.

$$\mathbf{r} = \alpha.\mathbf{X} + \beta.\mathbf{Y} \quad \text{with} \quad \alpha, \beta \in \mathbb{R}. \tag{9}$$

This results the affine coupling κ mentioned in Theorem 1, namely:

$$\kappa: \ (x_i, y_i) \longmapsto r_i = \alpha x_i + \beta y_i \quad \text{with} \quad \alpha = \tfrac{1}{x_2} \quad \text{and} \quad i = 2, \ldots, 5. \tag{10}$$

Note that the planes $\mathscr{P}_1$ and $\mathscr{P}_2$ do not necessarily coincide in this case.

Non-planar case For this case we can only prove (cf. [12]) the following theorem:

Theorem 2. *Non-architecturally singular linear pentapods with a non-planar base possessing a singularity polynomial, linear in position variables do not exist.*

In total the results of Section 2.1 show that the manipulator given in Theorem 1 is the only one with a singularity variety linear in position variables. This manipulator design was already known to the authors of [5][2], who also pointed out that the forward kinematics of these pentapods can be solved quadratically.

2.2 Linear in *u*, *v* and *w*

In this section we determine all non-architecturally singular designs where the singularity polynomial $\det(\mathbf{S}) = 0$ is only linear in orientation variables. As in Section 2.1 we distinguish between linear pentapods with planar and non-planar bases.

Planar case Under the planar condition ($z_4 = z_5 = 0$) the *undesired polynomial* is:

$$\left[\det\left(\mathbf{S}\,^{3,7}_{1,3}\right) + \det\left(\mathbf{S}\,^{4,6}_{1,3}\right)\right] vw + \left[\det\left(\mathbf{S}\,^{2,7}_{1,3}\right) - \det\left(\mathbf{S}\,^{4,5}_{1,3}\right)\right] uw + \det\left(\mathbf{S}\,^{2,6}_{1,3}\right) uv - \det\left(\mathbf{S}\,^{4,7}_{1,3}\right) w^2 = 0. \tag{11}$$

Theorem 3. *A non-architecturally singular linear pentapod with a planar base has a singularity polynomial linear in orientation variables in the following cases (using a combinatorial classification):*

1. *M_2, M_3, M_4, M_5 are collinear,*
2. *$m_1 = m_i$ and M_j, M_k, M_l are collinear with pairwise distinct $i, j, k, l \in \{2, 3, 4, 5\}$,*
3. *$m_1 = m_i = m_j$ with pairwise distinct $i, j \in \{2, 3, 4, 5\}$.*

[2]Note that Theorems 1 and 2 cannot be concluded from [5], as the authors restricted to the planar case with no four anchor points aligned.

Fig. 1. Three possible designs mentioned in Theorem 3. It can be shown by a series of Δ-transforms [15], that the singularity loci of all the three combinatorial cases are identical.

Proof. Eq. (11), independently of the orientation variables, gives $\det\left(\mathbf{S}\,{}^{4,7}_{1,3}\right)=0$. Once again, by resorting to the literature of *brackets*, $\det\left(\mathbf{S}\,{}^{4,7}_{1,3}\right)=0$ if and only if the following holds:

$$[\mathrm{r},\mathrm{Y},\mathrm{rX},\mathrm{rY}]=[\mathrm{r},\mathrm{X},\mathrm{rX},\mathrm{rY}]=0. \tag{12}$$

Now, name the plane characterized by the points r, rX and rY as $\mathscr{P}$. If the points r, rX and rY are not collinear then the plane $\mathscr{P}$ is defined *uniquely* and hence by Eq. (12) X and Y are also on $\mathscr{P}$ which by Lemma 3 results in an architecture singularity.

On the other hand if the points r, rX and rY are collinear then there is the possibility of having the points X and Y on two different planes which does not necessarily lead to an *architectural singularity*. Under this assumption, we get $\mathbf{r}\in\operatorname{span}\{\mathbf{rX},\mathbf{rY}\}$; i.e.

$$\mathbf{r}=\alpha.\mathbf{rX}+\beta.\mathbf{rY} \quad \text{with} \quad \alpha,\beta\in\mathbb{R} \tag{13}$$

where $(\alpha,\beta)\neq(0,0)$ holds. Now having Eq. (13) in mind, the following possibilities arise for a non-architecturally singular design:

1. $\forall\, i\in\{2,...,5\}, r_i \neq 0$. Then Eq. (13) can be rewritten as $\mathbf{1}=\alpha.\mathbf{X}+\beta.\mathbf{Y}$ which means that the base points M_2, M_3, M_4, M_5 are collinear (Fig. 1-right).
2. $\exists!\ i\in\{3,4,5\}$ such that $r_i=0$. This case yields $m_1=m_i$ and M_2, M_j, M_k are collinear with pairwise distinct $i,j,k\in\{3,4,5\}$ (Fig. 1-center).
3. $\exists\, i$ and $j\in\{3,4,5\}$, where $i\neq j$ such that $r_i=r_j=0$. Now we get $m_1=m_i=m_j$ with pairwise distinct $i,j\in\{3,4,5\}$ (Fig. 1-left).

Non-planar case For this case we can only prove (cf. [12]) the following theorem:

Theorem 4. *Non-architecturally singular linear pentapods with a non-planar base possessing a singularity polynomial linear in orientation variables do not exist.*

2.3 Quadratic

In this section we study linear pentapods where the singularity polynomial is only quadratic in total. We are only able to prove (cf. [12]) the following negative result:

Theorem 5. *Non-architecturally singular linear pentapods possessing a singularity polynomial, which is quadratic in pose variables, do not exist.*

3 Kinematic redundant designs

A certain drawback of parallel robots is the limitation of their singularity-free workspace, which can be overcome by the concept of redundancy. A review on the different types of redundancy for parallel robots with S$\underline{\text{P}}$S-legs including a discussion of their pros and cons is given by the authors in [16]. Following these arguments, we are preferring the concept of kinematic redundancy by reconfiguring the base anchor points of the the pentapod by additional joints. In the case of a given path of the platform, the kinematic redundant dofs can be used to avoid singularities (if possible[3]) and to increase the performance of the manipulator during the prescribed motion [3].

3.1 Design 1

This design, displayed in Fig. 2-left, is based on the idea to change the coefficient β of the affine coupling κ given in Eq. (10) by a reconfiguration of the base. This can be achieved by a suitable sliding of the base points. The fibers of the singular affine transformation κ from the base plane to the platform correspond to parallel lines in the base plane. It is well known (cf. Section 4.3 of [4]) that a reconfiguration of a base point along its corresponding fiber does not change the singularity variety. Therefore it suggests itself to mount the sliders orthogonal to the fiber-direction. This sliding gives the first degree of kinematic redundancy.

Remark 2. The linear pentapod given in Fig. 2-left has been designed in a symmetric way, such that the sliders of M_i and M_{i+1} (for $i = 2,4$) have to move with the same velocity (but in opposite directions). Note that one can drive all sliders of $M_2,\ldots,M_5$ with only one motor and a fixed gearing, as the ratio of the velocities of the sliders of M_2 and M_4 is constant. Moreover it can easily be checked, that the symmetric design proposed in Fig. 2-left, can never be architecturally singular in practice. $\diamond$

The second degree of kinematic redundancy is achieved by the sliding of the first base point in fiber-direction. This will not affect the singularity surface, but it can be used to increase the performance of the manipulator during an end-effector motion [3].

3.2 Design 2

This design, based on item 1 of Theorem 3 and displayed in Fig. 2-right, is also a 2-dof kinematically redundant pentapod with planar base, which has the property that its singular polynomial is linear in orientation for all possible configurations. The base points $M_2,\ldots,M_5$ are collinearly mounted on a rod g, which slides (active joint) along a circular rail on the ground and is connected over a U-joint (passive joint) with the ceiling. Therefore the rod g generates during the motion a right circular cone. For a better understanding of the redundant dofs, we have a look at the singular-invariant replacement of legs keeping the given platform anchor points:

[3] The singularity variety is a hypersurface in the mechanism's configuration space; thus two points of the configuration space can be separated by this hypersurface.

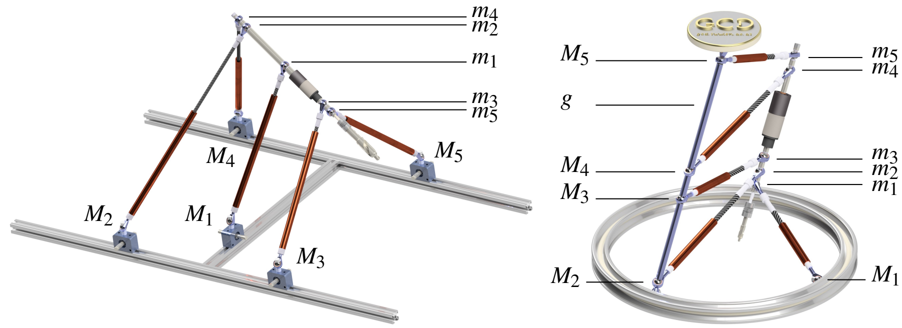

Fig. 2. Left: Kinematic redundant linear pentapod of Section 3.1 with a linear singularity variety in position variables. Right: Kinematic redundant linear pentapod of Section 3.2 with a linear singularity variety in orientation variables. The suggested design, where the upper part is mounted on the ceiling, can be of interest for e.g. the milling of an object without any need of its repositioning, as the manipulator can go around the object by 360 degrees.

- ⋆ As this linear pentapod contains a line-line component (cf.[15]), one can relocate the base anchor points of the legs $m_2M_2, \ldots, m_5M_5$ arbitrarily on g (assumed that the resulting manipulator is not architecturally singular).

  Remark 3. One can additionally allow a sliding (by active joints) of the base points along the rod g (yielding further degrees of kinematic redundancy) but this will not change the singularity variety. These reconfigurations can only be used to improve the performance of the manipulator. ◇
- ⋆ The base point of the first leg can be replaced by any point of the plane spanned by M_1 and g (assumed that the resulting manipulator is not architecturally singular). Therefore a sliding of M_1 along the circular rail changes the singularity variety.

3.3 Design 3

This design, based on item 2 of Theorem 3 and displayed in Fig. 3, is a 3-dof kinematically redundant pentapod with planar base, which has the property that its singular polynomial is linear in orientation for all possible configurations. The anchor points M_1 and M_2 can slide along a circular rail (two active joints). The third degree of kinematic redundancy is obtained by the rotation of the rod g on which the collinear points M_3, M_4, M_5 are mounted. For a better understanding of the redundant dofs, we study again the singular-invariant leg-replacements keeping the given platform anchor points:

- ⋆ One can relocate the base anchor points of the legs m_3M_3, m_4M_4, m_5M_5 arbitrarily on g (assumed that the resulting manipulator is not architecturally singular). Therefore also Remark 3 holds in this context.
- ⋆ The base points of the first and second leg can be replaced by any two points of the carrier plane of the circular rail (assumed that the resulting manipulator is not architecturally singular). As a consequence the sliding of M_1 and M_2 along the circular rail does not change the singularity variety. Therefore these two redundant dofs can only be used to improve the performance of the manipulator.

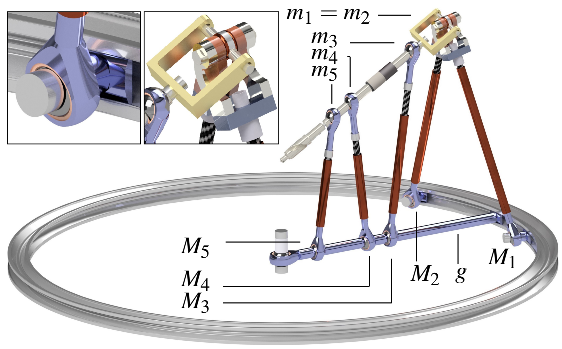

Fig. 3. Kinematic redundant linear pentapod of Section 3.3 with a linear singularity variety in orientation variables. This design also allows a milling by 360 degrees around the object. Moreover, detailed views of the circular slider of M_2 and the double joint $m_1 = m_2$ are provided. In this context it should be noted that a design, based on item 3 of Theorem 3, is not suited for technical realization due to the triple joint at the platform.

4 Conclusions

In this paper we computed linear pentapods with a simplified singularity variety. In detail we determined all non-architecturally singular designs where the singularity polynomial is linear in position variables (cf. Section 2.1) or orientation variables (cf. Section 2.2). Moreover we were able to prove that linear pentapods with a quadratic singularity polynomial do not exist (cf. Section 2.3). Finally three kinematic redundant linear pentapods with a simplified singularity variety were proposed in Section 3.

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Appendix

Kinematic singularities occur whenever the Jacobian matrix **J** becomes rank deficient, where **J** can be written as follows (cf. [2]):

$$\mathbf{J} = \begin{pmatrix} \mathbf{l}_1 & \ldots & \mathbf{l}_5 \\ \hat{\mathbf{l}}_1 & \ldots & \hat{\mathbf{l}}_5 \end{pmatrix}^T \text{ with } \mathbf{l}_j = \begin{pmatrix} p_x + r_j u - x_j \\ p_y + r_j v - y_j \\ p_z + r_j w - z_j \end{pmatrix}, \ \hat{\mathbf{l}}_j = \begin{pmatrix} z_j(p_y + r_j v) - y_j(p_z + r_j w) \\ x_j(p_z + r_j w) - z_j(p_x + r_j u) \\ y_j(p_x + r_j u) - x_j(p_y + r_j v) \end{pmatrix}.$$

As $(\mathbf{l}_j, \hat{\mathbf{l}}_j)$ are the Plücker coordinates of the jth leg, the condition $rk(\mathbf{J}) < 5$ is equivalent with the statement that the five legs belong to a linear line congruence (cf. [17]). Now the idea is to add a sixth line in a way that it does not belong to this line congruence for all poses of ℓ. The simplest way for doing that is to consider the ideal line of a plane perpendicular to ℓ, which has Plücker coordinates: $(\mathbf{l}_6, \hat{\mathbf{l}}_6) := (0,0,0,u,v,w)$. This line cannot belong to the line congruence because it does not intersect the linear platform ℓ. Therefore $rk(\mathbf{J}) < 5$ is equivalent to $rk(\mathbf{J}_+) < 6$ and $rk(\mathbf{S}^*) < 7$, respectively, with

$$\mathbf{J}_+ = \begin{pmatrix} \mathbf{l}_1 & \dots & \mathbf{l}_6 \\ \hat{\mathbf{l}}_1 & \dots & \hat{\mathbf{l}}_6 \end{pmatrix}^T \quad \text{and} \quad \mathbf{S}^* = \begin{pmatrix} 1 & \mathbf{o} \\ \mathbf{o} & \mathbf{J}_+ \end{pmatrix}. \tag{14}$$

As $\mathbf{J}_+$ and $\mathbf{S}^*$ are square matrices, the singularities are characterized by $\det(\mathbf{J}_+) = \det(\mathbf{S}^*) = 0$. By applying row and column operations to $\mathbf{S}^*$, we obtain $\mathbf{S}$ of Eq. (1).

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