# On flexes associated with higher-order flexible bar-joint frameworks


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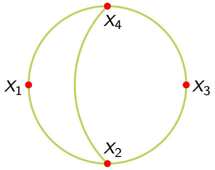
# Fundamentals

# Bar-joint framework

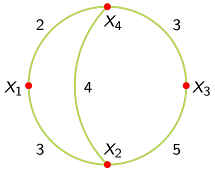
## Graph $G$ of a framework

consists of $w$ knots $X_1, \ldots, X_w$, which are connected by $e$ edges ($\Rightarrow$ combinatorial structure).



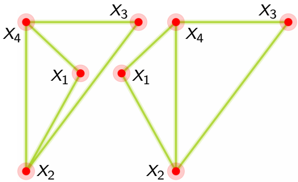

## Inner geometry

is determined by assigning to each edge a non-zero length ($\Leftrightarrow$ fixing intrinsic metric).




## Realization $G(\mathbf{X})$

with $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_w)$ corresponds to the embedding of the framework with fixed inner geometry into the Euclidean $s$-space. Let's assume $s = 2$.



# Algebraic definitions for flexibility and rigidity

The relation that two knots $X_i$ and $X_j$ are connected by an edge of length $L_{ij}$ can also be expressed algebraically as $\|\mathbf{x}_i - \mathbf{x}_j\|^2 - L_{ij}^2 = 0$.

This implies $e$ quadratic conditions $c_1, \ldots, c_e$ in $m = 2w - 3$ unknowns (after eliminating isometries of the complete framework) constituting an algebraic variety $V(c_1, \ldots, c_e)$.

**Definition:** *A realization is* **flexible**

if it belongs to a (real) positive-dimensional component of $V(c_1, \ldots, c_e)$.

**Definition:** *A realization is* **rigid**

if it corresponds to a real isolated solution of $V(c_1, \ldots, c_e)$.

# Infinitesimal flexibility and rigidity

We can compute in a realization the tangent-hyperplane to each of the hypersurfaces $c_i = 0$ in $\mathbb{R}^m$ for $i = 1, \ldots, e$. The normal vectors $\nabla c_i$ of these tangent-hyperplanes constitute the columns of the $m \times e$ **rigidity matrix** $\mathbf{R}_{G(\mathbf{X})}$ of the realization $G(\mathbf{X})$; i.e.

$$\mathbf{R}_{G(\mathbf{X})} = (\nabla c_1, \nabla c_2, \ldots, \nabla c_e)$$

For $rk(\mathbf{R}_{G(\mathbf{K})}) = m$ the realization $G(\mathbf{K})$ is **infinitesimal rigid**.

For $rk(\mathbf{R}_{G(\mathbf{K})}) < m$ the realization $G(\mathbf{K})$ is **infinitesimal flexible**; i.e. the hyperplanes have a positive-dimensional affine subspace in common. Therefore the intersection multiplicity of the $e$ hypersurfaces is at least two in an infinitesimal flexible realization.

# Flexion order of a bar-joint framework

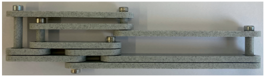
According to Nawratil* the flexion order of a bar-joint framework can be defined as follows:

**Definition of the flexion order of a bar-joint framework**

If a configuration does not belong to a continuous flexion of the framework then we define its flexion order $r$ by the number of coinciding framework realizations minus 1.
For configuration belonging to a continuous flexion we have $r = \infty$.

**Open Question:** What are the flexes associated with a bar-joint framework of flexion order $r$?

**Example:** Bar-joint framework with $r = 23$.
What are the associated flexes?



* A global approach for the redefinition of higher-order flexibility and rigidity. Mechanism and Machine Theory **205**:105853 (2025)

# Review

# Review on higher-order flexes

According to Stachel* the *classical* definition of a higher-order flex can be written as follows:

**Classical Definition:** *A framework has a $n^{th}$-order flex*

if for each vertex $\mathbf{x}_i$ $(i = 1, \ldots, w)$ there is a polynomial function

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}t + \ldots + \mathbf{x}_{i,n}t^n \quad \text{with} \quad n > 0$$

such that

1. the replacement of $\mathbf{x}_i$ by $\mathbf{x}_i'$ in the equations $c_1, \ldots, c_e$ gives stationary values of multiplicity $\geq n+1$ at $t = 0$; i.e.

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 - L_{ij}^2 = 0 \quad \Longrightarrow \quad \|\mathbf{x}_i' - \mathbf{x}_j'\|^2 - L_{ij}^2 = o(t^n)$$

2. the velocity vectors $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ do not originate from a rigid body motion (incl. standstill) of the complete framework.

* **Stachel, H.:** A proposal for a proper definition of higher-order rigidity. (Slides) Tensegrity Workshop, La Vacquerie, France (2007)

# Review on higher-order flexes

Based on the classical definition of $n^{th}$-order flex one can define $n^{th}$-order rigidity as follows according to Connelly & Servatius*:

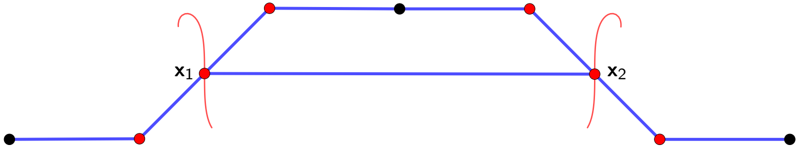
**Classical Definition:** *A framework is $n^{th}$-order rigid*

if every $n^{th}$-order flex has $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ trivial as a first-order flex; i.e. it originates from a rigid body motion (incl. standstill) of the complete framework.

The **double-Watt mechanism** of Connelly & Servatius* raises some problems concerning these classical definitions, as they attest the mechanism in a certain configuration a 3rd-order rigidity which conflicts with its continuous flexibility; i.e. a proper definition should imply rigidity from $n^{th}$-order rigidity.

* **Connelly, R., Servatius, H.:** Higher-order rigidity - What is the proper definition? Discrete & Comp. Geometry **11**:193–200 (1994)

# Double-Watt mechanism of Connelly & Servatius




**The dimensions of each Watt mechanism**

The arms have length 1 and the couplers length $\sqrt{2}$. The midpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ of both couplers are connected by a bar of length 3.

The problematic configuration corresponds to a cusp in the configuration space; i.e. the mechanism has an instantaneous standstill.

Further cusp mechanisms were given by Lopez-Custodio et al. in


**Lopez-Custodio, P.C., Müller, A., Rico, J.M., Dai, J.S.:** A synthesis method for 1-dof mechanisms with a cusp in the configuration space. Mechanism and Machine Theory **132**:154–175 (2019)

# Stachel's attempt to resolve the dilemma

Stachel's approach follows the more general notation of $(k, n)$-flexibility suggested by Sabitov and was presented in

**Stachel, H.:** A proposal for a proper definition of higher-order rigidity. (Slides) Tensegrity Workshop, La Vacquerie, France (2007)

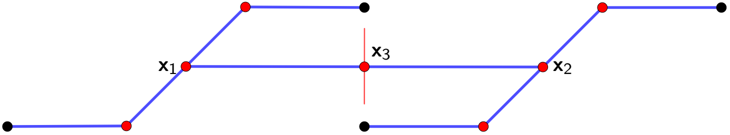
$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}t + \ldots + \mathbf{x}_{i,n}t^n \quad \Longrightarrow$$
$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,k}t^k + \ldots + \mathbf{x}_{i,n}t^n \quad \text{with} \quad n \geq k > 0$$

where $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{w,k}$ are non-trivial.
In addition the **(k,n)-flex has to be irreducible**; this means that the flex does not result from a polynomial parameter substitution

$$t = \bar{t}^q(a_0 + a_1\bar{t} + a_2\bar{t}^2 + \ldots) \quad \text{with} \quad a_0 \neq 0 \quad \text{and} \quad q > 1$$

of a lower-order flex. With this approach Stachel was able to show that the double-Watt mechanism has a $(2, \infty)$-flex.

# New dilemma: Extended Double-Watt mechanism




Stachel's proposal was only presented at the Tensegrity Workshop in 2007. It remained unpublished as another dilemma arose; namely no unique $(k, n)$-flex can be identified for another double-Watt mechanism extended by a linear point-guidance of the coupler midpoint;

**Stachel, H.:** A (3,8)-flexible bar-and-joint framework? (Slides) AIM Workshop rigidity & polyhedral combinatorics, Palo Alto, USA (2007)

For the resulting rigid framework Stachel ended up with an infinite sequence of irreducible $(k, 3k-1)$-flexes for $k = 1, 2, \ldots$.

**Which is the correct (k,n)-flex? The problem is not yet settled!**

# Solution

# Preliminary considerations

The problem in Stachel's approach is that the correct value for $k$ is unknown. **What is the meaning of k?**

In the limit case $n = \infty$ the flexes correspond to branches of algebraic curve, which can be locally parametrized by Puiseux series; cf.

**Burau, W.:** Algebraische Kurven und Flächen I & II. Walter De Gruyter & Co. (1962)

**Semple, J.G., Kneebone, G.T.:** Algebraic Curves. Oxford University Press (1959)

**Walker, R.J.:** Algebraic Curves. Springer (1978)

This theory is well established for planar algebraic curves, but it can also be extended to the non-planar case; e.g.

**Alonso, M.W., et al.:** Local Parametrization of Space Curves at Singular Points. Computer Graphics and Mathematics, 61–90, Springer (1992)

**Jensen, A.N., Markwig, H., Markwig, T.:** An algorithm for lifting points in a tropical variety. Collectanea Mathematica **59**:129–165 (2008)

**Maurer, J.:** Puiseux expansion for space curves. Manuscripta Mathematica **32**:91–100 (1980)

# Minimal parametrization: Algebraic planar curves

Each branch through a point (w.l.o.g. the origin) of a planar algebraic curve $C$, which is given by the zero-set of a polynomial $P(x, y)$, can be parametrized by a pair of convergent power series.
This results from the **Theorem of Puiseux** that

$$P(x, y) = \alpha_0(x) + \alpha_1(x)y + \ldots + \alpha_n(x)y^n = \prod_{u=1}^{n} (y - S_u(x))$$

where $S_u(x)$ is the **Puiseux series** of the form

$$S_u(x) = \beta_{u,1}x^{\frac{\nu_{u,1}}{\nu_{u,0}}} + \beta_{u,2}x^{\frac{\nu_{u,2}}{\nu_{u,0}}} + \ldots, \quad 0 < \nu_{u,0},\, 0 < \nu_{u,1} < \nu_{u,2} < \ldots$$

and relative prime $\nu_{u,0}, \nu_{u,1}, \nu_{u,2}, \ldots$ obtained by Newton diagrams.
From this we get the **minimal parametrization** of the branch as:

$$x_u(t) = t^{\nu_{u,0}}, \quad y_u(t) = \beta_{u,1}t^{\nu_{u,1}} + \beta_{u,2}t^{\nu_{u,2}} + \ldots.$$

with order $\min(\nu_{u,0}, \nu_{u,1})$.

# Example: Planar quartic curve

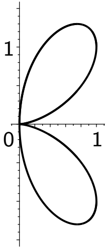
We study the planar quartic curve $C$ with

$$P(x,y) = (x^2+y^2-2x)^2 + 4x^3 - 4x^2$$

We get $P(x,y) = \prod_{u=1}^{4}(y - S_u(x))$ with

$$S_{1,3}(x) = \pm 2x^{\frac{1}{2}} \mp \tfrac{1}{2}x^{\frac{3}{2}} \mp \tfrac{1}{8}x^{\frac{5}{2}} \mp \tfrac{1}{16}x^{\frac{7}{2}} \mp \tfrac{5}{128}x^{\frac{9}{2}} \mp \dots$$

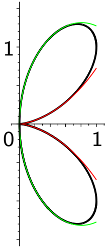
$$S_{2,4}(x) = \pm \tfrac{1}{2}x^{\frac{3}{2}} \pm \tfrac{1}{8}x^{\frac{5}{2}} \pm \tfrac{1}{16}x^{\frac{7}{2}} \pm \tfrac{5}{128}x^{\frac{9}{2}} \pm \dots$$




ending up with the minimal parametrizations

$$x_1(t) = t^2, \quad y_1(t) = 2t - \tfrac{1}{2}t^3 - \tfrac{1}{8}t^5 - \tfrac{1}{16}t^7 - \tfrac{5}{128}t^9 - \dots$$

$$x_2(t) = t^2, \quad y_2(t) = \tfrac{1}{2}t^3 + \tfrac{1}{8}t^5 + \tfrac{1}{16}t^7 + \tfrac{5}{128}t^9 + \dots$$




which are not unique for $\nu_{u,0} > 1$ as the substitution of $t$ by $\epsilon t$ yields and equivalent parametrization; where $\epsilon$ denotes a $\nu_{u,0}$-th root of unity.

# Minimal parametrization: Algebraic space curves

For the minimal parametrizations of the branches of a space curve we use the method given in

**Melanova, H.:** Geometric Invariants for the Resolution of Curve Singularities and for the Problem of the Moduli Space of n Points on the Projective Line. Dissertation (Supervisor: Herwig Hauser), University of Vienna (2020)

The strategy works in the following steps:

i. Project a space curve to all possible coordinate planes containing a selected coordinate axis; e.g. by means of resultant method.

ii. Construct the minimal parametrizations of these planar curves.

iii. The minimal parametrization of the space curve can be reconstructed from the planar ones.

# Removal approach for isostatic bar-joint frameworks

The realization is called **isostatic** (minimally rigid) if the removal of any edge constraint $c_i$ will make the realization flexible ($\Leftrightarrow m = e$).

From standpoint of kinematics following procedure is quite natural:

**a)** **Remove the i$^{th}$ bar** of an isostatic bar-joint framework for $i \in \{1, \ldots, e\}$ and to consider the **resulting 1-dof mechanism**.

**b)** Compute in the configuration $\mathbf{X}$ of interest the branches of the 1-dimensional configuration curve generated by the ideal

$$\langle c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_e \rangle. \tag{1}$$

**c)** Check up to which order each branch is compatible with the removed condition $c_i = 0$, which contains $\mathbf{X}$ as regular point, by determining the intersection multiplicity $n + 1$. Then this branch implies a $n^{th}$-order flex.

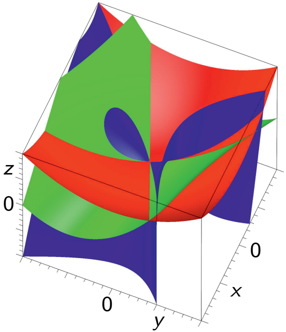
# Example

Let us consider three quadrics $C_i \in \mathbb{R}^3$:

$c_1(x, y, z) := x^2 + y^2 - 2z$

$c_2(x, y, z) := y^2 + xy - z$

$c_3(x, y, z) := 2x^2 - 3xy - 2y^2 - 2yz + z$




Note that the three regular quadrics intersect in the origin with multiplicity 4 ($\Rightarrow r = 3$). It can easily be seen that these quadrics have a common tangent plane $z = 0$ at the origin, which is also a double point of each three possible intersection curves.

By applying only the removal procedure one would end up with the conclusion that there are always two linear branches, which both imply flexes of order 1 $\Longrightarrow$ **(1,1)-flexes**

# Removal approach for isostatic bar-joint frameworks

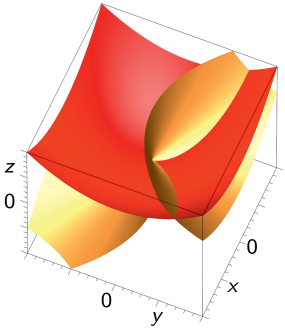
This kinematic motivated **removal procedure cannot generate a complete picture of the flexes**, which becomes clear by looking at the problem from the standpoint of algebraic geometry.
The ideal of Eq. (1) can be generalized to

$$\langle c_1 + \lambda_1 c_i, \ldots, c_{i-1} + \lambda_{i-1} c_i, c_{i+1} + \lambda_{i+1} c_i, \ldots, c_e + \lambda_e c_i \rangle \quad (2)$$

where the $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_e \in \mathbb{R}$ imply a $(e-1)$-parametric set of curves, whose branches have to be intersected with the hypersurface $c_i = 0$ containing the origin as regular point.
The resulting intersection multiplicity $n+1$ yields again the order $n$ of the flex implied by the corresponding branch.
Note that if we add $c_i = 0$ to the ideal of Eq. (2), then the resulting ideal is equivalent to the initial one $\langle c_1, \ldots, c_e \rangle$ **including also its intersection multiplicity**.

# Example: Continuation

Within the bundle of quadrics spanned by $C_1$, $C_2$, $C_3$ there also exists a **pencil of cones** $\mathcal{P}$ having their apexes in the origin. Moreover $\mathcal{P}$ touches the plane $z = 0$ in a pencil of lines through the origin.




Therefore each cone of $\mathcal{P}$ intersect $C_1$ in a quartic curve having a cusp in the origin in direction of the cone's generator contained in $z = 0$. This is illustrated for the cone $C_4 \in \mathcal{P}$ given by $c_4 = 0$ with $c_4(x, y, z) := x^2 - 2yz$. Hence there exists a **pencil of (2,3)-flexes**.

Higher-order cusps ($k > 2$) are not possible due to degree reasons.

# Removal approach for general bar-joint frameworks

## Modified Definition of (k,n)-flexes

A bar-joint framework, which is not continuous flexible, has a 1-parametric $(k, n)$-flex if for each vertex $\mathbf{x}_i$ $(i = 1, \ldots, w)$ there is a polynomial function

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,k} t^k + \ldots + \mathbf{x}_{i,n} t^n \quad \text{with} \quad n \geq k > 0$$

such that

1. the replacement of $\mathbf{x}_i$ by $\mathbf{x}_i'$ in the equations $c_1, \ldots, c_e$ gives stationary values of multiplicity $\geq n + 1$ at $t = 0$;
2. the vectors $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{w,k}$ are non-trivial;
3. $\mathbf{x}_i'$ can be extended to a minimal parametrization of a branch of order $k$ of an algebraic curve, which corresponds to a 1-dim irreducible component of a variety determined by an ideal, whose generators are contained in the linear family of quadrics spanned by $c_1, \ldots, c_e$.

# Remarks on the modified definition of (k,n)-flexes

1. The replacement of Stachel's irreducability condition by item 3 resolves the dilemma of ending up with an infinite series of possible $(k, n)$-flexes, as $k$ cannot be greater than $r$.
2. We speak more precisely of a "1-parametric flex" as one can also think of $p$-parametric flexes with $p > 1$. Their definition can be done similarly and their study can be based on local approximations of $p$-surfaces by multivariate Puiseux series; cf.


**Aroca, F., Ilardi, G., Lopez de Medrano, L.:** Puiseux power series solutions for systems of equations. Int. Journal of Mathematics **21**:1439–1459 (2010)

**Buchacher, M.:** The Newton-Puiseux algorithm and effective algebraic series. arXiv 2209.00875 (2022)

**McDonald, J.:** Fiber polytopes and fractional power series. Journal of Pure and Applied Algebra **104**:213–233 (1995)

**McDonald, J.:** Fractional power series solutions for systems of equations. Discrete & Computational Geometry **27**:501–529 (2002)

# Remarks on the modified definition of (k,n)-flexes

3. We added also the assumption that the framework is not continuous flexible, but the redefinition also holds for frameworks with a 1-dim mobility ($\Leftrightarrow n = \infty$).
4. Item 3 of the redefinition implies a **global construction** for determining $(k, n)$-flexes instead of a local mobility analysis.
5. The algebraic approach for the computation of the $(k, n)$-flexes operates over $\mathbb{C}$ but it also allows to **take reality issues into account** by using only the real part of the minimal parametrizations of the branches. This can be used to determine the highest real flex $(k_{\max}, n_{\max})$, which is of interest as it complements the flexion order $r$.

   For the discussed example we get $(r; k_{\max}, n_{\max}) = (3; 2, 3)$.
   The original double-Watt mechanism has the triple $(\infty; 2, \infty)$.

# Example: Immobile 4-bar mechanism


The three quadratic constraints read as follows:

$$c_1: \ \|M_1 - F_1\|^2 - 1^2 = 0, \quad c_2: \ \|M_2 - M_1\|^2 - 3^2 = 0, \quad c_3: \ \|M_2 - F_2\|^2 - 2^2 = 0.$$

As equation $c_1 = 0$ only depends on the variables $a$ and $b$, it already corresponds to the projection of the space curve to the $ab$-plane; i.e. $P(a, b) = c_1$. Its minimal parametrization is given by

$$a(t) = -\tfrac{1}{2}t^2 - \tfrac{1}{8}t^4 + \ldots, \quad b(t) = t. \tag{3}$$

For the projection of the space curve to the $bc$-plane we first eliminate $a$ by means of resultant from the equations $c_1 = c_2 = 0$. From the resulting expression $Res(c_1, c_2; a)$ and $c_3$ we eliminate $d$ again by means of resultant yielding:

# Example: Immobile 4-bar mechanism

$$P(b,c) := 144b^4c^2 + 144b^2c^4 + 384b^4c + 456b^2c^3 + 256b^4 + 1376b^2c^2 + 1225c^4 + 1024b^2c + 2800c^3 + 1600c^2.$$

Its minimal parametrization splits up into conjugate complex branches:

$$b(t) = t, \quad c(t) = \frac{-8 \pm 6I}{25}t^2 + \frac{-79 \pm 3I}{1250}t^4 + \dots. \qquad (4)$$

Finally, we project the space curve to the *bd*-plane. This can be achieved by eliminating $c$ from $Res(c_1, c_2; a)$ and $c_3$ by means of resultant, which implies:

$$P(b,d) := 144b^4d^2 - 288b^3d^3 + 144b^2d^4 + 1024b^4 - 1408b^3d + 1968b^2d^2 - 2080bd^3 + 1225d^4 + 11520b^2 - 5760bd + 7200d^2.$$

Its minimal parametrization splits up into conjugate complex branches:

$$b(t) = t, \quad d(t) = \frac{2 \pm 6I}{5}t + \frac{6 \pm 33I}{250}t^3 + \dots. \qquad (5)$$

# Example: Immobile 4-bar mechanism


Then Eqs. (3, 4, 5) imply the minimal parametrization of the two conjugate complex linear branches given by $b(t) = t$ and

$$a(t) = \frac{-1}{2}t^2 - \frac{1}{8}t^4 + \dots \quad c(t) = \frac{-8 \pm 6I}{25}t^2 + \frac{-79 \pm 3I}{1250}t^4 + \dots \quad d(t) = \frac{2 \pm 6I}{5}t + \frac{6 \pm 33I}{250}t^3 + \dots$$

through the origin, which is the only real point.

By plugging the real part of this parametrization into $c_1, c_2, c_3$ it can be seen that at least $t^2$ factors out from the resulting three expressions. This implies the triple $(r; k_{\max}, n_{\max}) = (\infty; 1, 1)$.

# Future research & Acknowledgment

For an efficient computation of the $(k, n)$-flexes we plan to resort to the powerful mean of tropical geometry, cf.

**Maclagan, D., Sturmfels, B.:** Introduction to Tropical Geometry. American Mathematical Society (2015)

**Jensen, A.N., Markwig, H., Markwig, T.:** An algorithm for lifting points in a tropical variety. Collectanea Mathematica **59**:129–165 (2008)

which was also demonstrated in the kinematic context; cf.

**Nayak, A.:** C-Space Analysis Using Tropical Geometry. Proceedings of 2nd IMA Conference on Mathematics of Robotics, 98–106, Springer (2022)

This approach is dedicated to future research as well as the study of the already mentioned $p$-parametric flexes with $p > 1$.

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# Thanks