Planar Stewart Gough platforms with a type II DM self-motion

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Abstract. Due to previous publications of the author, it is already known that one-parametric self-motions of general planar Stewart Gough platforms can be classified into two so-called Darboux Mannheim (DM) types (I and II). Moreover, the author also proved the necessity of three conditions for obtaining a type II DM self-motion. Based on this result we determine in the article at hand, all general planar Stewart Gough platforms with a type II DM self-motion. This is an important step in the solution of the famous Borel Bricard problem.

Mathematics Subject Classification (2010). 53A17; 70B15.

Keywords. Self-motion, Stewart Gough platform, Borel Bricard problem.

1. Introduction

The geometry of a planar Stewart Gough (SG) platform is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, 0)^T$ with respect to the fixed system Σ_0 and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, 0)^T$ with respect to the moving system Σ. By using Study parameters $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$ for the parametrization of Euclidean displacements, the coordinates $\mathbf{m}'_i$ of the platform anchor points with respect to Σ_0 can be written as $K\mathbf{m}'_i = \mathbf{R}\,\mathbf{m}_i + (t_1, t_2, t_3)^T$ with

$$t_1 = 2(e_0f_1 - e_1f_0 + e_2f_3 - e_3f_2), \qquad t_2 = 2(e_0f_2 - e_2f_0 + e_3f_1 - e_1f_3),$$

$$t_3 = 2(e_0f_3 - e_3f_0 + e_1f_2 - e_2f_1), \qquad K = e_0^2 + e_1^2 + e_2^2 + e_3^2 \neq 0 \quad \text{and}$$

$$\mathbf{R} = (r_{ij}) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}.$$

Now all points of the real 7-dimensional space $P^7_{\mathbb{R}}$, which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to a Euclidean displacement, with exception of the subspace $e_0 = \ldots = e_3 = 0$ of Ψ, as these points cannot fulfill the normalizing condition $K = 1$.

If the geometry of the manipulator is given as well as the six leg lengths, then the SG platform is in general rigid, but it can even happen that the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion. Note that such motions are also solutions to the famous Borel Bricard problem (cf. [1, 3, 5, 12, 25]). This still unsolved problem was posed 1904 by the French Academy of Science for the *Prix Vaillant*.

In this article we present all general planar SG platforms with a type II DM self-motion. All these manipulators steam either from solution E (cf. Corollary 5.4) or solution G (cf. Theorem 5.6) of the determined type II DM self-motions (cf. Section 3). The solution set of SG platforms implied by solution E is 12-dimensional and can be seen as generalization of line-symmetric Bricard octahedra (cf. Remark 5.5). The solution set of manipulators implied by solution G is 11-dimensional and consists of special polygon platforms (cf. Remark 5.7). Moreover, we show that the type II DM self-motions of all presented SG platforms are line-symmetric and octahedral (cf. Theorems 4.2 and 5.6).

1.1. Types of self-motions

In this subsection and section 2 we give a very short review of the results and ideas stated in [18], where more details and also some concrete examples can be found.

It is already known, that manipulators which are singular in every possible configuration, possess self-motions in each pose. These manipulators are so-called architecturally singular SG platforms [14] and they are well studied: For the characterization of architecturally singular planar SG platforms we refer to [8, 16, 22, 27]. For the non-planar case we refer to [9, 17]. Therefore we are only interested in the computation of self-motions of non-architecturally singular SG platforms. Until now only few self-motions of this type are known, as their computation is a very complicated task. A detailed review of these self-motions was given by the author in [21] (see also [6]).

Moreover, it is known that if a planar SG platform with anchor points $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is not architecturally singular, then there exists at least a one-parametric set $\mathcal{L}$ of legs, which can be attached to the given manipulator without restricting the forward kinematics [7, 15, 21]. The underlying linear system of equations is given in Eq. (30) of [15]. As the solvability condition of this system is equivalent to the criterion given in Eq. (12) of [2], also the singularity surface of the manipulator does not change by adding legs of $\mathcal{L}$. Moreover, it was shown that in general the base anchor points M_i as well as the corresponding platform anchor points m_i of $\mathcal{L}$ are located on planar cubic curves C and c, respectively.

Assumption 1.1. *We assume that there exist such cubics* c *and* C *(which can also be reducible) in the Euclidean domain of the platform and the base, respectively.*

Now, we consider the complex projective extension $P^3_{\mathbb{C}}$ of the Euclidean 3-space E^3, i.e.

$$a_i = \frac{x_i}{w_i}, \quad b_i = \frac{y_i}{w_i}, \quad A_i = \frac{X_i}{W_i}, \quad B_i = \frac{Y_i}{W_i}. \tag{1.1}$$

Note that ideal points are characterized by $w_i = 0$ and $W_i = 0$, respectively. Therefore we denote in the remaining part of this article the coordinates of anchor points, which are ideal points, by x_i, y_i and X_i, Y_i, respectively. For all other anchor points we use the coordinates a_i, b_i and A_i, B_i, respectively.

The correspondence between the points of C and c in $P^3_{\mathbb{C}}$, which is determined by the geometry of the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$, can be computed according to [7, 15] or [2] under consideration of Eq. (1.1). As this correspondence has not to be a bijection, a point $\in P^3_{\mathbb{C}}$ of c resp. C is in general mapped to a non-empty set of points $\in P^3_{\mathbb{C}}$ of C resp. c. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{\,\}$.

In $P^3_{\mathbb{C}}$ the cubic C has three ideal points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$, where at least one of these points (e.g. U_1) is real. The remaining points U_2 and U_3 are real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{u}_1\}, \{\mathsf{u}_2\}, \{\mathsf{u}_3\}$ of c ($\Rightarrow \{\mathsf{u}_1\}$ contains real points). We denote the ideal points of c by $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$, where again one (e.g. u_4) has to be real. The remaining points u_5 and u_6 are again real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{U}_4\}, \{\mathsf{U}_5\}, \{\mathsf{U}_6\}$ of C ($\Rightarrow \{\mathsf{U}_4\}$ contains real points).

Assumption 1.2. *For guaranteeing a general case, we assume that each of the corresponding locations* $\{\mathsf{u}_1\}$, $\{\mathsf{u}_2\}$, $\{\mathsf{u}_3\}, \{\mathsf{U}_4\}$, $\{\mathsf{U}_5\}$, $\{\mathsf{U}_6\}$ *consists of a single point. Moreover, we assume that no 4 collinear platform anchor points* u_j *or base anchor points* U_j *(*$j = 1, \ldots, 6$*) exist.*

Example. The cubics C and c of an octahedral manipulator (C and c split up into three lines) are illustrated in Fig. 1. Moreover, the points u_i and U_i ($i = 1, \ldots, 6$) are displayed as well as as some additional legs of $\mathcal{L}$.

Now the basic idea can simply be expressed by attaching the special legs $\overline{\mathsf{u}_i\mathsf{U}_i} \in \mathcal{L}$ with $i = 1, \ldots, 6$ to the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$. The attachment of the special leg $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i \in \{1, 2, 3\}$ corresponds with the so-called Darboux constraint, that the platform anchor point u_i moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i. Moreover, the attachment of the special leg $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i \in \{4, 5, 6\}$ corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to u_i slides through the point U_i.

By removing the originally six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ with $i = 1, \ldots, 6$ we remain with the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$, which is uniquely determined due to Assumption

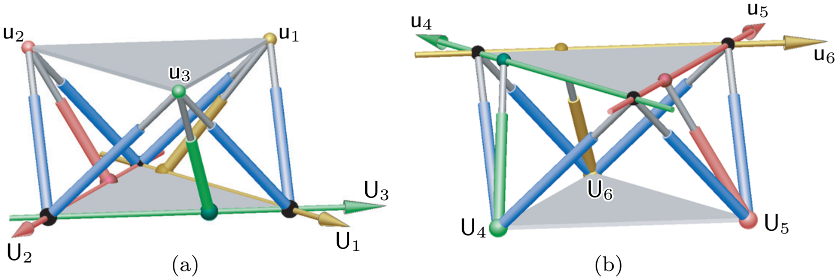

FIGURE 1. Octahedral manipulator: (a) Cubic C of base anchor points. (b) Cubic c of platform anchor points.

1.1 and 1.2. Moreover, under consideration of Assumption 1.1 and 1.2, the following statement holds (cf. [18]):

Theorem 1.3. *The manipulator* $\mathsf{u}_1, \ldots, \mathsf{U}_6$ *is redundant and therefore architecturally singular. Moreover, all anchor points of the platform* $\mathsf{u}_1, \ldots, \mathsf{u}_6$ *and as well of the base* $\mathsf{U}_1, \ldots, \mathsf{U}_6$ *are distinct.*

It was also proven in [18] that there only exist type I and type II Darboux Mannheim (DM) self-motions, where the definition of types reads as follows:

Definition 1.4. Assume $\mathcal{M}$ is a one-parametric self-motion of a SG platform $\mathsf{m}_1, \ldots, \mathsf{M}_6$, which is not architecturally singular. Then $\mathcal{M}$ is of the type n DM if the corresponding architecturally singular manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ has an n-parametric self-motion.

2. Type II DM self-motions

In the remainder of the article we focus on type II DM self-motions. The author [18] was already able to compute the set of equations yielding a type II DM self-motion explicitly. This was only possible by the usage of the analytical versions of the Darboux and Mannheim constraints, which are repeated next:

Darboux constraint: The constraint that the platform anchor point u_i ($i = 1, 2, 3$) moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i can be written as (cf. [18])

$$\Omega_i : \ \overline{X}_i(a_i r_{11} + b_i r_{12} + t_1) + \overline{Y}_i(a_i r_{21} + b_i r_{22} + t_2) + L_i K = 0,$$

with $X_i, Y_i, a_i, b_i, L_i \in \mathbb{C}$. This is a homogeneous quadratic equation in the Study parameters where $\overline{X}_i$ and $\overline{Y}_i$ denote the conjugate complex of X_i and Y_i, respectively.

Mannheim constraint: The constraint that the plane orthogonal to u_i ($i = 4, 5, 6$) through the platform point $(g_i, h_i, 0)$ slides through the point U_i of the fixed system can be written as (cf. [18])

$$\begin{aligned}\Pi_i : \overline{x}_i[A_i r_{11} + B_i r_{21} - g_i K - 2(e_0 f_1 - e_1 f_0 - e_2 f_3 + e_3 f_2)]+\\ \overline{y}_i[A_i r_{12} + B_i r_{22} - h_i K - 2(e_0 f_2 + e_1 f_3 - e_2 f_0 - e_3 f_1)] = 0,\end{aligned}$$

with $x_i, y_i, A_i, B_i, g_i, h_i \in \mathbb{C}$. This is again a homogeneous quadratic equation in the Study parameters where $\overline{x}_i$ and $\overline{y}_i$ denote the conjugate complex of x_i and y_i, respectively.

The content of the following lemma was also proven in [18]:

Lemma 2.1. *Without loss of generality (w.l.o.g.) we can assume that the algebraic variety of the two-parametric self-motion of the manipulator* $\mathsf{u}_1, \ldots, \mathsf{U}_6$ *is spanned by* $\Psi, \Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5$. *Moreover, we can choose following special coordinate systems in* Σ_0 *and* Σ *w.l.o.g.:* $X_1 = Y_2 = Y_3 = x_4 = y_5 = 1$ *and* $a_1 = b_1 = y_4 = A_4 = B_4 = Y_1 = h_4 = g_5 = 0$.

Moreover, the author has already proven the following result (see [19, 20]):

Theorem 2.2. *The corresponding manipulator* $\mathsf{u}_1, \ldots, \mathsf{U}_6$ *of a general planar SG platform (fulfilling Assumptions* 1.1, 1.2 *and Lemma* 2.1*) with a type II DM self-motion has to fulfill either the three conditions*

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = \overline{X}_2 a_2 - \overline{X}_3 a_3 + b_2 - b_3 = \overline{X}_2 b_2 - \overline{X}_3 b_3 - a_2 + a_3 = 0, \tag{2.1}$$

or the three conditions

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = \overline{X}_2 a_2 - \overline{X}_3 a_3 - b_2 + b_3 = \overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2 - a_3 = 0. \tag{2.2}$$

Based on Theorem 2.2 we determine all type II DM self-motions in section 3. In section 4 we investigate these self-motions in more detail and in section 5 we present all SG platforms with a type II DM self-motion. Finally, we close the paper with an concrete example.

Before we can start with section 3, one more preparatory work has to be done: It was pointed out by Karger [12], that the rotations about the x-, y-, and z-axis with angle π, which are represented in the Euler parameter space by the transformation:

$$\begin{aligned}(e_0, e_1, e_2, e_3) &\mapsto (-e_1, e_0, -e_3, e_2),\\ (e_0, e_1, e_2, e_3) &\mapsto (-e_2, e_3, e_0, -e_1),\\ (e_0, e_1, e_2, e_3) &\mapsto (-e_3, -e_2, e_1, e_0),\end{aligned} \tag{2.3}$$

represent equivalences for planar SG platforms (either change of the orientation of the normal of the plane of the platform or of the base or change of the orientation of axes in the plane of the platform or in the plane of the base).

As the two triples of necessary conditions given in Eqs. (2.1) and (2.2) are connected by a rotation about the x-axis with angle π (see [20]), we can restrict ourselves to one solution. W.l.o.g. we choose the conditions given in Eq. (2.2). Note that a geometric interpretation of these necessary conditions was given in [20].

3. Determination of type II DM self-motions

In order to determine all type II DM self-motions, we distinguish three main cases, which are given in the subsections 3.1, 3.2 and 3.3, respectively.

3.1. $(e_0e_2 - e_1e_3)e_0e_3 \neq 0$

Under this assumption we can solve the linear system of equations Ψ, Ω_1, Ω_2, Π_4 for $f_0, \ldots, f_3$ and plug the obtained expressions in the remaining two equations. This yields in general two homogeneous polynomials $\Omega[40]$ and $\Pi[96]$ in the Euler parameters of degree 2 and 4, respectively. The number in the square brackets gives the number of terms.

1. $X_3 \neq \pm i$: Under this assumption we can solve the three equations given in Eq. (2.2) for a_3, b_3 and L_3. As for $a_2 = b_2\overline{X}_3$ the condition $\Omega = 0$ cannot be fulfilled without contradiction (w.c.) we can assume $b_2\overline{X}_3 - a_2 \neq 0$. Under this assumption we can solve Ω for e_0 which yields
$$e_0 = \frac{(V - b_2 - a_2\overline{X}_3)e_3}{b_2\overline{X}_3 - a_2} \quad \text{with} \quad V = \pm\sqrt{(a_2^2 + b_2^2)(\overline{X}_3^2 + 1)}. \tag{3.1}$$
Now we plug the expression for e_0 into Π, which yields in the numerator $e_3F[650]$. As $e_3 = 0$ yields a contradiction, $F = 0$ has to vanish independently of the remaining Euler parameters. We denote the coefficients of $e_1^ie_2^je_3^k$ of F by F_{ijk}. Then we can express h_5 from $F_{300} = 0$, B_5 from $F_{030} = 0$, A_5 from $F_{210} = 0$ and g_4 from $F_{120} = 0$. Now the numerator of F_{012} factors into $(V - b_2 - a_2\overline{X}_3)G_1[192]$ and the numerator of the last remaining coefficient F_{102} factors into $(b_2\overline{X}_3 - a_2)G_2[192]$. Therefore $G_1 = G_2 = 0$ has to hold. Now $G_1 + G_2 = 0$ can only vanish w.c. for:
    a. $b_2 = 0$: Now the remaining condition $G_1 = 0$ can only vanish w.c. for:
        i. $x_5 = -X_3$: This yields solution A.
        ii. $\overline{x}_5 = \frac{a_3}{b_3}$: This yields solution B.
    b. $\overline{X}_2 = \frac{(V+a_2-a_2\overline{X}_3-b_2-b_2\overline{X}_3)(V-a_2-a_2\overline{X}_3-b_2+b_2\overline{X}_3)}{2(V-b_2-a_2\overline{X}_3)(b_2\overline{X}_3-a_2)}$ and $b_2 \neq 0$: Now the remaining condition $G_1 = 0$ can only vanish w.c. for:
        i. $x_5 = -X_2$: This yields solution C.
        ii. $\overline{x}_5 = \frac{a_2}{b_2}$: This yields solution D.
2. $X_3 = -i$: This immediately implies $X_2 = i$. Then we can solve the three equations given in Eq. (2.2) for a_2, a_3 and L_3, which yields:
$$L_3 = L_2 + 2iL_1, \quad a_2 = ib_2, \quad a_3 = -ib_3.$$

As for $b_2 = b_3$ the condition $\Omega = 0$ cannot be fulfilled w.c. we can assume $b_2 \neq b_3$. Under this assumption we can solve Ω for e_0 which yields

$$e_0 = \frac{(ib_2 + ib_3 - 2V)e_3}{b_2 - b_3} \quad \text{with} \quad V = \pm\sqrt{-b_2 b_3}.$$

Note that $(ib_2 - V)$ or $(ib_3 - V)$ cannot vanish w.c., and therefore we can assume $(ib_2 - V)(ib_3 - V) \neq 0$. Now we plug the expression for e_0 into Π, which yields in the numerator $e_3F[325]$. As $e_3 = 0$ yields a contradiction, $F = 0$ has to vanish independently of the remaining Euler parameters. We denote the coefficients of $e_1^i e_2^j e_3^k$ of F by F_{ijk}. Then we can express h_5 from $F_{300} = 0$ and B_5 from $F_{030} = 0$.

a. $x_5 \neq \pm i$: Under this assumption we can express A_5 from $F_{210} = 0$. Then the numerator of $F_{120} = 0$ factors into $(ib_2 - V)(ib_3 - V)G[11]$. We compute L_2 from $G = 0$, but this yields $A_5 = B_5 = 0$, a contradiction.
b. $x_5 = -i$: Now we can express L_2 from $F_{210} = 0$. Then F_{012} can only vanish w.c. for $A_5 = ib_2$. Finally, $F_{102} = 0$ yields the contradiction.
c. $x_5 = i$: Again we can express L_2 from $F_{210} = 0$. Then the difference of the only non-contradicting factors of F_{012} and F_{102} yields $ib_2(ib_2 - V)^2$. This expression cannot vanish w.c..

3. $X_3 = i$: It can be proven analogously to item 2, that there exist no solution.

3.2. $e_0e_2 - e_1e_3 \neq 0$, $e_0e_3 = 0$

There are two possible cases, namely $e_0 = 0$, $e_1e_3 \neq 0$ or $e_3 = 0$, $e_0e_2 \neq 0$. As these two cases correspond to each other by a rotation about the z-axis with angle π (cf. Eq. (2.3)), we can restrict ourselves to the discussion of the case $e_0 = 0$, $e_1e_3 \neq 0$. Notwithstanding this case[1] was already discussed by the author in [18], we repeat its short discussion for the sake of completeness:

W.l.o.g. we can solve the three equations of Eq. (2.2) for a_2, a_3 and L_3. Then Ω equals $(b_2 - b_3)e_3^2$, which implies $b_2 = b_3$. Now the numerator of Π factors into $e_3F[34]$. As $e_3 = 0$ yields a contradiction, $F = 0$ has to vanish independently of the remaining Euler parameters. We denote the coefficients of $e_1^i e_2^j e_3^k$ of F by F_{ijk}. Then F_{012} implies $L_1 = -g_4$. Then we can express L_2 from $F_{300} = 0$ and A_5 from $F_{210} = 0$. Now $F_{120} - F_{102}$ implies $B_5 = b_3$ and F_{120} can only vanish w.c. for:

a. $x_5 = X_2$: This yields solution E.
b. $x_5 = X_3$: This yields solution F.

3.3. $e_0e_2 - e_1e_3 = 0$

We split this case up into the following three subcases:

[1] Another approach for self-motions characterized by $e_0 = 0$ was given by Karger in [10, 11].

1. As $e_0 = e_1 = e_2 = e_3 = 0$ does not correspond with a Euclidean motion, we start the case study by considering the following four cases:

$$e_0 = e_1 = e_2 = 0, \;\; e_0 = e_1 = e_3 = 0, \;\; e_0 = e_2 = e_3 = 0, \;\; e_1 = e_2 = e_3 = 0.$$

   We only discuss the case $e_0 = e_1 = e_2 = 0$ in more detail because the other three are equivalent to this case with respect to a rotation of the platform about the x-, y-, and z-axis with the angle π (cf. Eq. (2.3)).

   Now $\Psi = 0$ implies $f_3 = 0$. Then $\Omega_1 = 0$ yields an expression for f_2 and $\Omega_2 = 0$ implies an expression for f_1. This cannot yield a two-parametric self-motion as only the homogeneous parameters e_3 and f_0 are free.
2. In this part we discuss the following four special cases:
   a. $e_0 = e_1 = 0$: Due to item 1 we can assume $e_2 e_3 \neq 0$. We can compute f_2 from $\Psi = 0$. Then Ω_1 implies $f_3 = -L_1 e_2/2$. Then Π_4 can only vanish w.c. for $g_4 = -L_1$. Moreover, we can express f_1 from Π_5. Finally, the coefficients of $e_2 f_0$ of Ω_2 and Ω_3 cannot vanish w.c..
   b. $e_2 = e_3 = 0$: This case is equivalent to the last one as these cases correspond to each other by a rotation about the y- resp. z-axis with the angle π (cf. Eq. (2.3)).
   c. $e_0 = e_3 = 0$: Due to item 1 we can assume $e_1 e_2 \neq 0$. We can compute f_1 from $\Psi = 0$. Then we can express f_0 from $\Pi_4 = 0$. Moreover, we can compute f_3 from $\Pi_5 = 0$. Now Ω_1, Ω_2 and Ω_3 have to vanish independently of the choice of the unknowns e_1, e_2, f_2. The coefficient of e_1^4 of Ω_2 implies an expression for h_5. Then we get L_2 from the coefficient of $e_1^1 e_2^3$ of Ω_2 and L_3 from the coefficient of e_1^4 of Ω_3. Then the coefficients of e_1^4 and e_2^4 of Ω_1 imply $L_1 = g_4 = 0$. Now we can compute a_2 from the coefficient of $e_1^1 e_2^3$ of Ω_1. Moreover, the coefficient of $e_1^3 e_2^1$ of Ω_1 implies $B_5 = \overline{x}_5 A_5$ and from the coefficient of $e_1^1 e_2^3$ of Ω_3 we get $a_3 = A_5(1 + \overline{x}_5^2) - \overline{X}_3 b_3$. Then the coefficient of $e_1^2 e_2^2$ of Ω_1 can only vanish w.c. for:
      (i) $x_5 = i$: Then $\Omega_2 = 0$ implies $X_2 = -i$ and from $\Omega_3 = 0$ we get $X_3 = i$. This yields solution G.
      (ii) $x_5 = -i$: Then $\Omega_2 = 0$ implies $X_2 = i$ and from $\Omega_3 = 0$ we get $X_3 = -i$. This yields solution H.
   d. $e_1 = e_2 = 0$: This case is equivalent to the last one as these cases correspond to each other by a rotation about the x-axis with the angle π (cf. Eq. (2.3)).
3. Due to the discussion of the special cases in item 1 and item 2, we now assume $e_0 e_1 e_2 e_3 \neq 0$. Therefore we can solve $e_0 e_2 - e_1 e_3 = 0$ for e_2. Moreover, we can solve $\Psi, \Omega_1, \Pi_4, \Pi_5$ for f_0, f_1, f_2, f_3.

   Now Ω_2 and Ω_3 have to vanish independently of the choice of the unknowns e_0, e_1, e_3. Therefore the coefficient of e_0^6 of Ω_2 implies $L_1 = g_4$. Then the coefficient of $e_0^5 e_3$ of Ω_2 yields an expression for L_2. Now we get $g_4 = 2a_2 - 2\overline{X}_2 b_2$ from the coefficient of $e_0^4 e_3^2$ of Ω_2. Moreover, we get $a_2 = \overline{X}_2 b_2$ from the coefficient of $e_1^2 e_3^4$ of Ω_2. Finally, the coefficient of $e_0^3 e_1^2 e_3$ of Ω_2 cannot vanish w.c..

This finishes the case study for the determination of all type II DM self-motions. A closer inspection of the obtained solutions A-G is given in section 4.

4. Discussion of the solutions A-G

We split up the discussion of the solutions into the following two parts:

4.1. Discussing solutions A-F

It can immediately be seen, that the solutions A and C correspond to each other just by a relabeling (switching index 2 and 3) of the anchor points. The same holds for the solutions B and D as well as for the solutions E and F. Therefore we can restrict ourselves to the solutions A, B and E.

For solution E it was already shown by the author in [18] that the self-motion is a line-symmetric motion. This follows directly from the property $e_0 = f_0 = 0$ under consideration of the result given by Selig and Husty [23].

In the following we show that the self-motions of the solutions A and B are also line-symmetric ones. We will prove this by using the characterization given in [23], that each motion which is located in the intersection of the Study quadric and a 5-plane, is a line-symmetric motion. Therefore we have to show that the Study parameters of the self-motions fulfill two linear equations.

Ad solution A. The first linear relation is the one of Eq. (3.1). Moreover, it is not difficult to see, that the linear relation $\nu_0 f_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 f_3 = 0$ holds with

$$\nu_0 = a_2^3 + a_2(a_2\overline{x}_5 + V)^2,\ \nu_2 = (a_2\overline{x}_5 + V)(V\overline{X}_2 + a_2\overline{x}_5\overline{X}_2 - a_2)a_2^2,$$
$$\nu_1 = -(a_2\overline{x}_5 + V)(V + a_2\overline{x}_5 + a_2\overline{X}_2)a_2^2,\ \nu_3 = a_2^2(a_2\overline{x}_5 + V) + (a_2\overline{x}_5 + V)^3.$$

Ad solution B. The first linear relation is again the one of Eq. (3.1). Moreover, the linear relation $\nu_0 f_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 f_3 = 0$ holds with

$$\nu_0 = a_2^3 + a_2(V - a_2\overline{X}_3)^2,\ \nu_2 = (V - a_2\overline{X}_3)(V\overline{X}_2 - a_2\overline{X}_2\overline{X}_3 - a_2)a_2^2,$$
$$\nu_1 = (a_2\overline{X}_3 - V)(V + a_2\overline{X}_2 - a_2\overline{X}_3)a_2^2,\ \nu_3 = a_2^2(V - a_2\overline{X}_3) + (V - a_2\overline{X}_3)^3.$$

As the self-motions of solution A, B and E are line-symmetric motions, we can apply Theorem 6 of Krames [13]. Therefore the cubics c and C in the platform and the base have to be congruent. This result implies that either the points U_2, U_3, u_5, u_6 are real or that U_2 and U_3 as well as u_5 and u_6 are conjugate complex.[2] Before we discuss these two cases separately, we repeat the following notation introduced in [18]:

[2]This can also be seen directly from the conditions computed in section 3.

Definition 4.1. A DM self-motion is called octahedral if the following triples of points are collinear for $i \neq j \neq k \neq i$ and $i,j,k \in \{1,2,3\}$:

$$(\mathsf{u}_i,\mathsf{u}_j,\mathsf{u}_6), \qquad (\mathsf{u}_i,\mathsf{u}_k,\mathsf{u}_5), \qquad (\mathsf{u}_j,\mathsf{u}_k,\mathsf{u}_4), \tag{4.1}$$

$$(\mathsf{U}_4,\mathsf{U}_5,\mathsf{U}_k), \qquad (\mathsf{U}_5,\mathsf{U}_6,\mathsf{U}_i), \qquad (\mathsf{U}_4,\mathsf{U}_6,\mathsf{U}_j). \tag{4.2}$$

The reason for this nomenclature is that all octahedral manipulators have such a point-configuration. For $i = 1$, $j = 2$ and $k = 3$ this can be seen in Fig. 1.

4.1.1. U_2, U_3, u_5, u_6 are real.

Ad solution A. It can immediately be seen that the following triples of points are collinear: $(\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_4)$, $(\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_5)$ and $(\mathsf{U}_2,\mathsf{U}_4,\mathsf{U}_5)$. As $\mathsf{u}_1,\ldots,\mathsf{U}_6$ is an architectural singular manipulator (cf. Theorem 1.3) we can apply Lemma 2 of [8], which implies that the triples $(\mathsf{U}_3,\mathsf{U}_5,\mathsf{U}_6)$, $(\mathsf{U}_1,\mathsf{U}_4,\mathsf{U}_6)$ and $(\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_6)$ are also collinear. These three collinearity conditions also determine the points U_6 and u_6 uniquely (as we can assume $y_6 = 1$ w.l.o.g.). Moreover, these collinearity conditions show that the self-motion is octahedral. Finally, it can easily be checked by direct computations that the triangles $\triangle(\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_3)$ and $\triangle(\mathsf{U}_5,\mathsf{U}_6,\mathsf{U}_4)$ are congruent.

Ad solution B. It can immediately be seen that the following triples of points are collinear: $(\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_4)$, $(\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_5)$ and $(\mathsf{U}_1,\mathsf{U}_4,\mathsf{U}_5)$. Again we can apply Lemma 2 of [8], which implies that the triples $(\mathsf{U}_3,\mathsf{U}_5,\mathsf{U}_6)$, $(\mathsf{U}_2,\mathsf{U}_4,\mathsf{U}_6)$ and $(\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_6)$ are also collinear. These three collinearity conditions again determine the points U_6 and u_6 and prove that the self-motion is octahedral. In this case the triangles $\triangle(\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_3)$ and $\triangle(\mathsf{U}_6,\mathsf{U}_5,\mathsf{U}_4)$ are congruent. This already shows that the solutions A and B are equivalent because they correspond to each other by a relabeling (switching indices 5 and 6) of the anchor points.

Ad solution E. It can immediately be seen that the triples of points $(\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_5)$, $(\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_4)$ and $(\mathsf{U}_3,\mathsf{U}_4,\mathsf{U}_5)$ are collinear. Again we can apply Lemma 2 of [8], which implies that the triples $(\mathsf{U}_2,\mathsf{U}_4,\mathsf{U}_6)$, $(\mathsf{U}_1,\mathsf{U}_5,\mathsf{U}_6)$ and $(\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_6)$ are also collinear. These three collinearity conditions again determine the points U_6 and u_6 and prove that the self-motion is octahedral. In this case the triangles $\triangle(\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_3)$ and $\triangle(\mathsf{U}_4,\mathsf{U}_5,\mathsf{U}_6)$ are congruent.

Moreover, also the solutions A and E are equivalent to each other. The only difference beside a relabeling (switching indices 4 and 6) of the anchor points, is that the coordinate systems of the fixed system and the moving system are chosen in a different way. Therefore we can restrict ourselves (due to equivalences) to solution E in the case that U_2, U_3, u_5, u_6 are real.

4.1.2. U_2 and U_3 as well as u_5 and u_6 are conjugate complex. In this case the solutions can be discussed analogously as in subsection 4.1.1. As $X_2 = \overline{X}_3$ and $x_5 = \overline{x}_6$ hold, we only have to interchange u_5 and u_6 as well as U_2 and U_3.

Also in this case the solutions A and B are equivalent because they correspond to each other by a relabeling (switching indices 5 and 6) of the anchor points. But now the solutions A and E are not equivalent, which can easily be seen as follows: In solution A the points $(\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_4)$ are collinear, where u_1 is a real finite point, u_4 a real point at infinity, and u_3 a complex point. But in solution E there do not exist such a triple of collinear points.

Moreover, we show in the following, that solution A does not yield a solution to our problem as $e_0 : \ldots : f_3 \notin P_{\mathbb{R}}^7$ holds. This can be seen as follows: In order that e_0 of Eq. (3.1) is a real number the imaginary parts of $\overline{X}_3$ and $J := \pm\sqrt{\overline{X}_3^2 + 1}$ have to be identical. Therefore the relation $J - \overline{J} = \overline{X}_3 - X_3$ has to hold. After squaring both sides this equation can be rewritten as $J\overline{J} = X_3\overline{X}_3 + 1$. If we square both sides again we will end up with the condition $(X_3 - \overline{X}_3)^2 = 0$. This already yields the contradiction as X_3 is a complex number.

Therefore we can also restrict ourselves to solution E in the case where U_2 and U_3 as well as u_5 and u_6 are conjugate complex.

4.2. Discussing solutions G and H

Clearly, also the solutions G and H correspond to each other by a relabeling of points. Therefore we can restrict ourselves to solution G.

It can immediately be verified that the triples $(\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_5)$ and $(\mathsf{U}_3, \mathsf{U}_4, \mathsf{U}_5)$ are collinear. Again we can apply Lemma 2 of [8], which implies that the triples $(\mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_6)$ and $(\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_6)$ are collinear. These conditions do not determine U_6 uniquely, because we only get $B_6 = iA_6$ from the collinearity condition of the points $(\mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_6)$. It can easily be seen from the rank condition of Röschel and Mick (cf. Remark 1 of [22]) that $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is architecturally singular if and only if $A_6 = -A_5 b_2/b_3$ holds. Note that in this case the self-motion has not to be octahedral.[3]

Moreover, $e_0 = e_3 = f_0 = f_3 = 0$ holds which already proves that the self-motion is a line-symmetric Schönflies motion (cf. [6]). Finally, it should be noted that the cubics c and C are cyclic, as $x_5 = X_3 = i$ and $x_6 = X_2 = -i$ hold.

Due to the results obtained in section 4 the following theorem holds:

Theorem 4.2. *All type II DM self-motions are line-symmetric motions. Moreover, all type II DM self-motions are either octahedral or Schönflies motions.*

[3]It is only octahedral if the additional condition $b_2 = b_3$ holds.

5. All SG platforms with a type II DM self-motion

In the following, we describe how all SG platforms with type II DM self-motion can be constructed: We start with a two-parametric DM self-motion. Due to section 4 we can restrict ourselves to the solutions E and G.

Now we attach an arbitrary leg to the manipulator $\mathsf{u}_1,\dots,\mathsf{U}_5$. We can choose freely a finite real point $\mathsf{M}_7=(A_7,B_7,0)$ in the planar base, a finite real point $\mathsf{m}_7=(a_7,b_7,0)$ in the planar platform and a real leg length R_7. Husty [4] showed that the condition for m_7 to be located on a sphere with center M_7 and radius R_7 can be expressed by the following homogeneous quadratic equation:

$$\begin{aligned}\Lambda_7:\quad &(A_7^2+B_7^2+a_7^2+b_7^2-R_7^2)K+4(f_0^2+f_1^2+f_2^2+f_3^2)\\&+2(e_3^2-e_0^2)(A_7a_7+B_7b_7)+2(e_2^2-e_1^2)(A_7a_7-B_7b_7)\\&+4[(f_0e_2-e_0f_2)(B_7-b_7)+(e_1f_3-f_1e_3)(B_7+b_7)\\&+(f_2e_3-e_2f_3)(A_7+a_7)+(f_0e_1-e_0f_1)(A_7-a_7)\\&+e_0e_3(A_7b_7-B_7a_7)-e_1e_2(A_7b_7+B_7a_7)]=0.\end{aligned}$$

We can set $B_7=0$ w.l.o.g. because the cubic C has to have a third real point on the x-axis beside the points U_1 and U_4. Moreover, the resulting manipulator $\mathsf{u}_1,\dots,\mathsf{U}_5,\mathsf{m}_7,\mathsf{M}_7$ is not architecturally singular as $(\mathsf{m}_7,\mathsf{M}_7)\neq(\mathsf{u}_6,\mathsf{U}_6)$ holds. As a consequence we can determine the one-parametric set $\mathcal{L}$ of legs, which can be attached to this manipulator $\mathsf{u}_1,\dots,\mathsf{U}_5,\mathsf{m}_7,\mathsf{M}_7$ without restricting the direct kinematics. This can be done as follows, where we use the symbol $\oplus$ as index in order to label the representative of $\mathcal{L}$:

Analogous considerations as in [7, 15] yield that the constraint $\Lambda_\oplus$ of the redundant leg has to be a linear combination of the following type:

$$\Upsilon:\ \mu_1\Omega_1+\mu_2\Omega_2+\mu_3\Omega_3+\mu_4\Pi_4+\mu_5\Pi_5+\mu_6\Lambda_7-\Lambda_\oplus=0.$$

Now the homogeneous quadratic equation Υ has to vanish independently of the Study parameters, where Υ has the following coefficients: $\Upsilon_{e_0e_3}$, $\Upsilon_{e_0f_1}$, $\Upsilon_{e_0f_2}$, $\Upsilon_{e_3f_1}$, $\Upsilon_{e_3f_2}$, $\Upsilon_{e_i^2}$, $\Upsilon_{e_1e_2}$, $\Upsilon_{e_1f_0}$, $\Upsilon_{e_1f_3}$, $\Upsilon_{e_2f_0}$, $\Upsilon_{e_2f_3}$ and $\Upsilon_{f_i^2}$ with $i=0,\dots,3$. Note that e.g. $\Upsilon_{e_0e_3}$ denotes the coefficient of e_0e_3 of Υ. As the following relations hold:

$$\begin{aligned}&\Upsilon_{e_0f_2}+\Upsilon_{e_3f_1}+\Upsilon_{e_1f_3}+\Upsilon_{e_2f_0}=0, &&\Upsilon_{e_0f_1}+\Upsilon_{e_3f_2}+\Upsilon_{e_2f_3}+\Upsilon_{e_1f_0}=0,\\&\Upsilon_{e_0f_2}-\Upsilon_{e_3f_1}-\Upsilon_{e_1f_3}+\Upsilon_{e_2f_0}=0, &&\Upsilon_{e_0f_1}-\Upsilon_{e_3f_2}-\Upsilon_{e_2f_3}+\Upsilon_{e_1f_0}=0,\\&\Upsilon_{e_0^2}-\Upsilon_{e_1^2}-\Upsilon_{e_2^2}+\Upsilon_{e_3^2}=0, &&\Upsilon_{f_0^2}=\Upsilon_{f_1^2}=\Upsilon_{f_2^2}=\Upsilon_{f_3^2},\end{aligned}$$

we can restrict to the following 10 coefficients:

$$\Upsilon_{e_0e_3},\ \Upsilon_{e_1e_2},\ \Upsilon_{e_0f_1},\ \Upsilon_{f_0^2},\ \Upsilon_{e_0^2},\ \Upsilon_{e_0f_2},\ \Upsilon_{e_1f_3},\ \Upsilon_{e_2f_3},\ \Upsilon_{e_1^2},\ \Upsilon_{e_2^2},\qquad(5.1)$$

in eleven unknowns $(a_\oplus,b_\oplus,A_\oplus,B_\oplus,R_\oplus,\mu_1,\dots,\mu_6)$. From now on everything can be done analogously to the method described in [7, 15] (see also [18]). Finally, we end up with the corresponding cubics c and C in the platform and the base, respectively.

Then we can choose any six legs from $\mathcal{L}$, where the corresponding anchor points do not cause an architecturally singular design. In this way all general planar SG platforms with a type II DM self-motion can be constructed.

In the following we investigate the SG platforms implied by solution E and G in more detail.

5.1. SG platforms implied by solution E

If we plug the expression for the f_i's into Λ_7 (under consideration of $e_0 = 0$) we get a homogeneous polynomial of degree six in e_1, e_2, e_3 with only 221 terms in its general form. Moreover, the unknown e_3 only appears with even powers and therefore the equation can in general be solved explicitly for e_3. As two of the six solutions have to be real, it follows immediately that there exist $e_0 : \ldots : f_3 \in P^7_{\mathbb{R}}$ if U_2, U_3, u_5, u_6 are real.

For the case that U_2 and U_3 as well as u_5 and u_6 are conjugate complex, we prove the following three theorems:

Theorem 5.1. *If* $\mathsf{M}_7 = \mathsf{U}_4$ *or* $\mathsf{m}_7 = \mathsf{u}_1$ *holds, then we cannot construct non-architecturally singular planar SG platforms with a type II DM self-motion.*

Proof. If $\mathsf{M}_7 = \mathsf{U}_4$ holds, then the cubic C has to split up into two conjugate complex lines $[\mathsf{U}_4, \mathsf{U}_2]$ and $[\mathsf{U}_4, \mathsf{U}_2]$ and in one real line S. As C and c are congruent due to Theorem 6 of Krames [13], c has to split up into the conjugate complex lines $[\mathsf{u}_1, \mathsf{u}_5]$ and $[\mathsf{u}_1, \mathsf{u}_6]$ and the real line $\mathsf{s} = [\mathsf{m}_7, \mathsf{u}_4]$.[4]

Therefore we can only add two real pencils of legs: One pencil is spanned by its vertex U_4 and an arbitrary point on s and the other one is spanned by its vertex u_1 and an arbitrary point on S. Any choice of six legs of these two pencils yields an architecturally singular manipulator.

Clearly, the same argumentation can be done for the case $\mathsf{m}_7 = \mathsf{u}_1$. □

In order to improve the readability of this article, the proofs of the following two theorems are given in the appendix.

Theorem 5.2. *For* $\mathsf{M}_7 \neq \mathsf{U}_4$ *and* $\mathsf{m}_7 \neq \mathsf{u}_1$ *the Study parameters* $e_0 : \ldots : f_3$ *can only be element of* $P^7_{\mathbb{R}}$

- ⋆ *if* $B_5 = b_2 = b_3 \in \mathbb{R}$ *and* $g_4 = -\frac{c_3}{c_1}$ *with* c_1 *and* c_3 *of Eq. (*5.5*) hold,*
- ⋆ *or in one of the solutions 1, 2 and 3. In these cases* c *and* C *are cyclic.*

Theorem 5.3. *From the solutions 1, 2 and 3 of Theorem* 5.2*, we cannot construct non-architecturally singular planar SG platforms with a type II DM self-motion.*

[4]This can also be checked by direct computations.

Corollary 5.4. *Due to Theorems* 5.1, 5.2 *and* 5.3 *we can only construct a non-architecturally singular planar SG platform with a type II DM self-motion from solution E ($b_2 = b_3 = B_5$, $x_5 = \overline{X}_2$, $a_2 = A_5 = B_5\overline{X}_3$, $a_3 = B_5\overline{X}_2$, $L_1 = -g_4$, $L_2 = -h_5$, $L_3 = (\overline{X}_2 - \overline{X}_3)g_4 - h_5$) for*

- $X_2, X_3, g_4, h_5, B_5 \in \mathbb{R}$ *or*
- $\overline{X}_3 = X_2 = r_1 + ic_1$, $h_5 = r_3 + ic_3$, $g_4 = -c_3/c_1$ *and* $B_5, r_1, c_1, r_3, c_3 \in \mathbb{R}$.

In the following we discuss the degrees of freedom of the two solutions of Corollary 5.4. It can easily be seen that the congruent cubics c and C do not depend on the choice of the unknowns g_4, h_5, R_7 resp. r_3, c_3, R_7. Therefore c and C are determined by the remaining six free parameters (X_2, X_3, B_5, A_7, a_7, b_7 resp. r_1, c_1, B_5, A_7, a_7, b_7). Moreover, we can choose freely six pairs of corresponding anchor points on c and C. Therefore we get a 12-dimensional solution set of planar SG platforms with a type II DM self-motion.

Remark 5.5. It was shown in [18] that all line-symmetric Bricard octahedra (cf. [24]) are included in the given 12-dimensional solution set. Therefore the manipulators of this set can be seen as generalization of these Bricard octahedra. Moreover, examples for both cases of Corollary 5.4 were already presented in [18].

5.2. SG platforms implied by solution G

If we substitute solution G into Λ_7 we get an expression with only 13 terms. Moreover, $\Lambda_7[13]$ is homogeneous of degree 4 in e_1, e_2 and f_2, whereby f_2 only appears with the power 2. As the coefficient of f_2^2 equals $e_1^2 + e_2^2$, it cannot vanish and therefore we can always solve $\Lambda_7[18]$ for f_2.

Theorem 5.6. *We can only construct a non-architecturally singular planar SG platforms with a type II DM self-motion from solution G ($x_5 = X_3 = i$, $X_2 = -i$, $a_2 = -ib_2$, $a_3 = ib_3$, $A_5 = iB_5$, $L_1 = L_2 = L_3 = g_4 = h_5 = 0$) if the self-motion is octahedral ($\Leftrightarrow b_2 = b_3$, cf. footnote 3).*

Proof. W.l.o.g. we can compute $R_\oplus$ from $\Upsilon_{e_2e_2}$.

1. $b_2 \neq b_3$: Under this assumption we can solve $\Upsilon_{f_0^2}$, $\Upsilon_{e_1f_3}$, $\Upsilon_{e_0f_2}$, $\Upsilon_{e_2f_3}$, $\Upsilon_{e_0f_1}$, $\Upsilon_{e_0^2}$ for $\mu_1, \ldots, \mu_6$.
   a. $\mathsf{m}_\oplus \neq \mathsf{u}_1$: Now we can express $A_\oplus$ and $B_\oplus$ from $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1e_2}$. Plugging the obtained expressions into $\Upsilon_{e_0e_3}$ yields the equation $C[46] = C_r[10] + iC_c[4]$ of c with $C_r, C_c \in \mathbb{R}$. In the first part we show that neither C_r nor C_c cannot vanish w.c. independently of the choice of $a_\oplus$ and $b_\oplus$.
   ad C_c) The coefficient of $a_\oplus^2$ implies $b_7 = 0$. Then the coefficient of $a_\oplus b_\oplus$ yields the contradiction.
   ad C_r) The coefficient of $b_\oplus^3$ already yields the contradiction.
   Therefore we are looking for real points of c. Clearly, such points have to fulfill $C_r = C_c = 0$. C_c can only vanish w.c. for:

i. $A_7 = 0$: Then $C_r = 0$ implies $b_7 = b_\oplus$. Moreover, $\mathsf{M}_7 = \mathsf{M}_\oplus = \mathsf{U}_4$ holds. Therefore we can add a pencil of lines with vertex $\mathsf{M}_7 = \mathsf{M}_\oplus$ ($\Rightarrow$ architecturally singular SG platform).

ii. $a_\oplus = 0$, $A_7 \neq 0$: Then we get $A_\oplus = A_7 b_7/b_\oplus$ and $B_\oplus = A_7 a_7/b_\oplus$. Moreover, C_r can only vanish w.c. for:

$$b_3 B_5 b_\oplus^2 + (A_7 a_7(b_3 + b_2) - 2b_3 b_7 B_5)b_\oplus - 2b_2 b_3 A_7 a_7 = 0$$

This only yields two solutions and therefore we cannot construct a 6-legged SG platform.

iii. $a_\oplus b_7 - a_\oplus a_7 = 0$, $a_\oplus A_7 \neq 0$: We distinguish following subcases:

⋆ $a_7 \neq 0$: Under this assumption we can express $b_\oplus$ from the last equation. Now C_r can only vanish w.c. for:
(α) $b_7 = 0$: We get $A_\oplus = A_7 a_7/a_\oplus$ which yields a regulus of additional legs ($\Rightarrow$ architecturally singular SG platform).
(β) $a_7 = a_\oplus$: In this case we get $\mathsf{m}_7 = \mathsf{m}_\oplus$ and $\mathsf{M}_7 = \mathsf{M}_\oplus$.

⋆ $a_7 = 0$: This already yields $b_7 = 0$. Then $C_r = 0$ implies $b_\oplus = 0$. Therefore we can add a pencil of lines with vertex $\mathsf{M}_\oplus = \mathsf{U}_4$ ($\Rightarrow$ architecturally singular SG platform).

b. $\mathsf{m}_\oplus = \mathsf{u}_1$: ($\Rightarrow a_\oplus = b_\oplus = 0$): $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1 e_2}$ can only vanish for:

i. $A_7 = 0$: Then $\Upsilon_{e_0 e_3}$ implies $B_\oplus = B_5 b_7/b_2$.

ii. $a_7 = b_7 = 0$: Then $\Upsilon_{e_0 e_3}$ implies $B_\oplus = 0$.

In both cases (i) and (ii) we can add a pencil of lines with vertex $\mathsf{m}_\oplus = \mathsf{u}_1$ ($\Rightarrow$ architecturally singular SG platform).

2. $b_2 = b_3$: Under this assumption we can solve $\Upsilon_{f_0^2}$, $\Upsilon_{e_1 f_3}$, $\Upsilon_{e_0 f_2}$, $\Upsilon_{e_2 f_3}$, $\Upsilon_{e_0 f_1}$, $\Upsilon_{e_0 e_3}$ for $\mu_1, \ldots, \mu_6$.

a. $\mathsf{m}_\oplus \neq \mathsf{u}_1$: Under this assumption we can express $A_\oplus$ and $B_\oplus$ from $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1 e_2}$ which yields:

$$A_\oplus = \frac{A_7(a_7 a_\oplus + b_7 b_\oplus)}{a_\oplus^2 + b_\oplus^2}, \quad B_\oplus = \frac{A_7(b_7 a_\oplus - a_7 b_\oplus)}{a_\oplus^2 + b_\oplus^2}. \tag{5.2}$$

Plugging the obtained expressions into $\Upsilon_{e_0^2}$ yields the equation of c:

$$\begin{aligned} B_5 b_\oplus^3 + B_5 b_\oplus a_\oplus^2 - B_5 b_7 a_\oplus^2 + (A_7 a_7 - B_5 b_7) b_\oplus^2 \\ -A_7 b_7 a_\oplus b_\oplus + b_3 A_7 b_7 a_\oplus - b_3 A_7 a_7 b_\oplus = 0 \end{aligned} \tag{5.3}$$

As this is a real cubic we obtain a solution.

b. $\mathsf{m}_\oplus = \mathsf{u}_1$: ($\Rightarrow a_\oplus = b_\oplus = 0$): $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1 e_2}$ can only vanish for:

i. $A_7 = 0$: Then $\Upsilon_{e_0 e_0}$ implies $B_\oplus = B_5 b_7/b_2$.

ii. $a_7 = b_7 = 0$: Then $\Upsilon_{e_0 e_0}$ implies $B_\oplus = 0$.

In both cases (i) and (ii) we can add a pencil of lines with vertex $\mathsf{m}_\oplus = \mathsf{u}_1$ ($\Rightarrow$ architecturally singular SG platform).

This finishes the proof of Theorem 5.6. □

In addition to the obtained solution of item (2a) the following should be noted: We can also solve $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1e_2}$ for $a_\oplus$ and $b_\oplus$. Then $\Upsilon_{e_0^2}$ yields the equation of C:

$$\begin{aligned}b_3B_\oplus^3 + b_3B_\oplus A_\oplus^2 - B_5b_7A_\oplus^2 + (A_7a_7 - B_5b_7)B_\oplus^2 \\ -A_7b_7A_\oplus B_\oplus + B_5A_7b_7A_\oplus - B_5A_7a_7B_\oplus = 0\end{aligned} \tag{5.4}$$

It can easily be seen that we get the equation (5.3) of c from equation (5.4) of C by setting $A_\oplus = a_\oplus B_5/b_3$ and $B_\oplus = b_\oplus B_5/b_3$. Therefore the cubics are similar to each other (they are only congruent for the special case $B_5 = b_3$).

Remark 5.7. On first sight this result seems to contradict Theorem 6 of Krames [13], but a closer look shows that the mapping given in Eq. (5.2) is for $\mathsf{m}_7 \neq \mathsf{u}_1$ a composition[5] of an inversion ι and a rotation. The center of ι is the origin and the radius of the inversion circle equals $\sqrt{\delta A_7}$ with $\delta = \sqrt{a_7^2 + b_7^2}$. Then $\iota(\mathsf{m}_\oplus)$ is rotated by the matrix

$$\begin{pmatrix} a_7/\delta & b_7/\delta \\ b_7/\delta & -a_7/\delta \end{pmatrix}.$$

Therefore we have a so-called *polygon platform* (cf. [26]). It is well known (cf. [1, 3, 6]) that all points of the platform and the base which are coupled by this mapping run on spherical paths (Schönflies Borel Bricard motion), and not only those on c and C. Therefore this is not a counterexample to Theorem 6 of [13].

In the following we discuss the degrees of freedom of solution G fulfilling the octahedral condition $b_2 = b_3$. It can easily be seen that the similar cubics c and C do not depend on R_7. Therefore c and C are determined by the remaining five free parameters (b_3, B_5, A_7, a_7, b_7). Moreover, we can choose freely six pairs of corresponding anchor points on c and C. Therefore we get an 11-dimensional solution set of planar SG platforms with a type II DM self-motion.

Example. Now we give an example for a SG platform with a type II DM self-motion implied by solution G fulfilling the octahedral condition $b_2 = b_3$. We get the cubics c and C printed in Fig. 2 for the following values: $B_5 = 9$, $b_7 = 7$, $A_7 = 13$, $a_7 = 5$, $b_3 = 15$, $R_7 = 21$. For this choice the length $R_\oplus$ is given by:

$$R_\oplus = \sqrt{A_\oplus^2 + B_\oplus^2 + 198 + 12506/(A_\oplus^2 + B_\oplus^2)}.$$

Moreover, we can parametrize these type II DM self-motions by setting $e_1 = \cos\varphi$ and $e_2 = \sin\varphi$. This yields:

$$f_1 = -\tan\varphi f_2 \quad \text{with} \quad f_2 = \pm\cos\varphi\sqrt{65\cos\varphi^2 + 91\cos\varphi\sin\varphi + 17},$$

for our concrete example.

[5] For $\mathsf{m}_7 = \mathsf{u}_1$ ($\Rightarrow a_7 = b_7 = 0$) we get $b_\oplus = 0$ from Eq. (5.3). In this case we can again only add a pencil of lines with vertex $\mathsf{M}_\oplus = \mathsf{U}_4$ ($\Rightarrow$ architecturally singular SG platform).

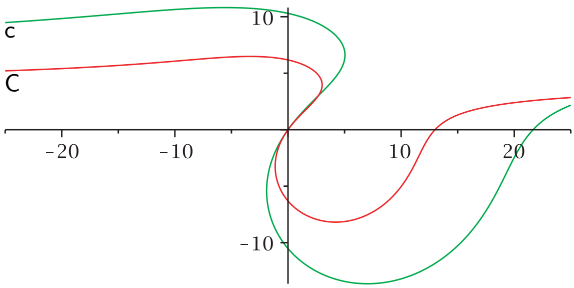

Figure 2. Similar cyclic cubics c and C.

Acknowledgment

This research is supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

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Appendix

Proof of Theorem 5.2

Proof. We consider the Darboux motion of the point u_1 with respect to the plane ε with homogeneous plane coordinates $[L_1 : 1 : 0 : 0]$. From the condition $\mathsf{u}_1 \in \varepsilon$ it follows immediately that $L_1 = -g_4 \in \mathbb{R}$ has to hold. Then we set

$$X_2 = r_1 + ic_1, \quad B_5 = r_2 + ic_2, \quad h_5 = r_3 + ic_3, \tag{5.5}$$

with $r_1, r_2, r_3, c_1, c_2, c_3 \in \mathbb{R}$. Moreover, $c_1 \neq 0$ has to hold. The imaginary part of the numerator of f_1, f_2 and f_3 can only vanish w.c. for:

$$\begin{aligned} &(r_1^2c_2 - c_3 + c_1^2c_2 - c_2 - g_4c_1)e_1^2 - (r_1^2c_2 + c_3 + c_1^2c_2 - c_2 + g_4c_1)e_2^2 \\ &- (r_1^2c_2 + c_3 + c_1^2c_2 + c_2 + g_4c_1)e_3^2 + 4e_1e_2r_1c_2 = 0. \end{aligned} \tag{5.6}$$

In this case also the imaginary part of Λ_7 vanishes. We stop the following case study if $A_7 = 0$ or $a_7 = b_7 = 0$ holds or if we obtain a solution:

Part A) $r_1^2c_2 + c_3 + c_1^2c_2 + c_2 + g_4c_1 \neq 0$**.** Under this assumption we can solve Eq. (5.6) for e_3. Then we plug the obtained expression into Λ_7. The numerator of the resulting expression factors into

$$(e_1c_1 + e_2i + r_1e_1i)(e_1c_1 + e_2i - r_1e_1i)N[893].$$

As this expression has to vanish independently of e_1 and e_2, it can immediately be seen that only $N = 0$ can yield a solution. We denote the coefficient of $e_1^ie_2^j$ of N by N_{ij}.

1. $c_2(c_3 + (r_1^2 + c_1^2 - 1)c_2 + c_1g_4) \neq 0$: Under this assumption we can express R_7 from the only non-contradicting factor of N_{04}.
   a. $b_7 \neq 0$: Now we can express A_7 from N_{13}.
      i. $r_1(c_3r_2 - c_2r_3 + g_4(r_1c_2 + r_2c_1)) \neq 0$: Under this assumption we can compute a_7 from N_{31}. Now the difference of the only non-contradicting factors of N_{40} and N_{22} cannot vanish w.c..
      ii. $r_1 = 0$: Then N_{31} can only vanish w.c. for:
         ⋆ $g_4 = 0$: Now $N_{22} = 0$ yields the contradiction.
         ⋆ $c_1 = \pm 1$, $g_4 \neq 0$: As for $c_2r_3 - c_3r_2 \mp r_2g_4 = 0$ the expression N_{40} cannot vanish w.c., we can assume $c_2r_3 - c_3r_2 \mp r_2g_4 \neq 0$. Under this assumption we can express a_7 from the only non-contradicting factor of N_{40}. We get solution 1.
         ⋆ $r_3 = \frac{r_2(c_3+g_4c_1)}{c_2}$: Now $N_{22} = 0$ yields the contradiction.
      iii. $r_3 = \frac{c_3r_2+g_4(r_1c_2+r_2c_1)}{c_2}$, $r_1 \neq 0$: $N_{31} = 0$ yields the contradiction.
   b. $b_7 = 0$: Now $N_{13} = 0$ factors into $(c_3r_2 - c_2r_3 + g_4(r_1c_2 + r_2c_1))T$ with

$$T = c_1g_4^2 + (c_2c_1^2 - 2r_1c_1r_2 - r_1^2c_2 - c_2 + c_3)g_4 + 2r_1(c_2r_3 - r_2c_3).$$

i. $r_3 = \frac{r_2(c_3+g_4c_1)}{c_2}$: Then N_{31} can only vanish w.c. for:
   - $\star$ $r_1 = 0$: Now the difference of the only non-contradicting factors of N_{40} and N_{22} can only vanish w.c. for:
     (α) $c_1 = \pm 1$: Then the only non-contradicting factor of N_{40} equals $c_2g_4^2 \pm 2g_4a_7A_7 + 2a_7A_7c_3 = 0$. This equation can be solved for g_4 w.l.o.g.. We get solution 2.
     (β) $g_4 = \pm 2\sqrt{a_7A_7}$: Now $N_{40} = 0$ yields the contradiction.
   - $\star$ $g_4 = \pm 2\sqrt{a_7A_7}$, $r_1 \neq 0$: $N_{40} = 0$ yields the contradiction.

ii. $T = 0$, $c_3r_2 - c_2r_3 + g_4(r_1c_2 + r_2c_1) \neq 0$: We distinguish two cases:
   - $\star$ $r_1 \neq 0$: Under this assumption we can express r_3 from $T = 0$. Then the difference of the only non-contradicting factors of N_{31} and N_{22} cannot vanish w.c..
   - $\star$ $r_1 = 0$: Now $T = 0$ implies $g_4 = 0$. Then N_{40} can only vanish w.c. for:
     (α) $c_3 = c_1^2c_2 - c_2$: Now $N_{22} = 0$ yields the contradiction.
     (β) $A_7 = \frac{c_1^2(c_2r_3-c_3r_2)^2}{2a_7c_2(c_1^2c_2-c_2+c_3)}$: Then N_{22} implies $c_1 = \pm 1$. We get solution 3.

2. $g_4 = \frac{c_2(1-r_1^2-c_1^2)-c_3}{c_1}$, $c_2 \neq 0$: Now we can compute r_3 from the only non-contradicting factor of N_{04}. We distinguish two cases:
   a. $r_1 \neq 0$: The only non-contradicting factor of N_{13} can be solved for R_7.
      i. $b_7 \neq 0$: Under this assumption we can compute A_7 from $N_{22} = 0$. Moreover, we can express a_7 from the only non-contradicting factor of N_{31}. Finally, N_{40} cannot vanish w.c..
      ii. $b_7 = 0$: Now $N_{22} = 0$ implies $c_3 = c_2(1 - r_1^2 - c_1^2)$. Then $N_{31} = 0$ yields the contradiction.
   b. $r_1 = 0$: In this case $N_{31} = 0$ implies $b_7 = 0$. Moreover, we can solve the only non-contradicting factor of N_{22} for R_7. Finally, $N_{40} = 0$ yields the contradiction.
3. $c_2 = 0$: Now N_{22} cannot vanish w.c..

Part B) $g_4 = \frac{r_1^2c_2+c_3+c_1^2c_2+c_2}{-c_1}$**.** Then Eq. (5.6) can only vanish for:

a. $e_2 = (-r_1 \pm c_1 i)e_1$: As (e_1, e_2) cannot be element of $\mathbb{R}^2$ we get a contradiction.
b. $c_2 = 0$: This yields the solution given in the first item of Theorem 5.2. It can easily be checked by direct computation, that in this case also the equation Λ_7 has only real coefficients with respect to the remaining Study parameters.

This finishes the proof of Theorem 5.2. □

Proof of Theorem 5.3

Proof. We compute the cubics c and C for the solutions 1, 2 and 3, respectively. W.l.o.g. we can compute $R_\oplus$ from $\Upsilon_{e_2e_2}$. Then we can solve $\Upsilon_{f_0^2}$, $\Upsilon_{e_1f_3}$, $\Upsilon_{e_0f_2}$, $\Upsilon_{e_2f_3}$, $\Upsilon_{e_0f_1}$, $\Upsilon_{e_0e_3}$ for $\mu_1, \ldots, \mu_6$. In the following we discuss solution

1: We distinguish two cases:
- a. $\mathsf{m}_\oplus \neq \mathsf{u}_1$: Under this assumption we can express $A_\oplus$ and $B_\oplus$ from $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1e_2}$. Plugging the obtained expressions into $\Upsilon_{e_0^2}$ yields the equation $C[46] = C_r[28] + iC_c[18]$ of c with $C_r, C_c \in \mathbb{R}$. In the first part we show that neither C_r nor C_c cannot vanish w.c. independently of the choice of $a_\oplus$ and $b_\oplus$.
  ad C_r) The coefficient of $b_\oplus^3$ implies $r_2 = 0$. Then the coefficient of $a_\oplus b_\oplus$ yields the contradiction.
  ad C_c) The coefficient of $b_\oplus^3$ already yields the contradiction.
  Therefore we are looking for real points of c. Clearly, such points have to fulfill $C_r = C_c = 0$.
  - i. $r_2(b_7 - b_\oplus) \neq 0$: Under this assumption we can express $a_\oplus$ from $C_r = 0$ (we obtain two solutions as this is a quadratic equation in $a_\oplus$). Then we plug the obtained expression into $C_c = 0$. After making the resulting equation square root free, it can easily be seen that it cannot be fulfilled w.c..
  - ii. $r_2 = 0$: We express $a_\oplus$ from $C_r = 0$, which is only linear in $a_\oplus$. Then $C_c = 0$ implies $b_7 = b_\oplus$ which yields $\mathsf{M}_7 = \mathsf{M}_\oplus$ and $\mathsf{m}_7 = \mathsf{m}_\oplus$.
  - iii. $b_7 = b_\oplus$: We can express $a_\oplus$ from $C_r = 0$, which is only linear in $a_\oplus$. Then C_c is already fulfilled identically. In this case we also get $\mathsf{M}_7 = \mathsf{M}_\oplus$ and $\mathsf{m}_7 = \mathsf{m}_\oplus$.
- b. $\mathsf{m}_\oplus = \mathsf{u}_1$ ($\Rightarrow a_\oplus = b_\oplus = 0$): Then $\Upsilon_{e_1e_2}$ cannot vanish w.c..

2: We distinguish two cases:
- a. $\mathsf{m}_\oplus \neq \mathsf{u}_1$: Under this assumption we can express $A_\oplus$ and $B_\oplus$ from $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1e_2}$. Now $\Upsilon_{e_0^2}$ can only vanish w.c. for:
  - i. $b_\oplus = 0$: In this case we get $B_\oplus = 0$ and $A_\oplus = A_7a_7/a_\oplus$. This is a projective correspondence between two lines of the platform and the base and therefore we get a regulus. If we choose six legs from this regulus we get an architecturally singular manipulator.
  - ii. $c_2C_r + iC_c = 0$ with $C_r = a_\oplus^2 + b_\oplus^2 - A_7a_7$ and $C_c = r_2(A_7a_7 - a_\oplus^2 - b_\oplus^2) - b_\oplus A_7a_7$, $b_\oplus \neq 0$: It can easily be seen that neither C_r nor C_c can vanish w.c. independently of the choice of $a_\oplus$ and $b_\oplus$.
    Therefore we set $C_r = C_c = 0$:
    - ⋆ $r_2 \neq 0$: Now $C_c + r_2C_r = 0$ yields the contradiction.
    - ⋆ $r_2 = 0$: Then $C_c = 0$ yields the contradiction.
- b. $\mathsf{m}_\oplus = \mathsf{u}_1$ ($\Rightarrow a_\oplus = b_\oplus = 0$): Then $\Upsilon_{e_1^2}$ cannot vanish w.c..

3: We distinguish two cases:

a. $\mathsf{m}_\oplus \neq \mathsf{u}_1$: Under this assumption we can express $A_\oplus$ and $B_\oplus$ from $\Upsilon_{e_1^2}$ and $\Upsilon_{e_1e_2}$. Now $\Upsilon_{e_0^2}$ can only vanish w.c. for:
   i. $b_\oplus = 0$: In this case we get $B_\oplus = 0$ and $A_\oplus = A_7a_7/a_\oplus$. This yields the same contradiction as in item (2ai).
   ii. $c_2C_r[5] + iC_c[8] = 0$, $b_\oplus \neq 0$: In the first part we show that neither C_r nor C_c cannot vanish w.c. independently of the choice of $a_\oplus$ and $b_\oplus$.
   ad C_r) The coefficient of $b_\oplus^2$ implies $r_2 = 0$. Then the coefficient of $b_\oplus$ yields the contradiction.
   ad C_c) The coefficient of $b_\oplus^2$ already yields the contradiction.
   Therefore we set $C_r = C_c = 0$:
   - $\star$ $r_2 \neq 0$: Now $C_c + r_2C_r = 0$ yields the contradiction.
   - $\star$ $r_2 = 0$: Then $C_c = 0$ yields the contradiction.

b. $\mathsf{m}_\oplus = \mathsf{u}_1$ ($\Rightarrow a_\oplus = b_\oplus = 0$): Then $\Upsilon_{e_1^2}$ cannot vanish w.c..

This finishes the proof of Theorem 5.3. □

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