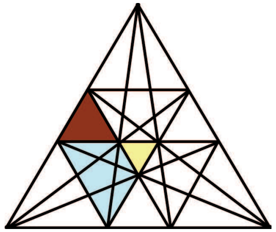
Review and Recent Results on Stewart Gough Platforms with Self-motions

Georg Nawratil

Institute of Geometry, TU Dresden, Germany

Research was supported by FWF (I408-N13)



Austrian Science Fund FWF

Table of contents

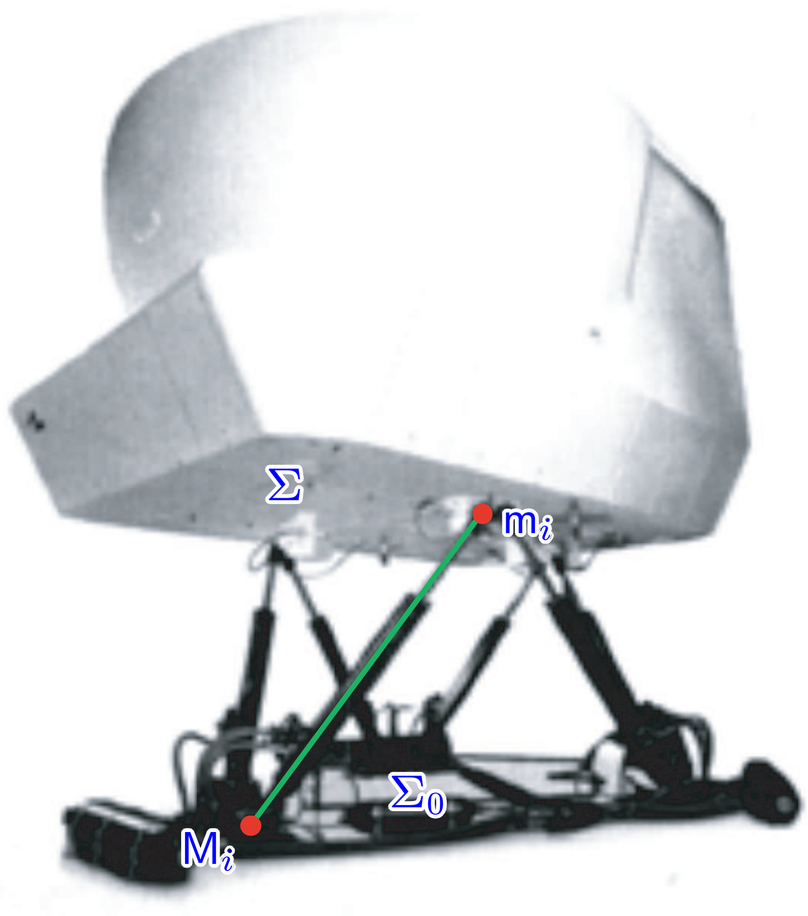
[1] What is a self-motion of a SGP?

The geometry of a SGP is given by the six base anchor points $\mathrm{M}_i \in \Sigma_0$ and by the six platform points $\mathrm{m}_i \in \Sigma$.

A SGP is called planar, if $\mathrm{M}_1, \ldots, \mathrm{M}_6$ are coplanar and $\mathrm{m}_1, \ldots, \mathrm{m}_6$ are coplanar.

M_i and m_i are connected with a S<u>P</u>S leg.

If all <u>P</u>-joints are locked, a SGP is in general rigid. But under particular conditions, the manipulator can perform an n-parametric motion $(n > 0)$, which is called self-motion.



[1] Historical background

All self-motions of SGPs are solutions to the still unsolved Borel Bricard (BB) problem, posed by the French Academy of Science for the Prix Vaillant 1904:

Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.

The papers of Borel [2] and Bricard [3] were awarded prizes, but both authors only presented partial solutions (see [4,5]). Known results before the year 1904 are e.g.:

(a) Chasles [6]: If points of two conics are in projective correspondence, then there exists a spatial motion, which keeps the corresponding points at fixed distance.

(b) Bricard [7]: The only non-trivial motion, where all points have spherical paths.

(c) Bricard [8]: All three types of flexible octahedra in the Euclidean 3-space.

[2] Review on SGPs with self-motions

It is known, that architecturally singular SGPs possess self-motions (over $\mathbb{C}$) in each pose. Their designs are well studied (see [18–21] for the planar case and [22,23] for the non-planar one), but less is known about their self-motions [24–26].

Therefore, we are only interested in self-motions of non-architecturally singular SGPs. Until now, only a few examples of this type are known:

- Husty and Zsombor-Murray [28] reported SGPs with a Schönflies self-motion of item (b). Planar SGPs of this type are also called *polygon platforms* (cf. [29]).
- Zsombor-Murray et al. [30] presented SGPs with a line-symmetric self-motion, which was already known to Borel [2], Bricard [3] and Krames [15].
- Husty and Karger [31] proved that the list of Schönflies Borel Bricard motions given by Borel [2] is complete.

[2] Review on SGPs with self-motions

- Karger [32] studied all self-motions of planar SGPs, where the platform and the base are affinely equivalent (for the special case of equiform and congruent platform and base see [33] and [34], respectively).
- Nawratil [35,36] gave a complete list of TSSM designs (planar 6–3 SGPs) with self-motions as by-product of the determination of all flexible octahedra in the projective extension of the Euclidean 3-space [37].
- Karger and Husty [39] classified all self-motions of the original SGP.
- Karger [40,41] presented a method for designing planar SGPs with self-motions of the type $e_0 = 0$, where e_0 denotes an Euler parameter.
- Geiß and Schreyer [42] gave a pure algebraic method for the computation of further SGPs, where the self-motion has a planar spherical image.

[3] Redundant planar SGPs

Mielczarek et al. [47] showed that the set Λ of additional legs, which can be attached to a given planar SG platform $\mathrm{m}_1, \ldots, \mathrm{M}_6$ without restricting the forward kinematics, is determined by a linear system of equations (Eq. (30) of [47]).

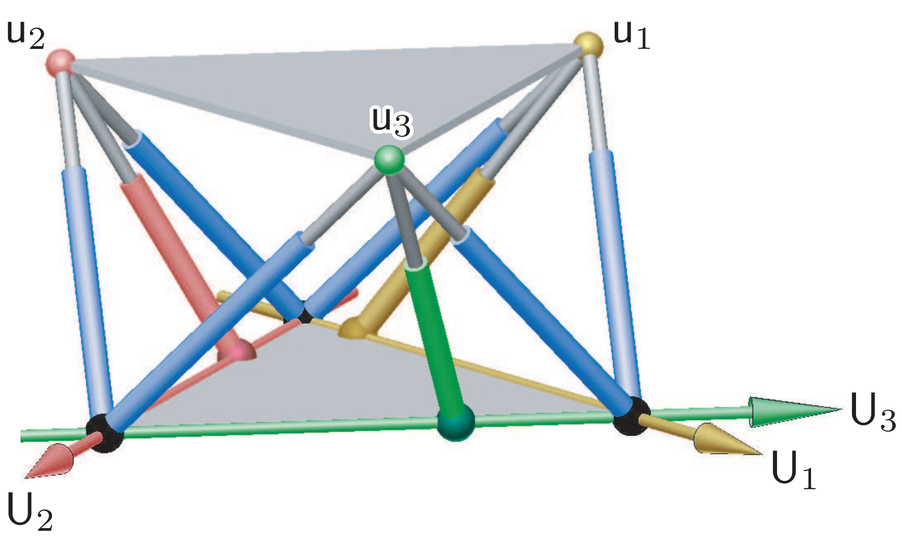
As the solvability condition of this system is equivalent with the criterion given in Eq. (12) of [48] also the singularity surface of the SGP does not change by adding legs of Λ.

Moreover, it was shown in [47] that in the general case Λ is 1-parametric and that the base anchor points as well as the corresponding platform anchor points are located on planar cubic curves C and c, respectively.

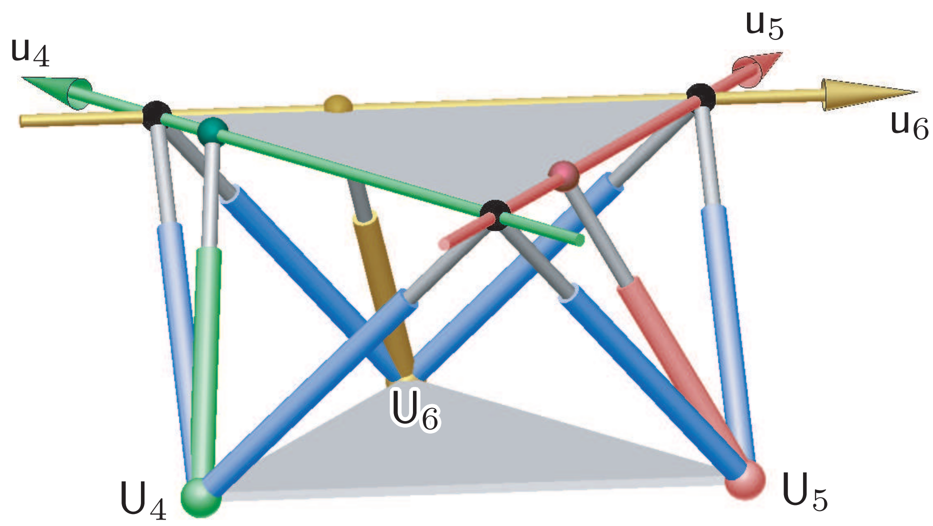
Assumption 1
We assume, that there exist such cubic curves c and C (which can also be reducible) in the Euclidean domain of the platform and the base, respectively.

$[3]$ Example: Octahedral SGP

Notation: U_1, U_2, U_3 are the ideal points of C. u_4, u_5, u_6 are the ideal points of c.



Cubic C of the octahedral SGP



Cubic c of the octahedral SGP

Moreover the yellow, red and green legs are additional legs belonging to the set Λ.

[3] Redundant planar SGPs

As the correspondence between c and C has not to be a bijection (see e.g. octahedral SGP), a point $\in P^3_{\mathbb{C}}$ of c resp. C is in general mapped to a non-empty set of points $\in P^3_{\mathbb{C}}$ of C resp. c. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{\,\}$.

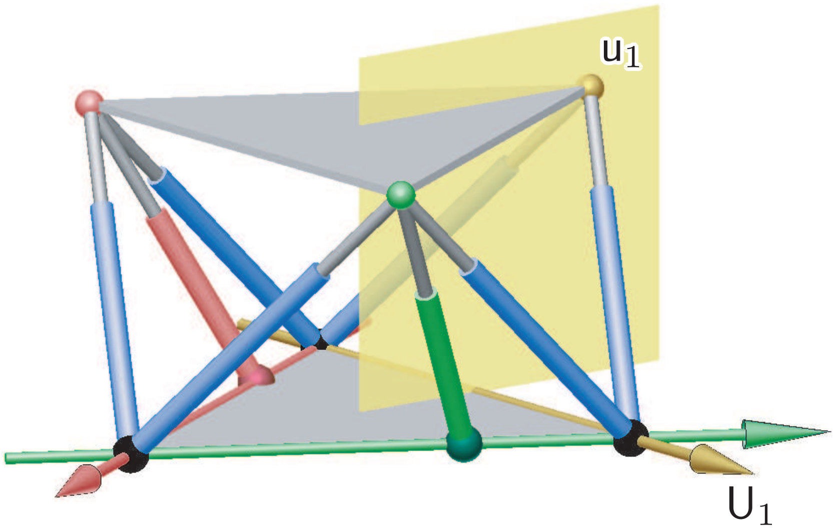
Assumption 2
For guaranteeing a general case, we assume that each of the corresponding locations $\{\mathsf{u}_1\}$, $\{\mathsf{u}_2\}$, $\{\mathsf{u}_3\}$, $\{\mathsf{U}_4\}$, $\{\mathsf{U}_5\}$, $\{\mathsf{U}_6\}$ consists of a single point. Moreover, we assume that no four collinear platform points u_i or base points U_i for $i = 1, \ldots, 6$ exist.

Due to Assumption 2, the six pairs of anchor points $(\mathsf{u}_i, \mathsf{U}_i)$ with $i = 1, \ldots, 6$ are uniquely determined.

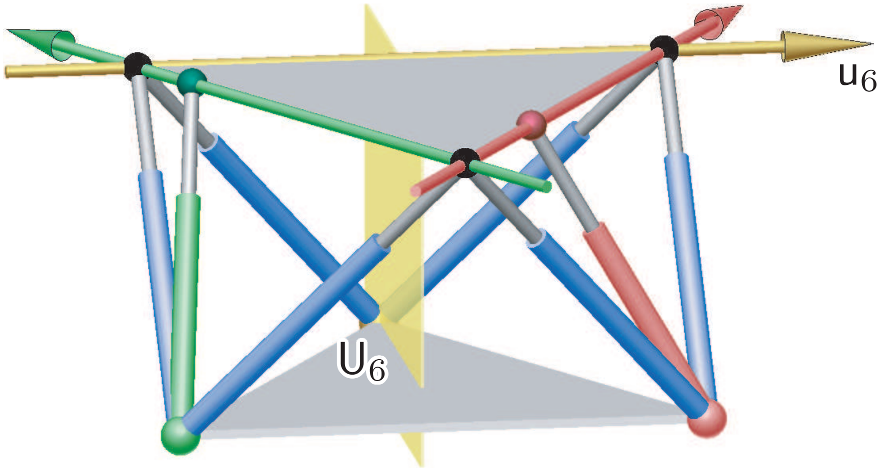
Basic idea: Attach the special "legs" $\overline{\mathsf{u}_i\mathsf{U}_i}$ with $i = 1, \ldots, 6$ to SGP $\mathsf{m}_1, \ldots, \mathsf{M}_6$.

[4] Darboux and Mannheim motion

The attachment of the "legs" $\overline{u_iU_i}$ with $i = 1, 2, 3$ corresponds with the so-called Darboux constraint, that u_i moves in a plane of the fixed system orthogonal to the direction of U_i.



The attachment of the "leg" $\overline{u_jU_j}$ with $j = 4, 5, 6$ corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to u_j slides through the point U_j.



$[4]$ Types of self-motions

By removing the originally six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ we remain with the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$.

Theorem 1 (Proof is given in Nawratil [43])
Given is a planar SGP $\mathsf{m}_1, \ldots, \mathsf{M}_6$, which is not architecturally singular and which fulfills Assumption 1 and 2. Then, the resulting manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is redundant and therefore architecturally singular.

Moreover, it was also proven in [43] that there only exist type I and type II Darboux Mannheim (DM) self-motions, where the definition of types reads as follows:

Definition 1
Assume $\mathcal{M}$ is a 1-parametric self-motion of a non-architecturally singular SGP $\mathsf{m}_1, \ldots, \mathsf{M}_6$. Then $\mathcal{M}$ is of type n DM (Darboux Mannheim) if the corresponding architecturally singular manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ has an n-parametric self-motion $\mathcal{U}$.

$[5]$ Planar SGPs with type II DM self-motions

Nawratil [44,45] proved the necessity of three conditions for obtaining a type II DM self-motion. Based on these conditions, all planar SGPs *fulfilling Assumption 1 and 2* with a type II DM self-motion were determined in [46]. They are either

i. generalizations of line-symmetric Bricard octahedra (12-dim. solution set) or

ii. special polygon platforms (11-dim. solution set).

Moreover, it was shown in [46] that the type II DM self-motions of all these SGPs are line-symmetric and octahedral, where the latter property is defined as follows:

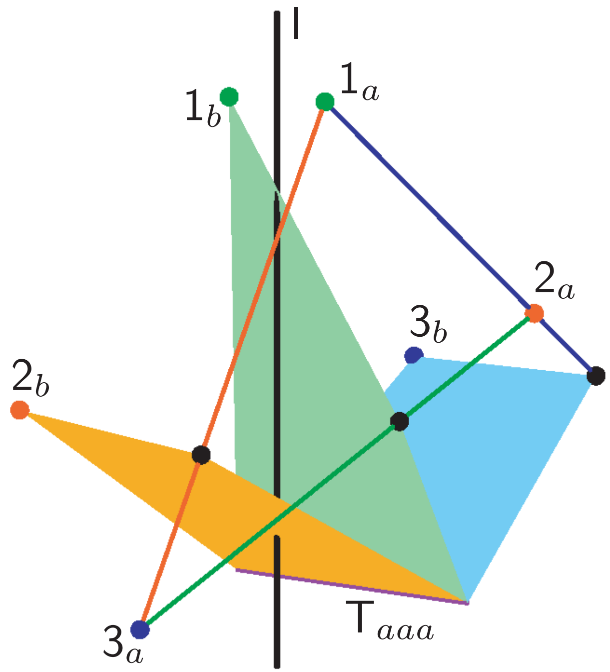
Definition 2

A DM self-motion is called octahedral if following triples of points are collinear for $i \neq j \neq k \neq i$ and $i, j, k \in \{1, 2, 3\}$:

$$(\mathsf{u}_i, \mathsf{u}_j, \mathsf{u}_6),\ (\mathsf{u}_i, \mathsf{u}_k, \mathsf{u}_5),\ (\mathsf{u}_j, \mathsf{u}_k, \mathsf{u}_4),\ (\mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_k),\ (\mathsf{U}_5, \mathsf{U}_6, \mathsf{U}_i),\ (\mathsf{U}_4, \mathsf{U}_6, \mathsf{U}_j).$$

[5] Line-symmetric Bricard octahedra

The geometric interpretation of the three necessary conditions identifies a new property of line-symmetric Bricard octahedra $\mathcal{O}$ with vertices $1_a, 1_b, 2_a, 2_b, 3_a, 3_b$, where v_a and v_b are symmetric with respect to the line l for $v \in \{1, 2, 3\}$. The following planes have a common line T_{ijk}:



$\star$ plane orthogonal to $[1_i, 2_j]$ though $3_{k'}$

$\star$ plane orthogonal to $[2_j, 3_k]$ though $1_{i'}$

$\star$ plane orthogonal to $[3_k, 1_i]$ though $2_{j'}$

with $i \neq i'$, $j \neq j'$, $k \neq k' \in \{a, b\}$.

New property: $\mathcal{O}$ possesses the 8 lines T_{ijk}.

[5] New results in context of Assumption 1 and 2

For the determination of all planar SGPs with a type II DM self-motion, where only Assumption 1 holds, we can assume w.l.o.g. that four platform or base anchor points are collinear, due to the following theorem:

Theorem 2
Given is a planar SGP fulfilling Assumption 1. If one cannot choose anchor points within $\{u_1\}$, $\{u_2\}$, $\{u_3\}$, $\{U_4\}$, $\{U_5\}$, $\{U_6\}$, such that four platform anchor points u_i or base anchor points U_i are collinear, then each of these corresponding locations has to consist of a single point.

Proof: For the proof, please see page 155 of the presented paper. □

Remark: A detailed study of these SGPs is dedicated to future research. ◇

$[5]$ New results in context of Assumption 1 and 2

Γ denotes the set of planar architecturally singular SGPs with no four points collinear, which do not belong to item (a) (cf. Chasles [6]).

Theorem 3
To any planar SGP $m_1, \ldots, M_6$ with exception of the set Γ, at least a 1-parametric set of additional legs Λ can be attached.

Proof: For the proof, please see pages 155–156 of the presented paper. □

Beside Theorem 3, only one non-trivial exceptional case of Assumption 1 is known:

For a planar SGP, where the platform and base are related by a projectivity κ (so-called planar projective SGP), the set Λ is 2-parametric, whereas the correspondence of anchor points of Λ is given by the projectivity κ itself (cf. Nawratil [52]).

Remark: A complete list of special cases is dedicated to future research. ◇

$[6]$ Planar projective SGPs with self-motions

s denotes the lines of intersection of the planar platform and the planar base in the projective extension of the Euclidean 3-space.

Definition 3
A self-motion of a non-architecturally singular planar projective SGP is called elliptic, if in each pose of this motion s exists with s $=$ sκ and the projectivity from s onto itself is elliptic.

Under consideration of this definition, the following result can be proven:

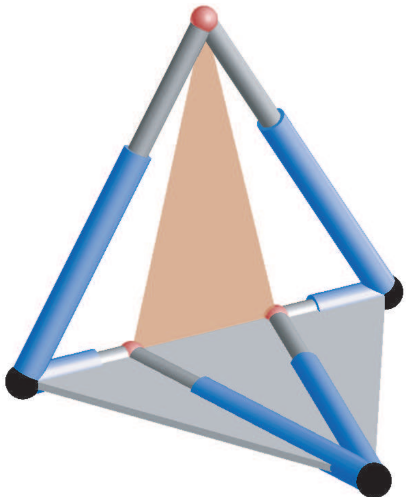
Theorem 4 (Proof is given in Nawratil [52])
Non-architecturally singular planar projective SGPs can only have either elliptic self-motions or pure translational ones. In the latter case κ has to be an affinity $\mathbf{t} + \mathbf{T}\mathbf{x}$, where the singular values s_1, s_2 of $\mathbf{T}$ with $0 < s_1 \leq s_2$ fulfill $s_1 \leq 1 \leq s_2$.

Remark: The study of elliptic self-motions is dedicated to future research. ◇

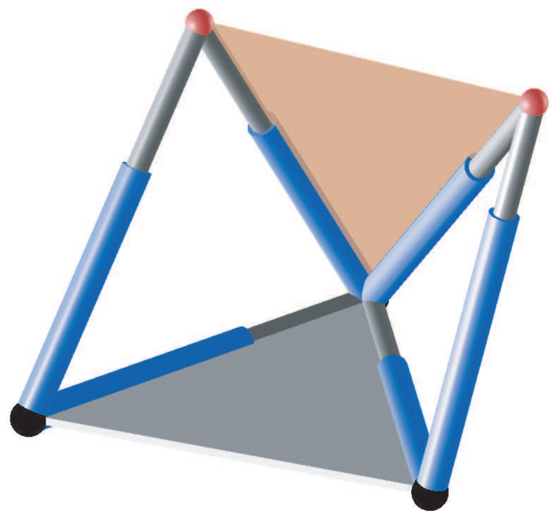
[7] Refining the classification of self-motions

Definition 4
Assume a planar SGP is given, where one can add the set Λ of legs. A self-motion of this manipulator is called degenerated if one can add further legs beside Λ to the planar manipulator without restricting the self-motion.

Butterfly motion



Spherical 4-bar motion



[7] Planar SGPs with type I DM self-motions

- ⋆ Degenerated rotational self-motions: The butterfly motion and the spherical 4-bar motion of the octahedral SGP [53] and the corresponding motions of the original SGP [39].
  Every SGP with four collinear anchor points possesses a butterfly motion.

- ⋆ Degenerated Schönflies self-motions: General polygon platforms.

- ⋆ Non-degenerated self-motions: Bricard octahedra of type 2 and 3.

In view of all known SGPs with self-motions, we have good reasons for the following central conjecture:

Conjecture 1
All non-degenerated 1-parametric self-motions of non-architecturally singular planar SGPs, fulfilling Assumption 1 and 2, are octahedral.

References and acknowledgements

All references refer to the list of publications given in the presented paper:

Nawratil, G.: *Review and Recent Results on Stewart Gough Platforms with Self-motions.* Applied Mechanics and Materials, Vol. **162** (Mechanisms, Mechanical Transmissions and Robotics, G. Gogu et al. eds.), pp. 151–160 (2012)

Acknowledgements
This research is supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.