Basic result on type II DM self-motions of planar Stewart Gough platforms

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$[1a]$ Stewart Gough Platform (SGP)

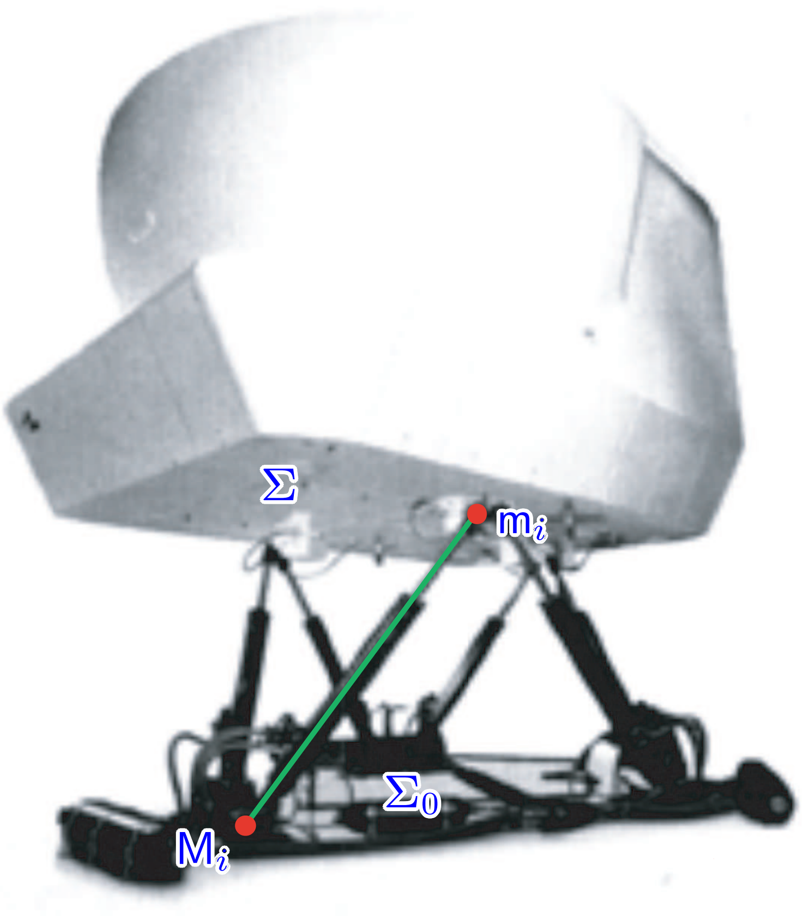
The geometry of a planar SGP is given by the six base anchor points M_i with

$\mathbf{M}_i := (A_i, B_i, 0)^T$ in the fixed space Σ_0,

and by the six platform points m_i with

$\mathbf{m}_i := (a_i, b_i, 0)^T$ in the moving space Σ.

M_i and m_i are connected with a S<u>P</u>S leg.



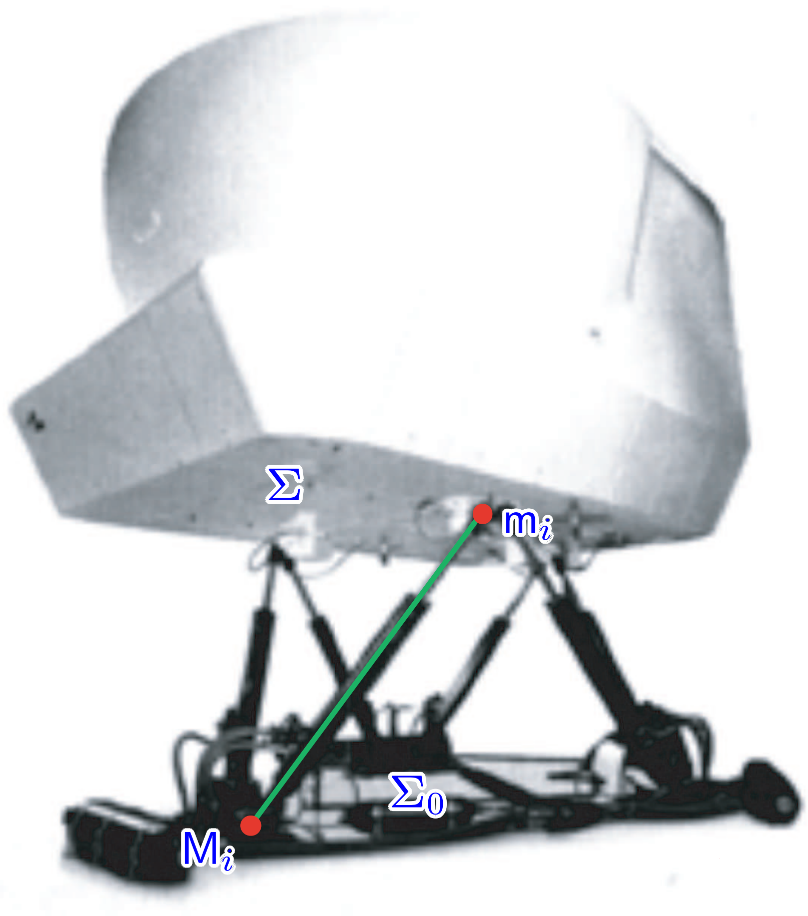
Theorem 1
A SGP is singular (infinitesimal flexible, shaky) if and only if the carrier lines of the six S<u>P</u>S legs belong to a linear line complex.

[1b] Self-motions and the Borel Bricard problem

If all P-joints are locked, a SGP is in general rigid. But, in some special cases the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion.

Note that in each pose of the self-motion, the SGP has to be singular. Moreover, all self-motions of SGPs are solutions to the famous Borel Bricard problem [1,3,4,12].

Borel Bricard problem (still unsolved)
Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.



$[1c]$ Architecturally singular SGPs

Manipulators which are singular in every possible configuration, are called architecturally singular.

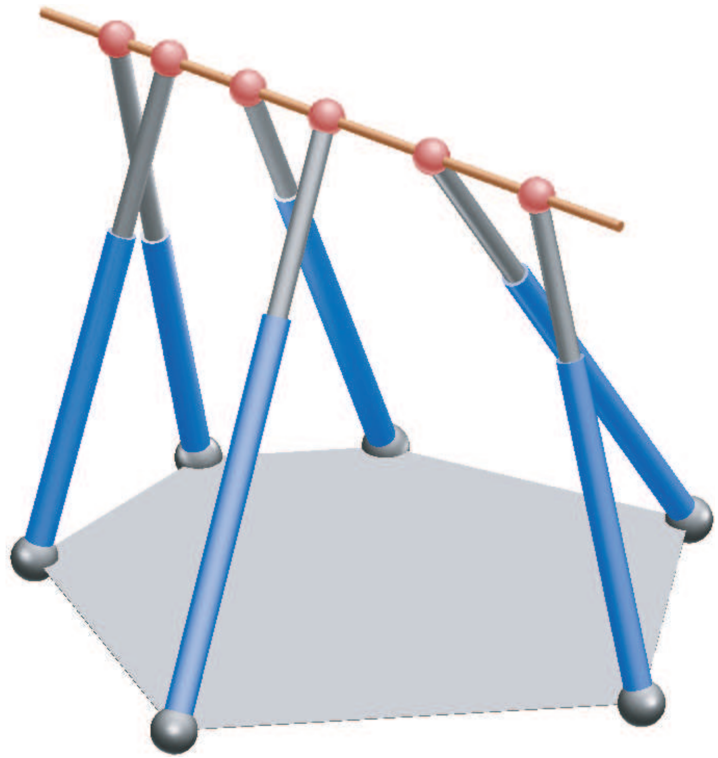
Architecturally singular SGPs are well studied:

⋆ For the planar case see [6,A,B,C],

⋆ For the non-planar case see [D,E].

It is well known, that architecturally singular SGPs possess self-motions in each pose.

Therefore we are only interested in self-motions of non-architecturally singular SGPs.



$[1d]$ Review on SGPs with self-motions

- Husty and Zsombor-Murray [F]: SGP with Schönflies self-motion
- Zsombor-Murray et al. [G]: SGP with line-symmetric self-motion
- Husty and Karger [H] proved that the list of Schönflies Borel Bricard motions given by Borel [1] is complete
- Karger and Husty [I]: Self-motions of the original SGP
- Karger [7,8] presented a method for designing planar SGPs with self-motions of the type $e_0 = 0$, where e_0 denotes an Euler parameter
- Nawratil [J] presented a complete list of TSSM self-motions (6-3 SGPs)

$[2a]$ Redundant planar SGPs

According to Husty [K], the "sphere constraint" that m_i is located on a sphere with center M_i can be expressed by a homogeneous quadratic equation Λ_i in the Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$.

Therefore the direct kinematic problem corresponds to the solution of the system $\Lambda_1, \ldots, \Lambda_6, \Psi$ where Ψ denotes the equation of the Study quadric.

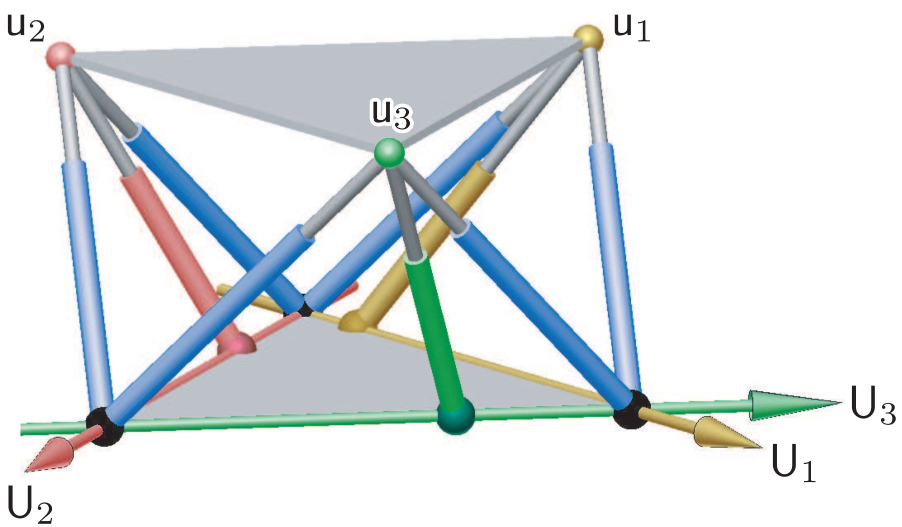
If a planar SGP is not architecturally singular, then at least a 1-parametric set of legs $\lambda_1\Lambda_1 + \ldots + \lambda_6\Lambda_6$ can be added without changing the direct kinematics [5,9].

As the solvability condition of the underlying linear system of equations (Eq. (30) of [9]) is equivalent with the criterion given in Eq. (12) of [2], also the singularity surface of the SGP does not change by adding legs of this 1-parametric set.

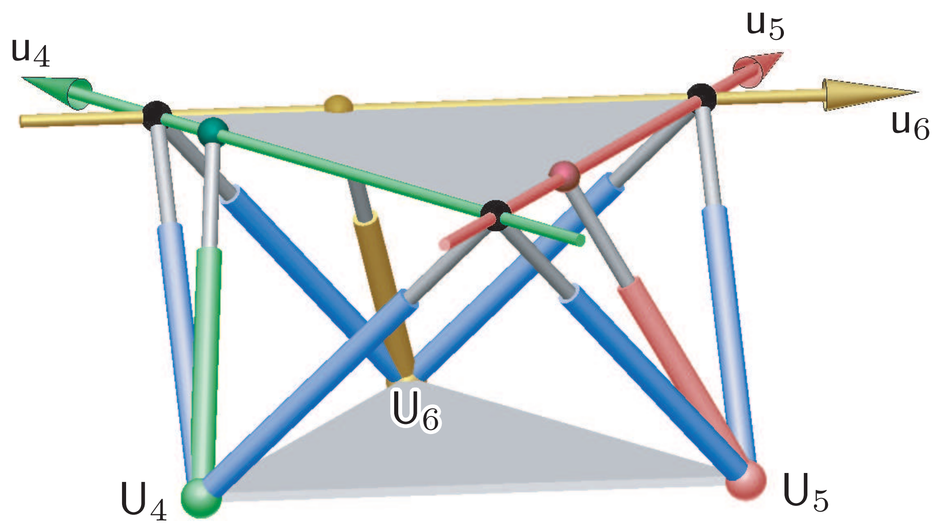
$[2a]$ Redundant planar SGPs

It was shown [5,9] that in general the base anchor points as well as the corresponding platform anchor points are located on planar cubic curves C resp. c.

U_1, U_2, U_3 are the ideal points of C. u_4, u_5, u_6 are the ideal points of c.



Cubic C of the octahedral SGP



Cubic c of the octahedral SGP

$[2b]$ Assumptions and basic idea

Assumption 1
We assume, that there exist such cubic curves c and C in the Euclidean domain of the platform and the base, respectively.

As the correspondence between c and C has not to be a bijection, a point $\in P^3_{\mathbb{C}}$ of c resp. C is in general mapped to an non-empty set of points $\in P^3_{\mathbb{C}}$ of C resp. c. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{\,\}$.

Assumption 2
For guaranteeing a general case, we assume that each of the corresponding locations $\{\mathrm{u}_1\}$, $\{\mathrm{u}_2\}$, $\{\mathrm{u}_3\}$, $\{\mathrm{U}_4\}$, $\{\mathrm{U}_5\}$, $\{\mathrm{U}_6\}$ consists of a single point. Moreover, we assume that no four collinear platform points u_i or base points U_i for $i = 1, \ldots, 6$ exist.

Basic idea: Attach the special "legs" $\overline{\mathrm{u}_i\mathrm{U}_i}$ with $i = 1, \ldots, 6$ to SGP $\mathrm{m}_1, \ldots, \mathrm{M}_6$.

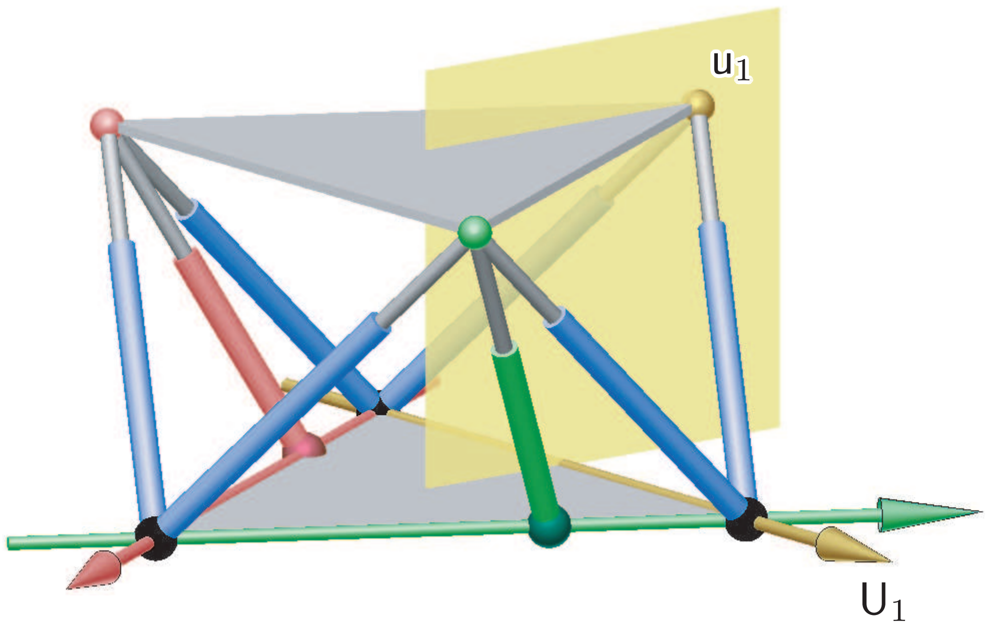
$[2c]$ Darboux constraint

The attachment of the "legs" $\overline{u_i U_i}$ with $i = 1, 2, 3$ corresponds with the so-called Darboux constraint, that the platform anchor point u_i moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i.

The Darboux constraint can be written as a homogeneous quadratic equation Ω_i in the Study parameters (for details see [10]).

Note that Ω_i depends only linearly on f_0, f_1, f_2, f_3.

Remark: Due to Assumption 2 not both points u_i and U_i can be ideal points. ◇



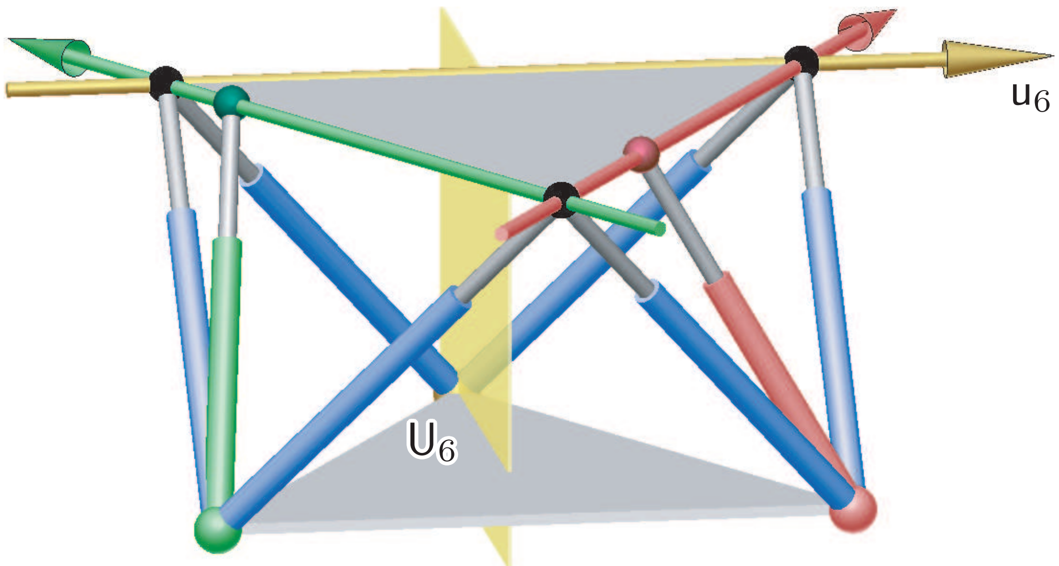
$[2c]$ Mannheim constraint

The attachment of the "leg" $\overline{\mathsf{u}_j\mathsf{U}_j}$ with $j = 4, 5, 6$ corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to u_j slides through the point U_j.

The Mannheim constraint can be written as a homogeneous quadratic equation Π_j in the Study parameters (for details see [10]).

Note that Π_j depends only linearly on f_0, f_1, f_2, f_3.

Remark: Due to Assumption 2 not both points u_j and U_j can be ideal points. $\diamond$



$[2d]$ Types of self-motions

Theorem 2 (Proof is given in [10])
Given is a planar SGP $\mathrm{m}_1, \ldots, \mathrm{M}_6$ which is not architecturally singular and which fulfills Assumption 1 and 2. Then the resulting manipulator $\mathrm{u}_1, \ldots, \mathrm{U}_6$ is redundant and therefore architecturally singular.

Definition 1
Assume $\mathcal{M}$ is a 1-parametric self-motion of a non-architecturally singular SGP $\mathrm{m}_1, \ldots, \mathrm{M}_6$. Then $\mathcal{M}$ is of type n DM (Darboux Mannheim) if the corresponding architecturally singular manipulator $\mathrm{u}_1, \ldots, \mathrm{U}_6$ has an n-parametric self-motion $\mathcal{U}$ (which includes $\mathcal{M}$).

Theorem 3 (Proof is given in [10])
All 1-parametric self-motions of non-architecturally singular planar SGPs fulfilling Assumption 1 and 2 are type I or type II DM self-motions.

$[3a]$ **Computation**

W.l.o.g. we can assume that the variety of a 2-parametric DM self-motion is spanned by $\Psi, \Omega_1, \Omega_2, \Omega_3$, Π_4, Π_5 (otherwise we can consider the inverse motion).

Lemma 1 (Proof is given in [10])
W.l.o.g. we can choose coordinate systems in Σ_0 and Σ with $X_2(X_2 - X_3)x_5 \neq 0$,

$$a_1 = b_1 = y_4 = A_4 = B_4 = Y_1 = h_4 = g_5 = 0, \quad X_1 = Y_2 = Y_3 = x_4 = y_5 = 1,$$

where $(0 : X_i : Y_i : 0)$ and $(0 : x_i : y_i : 0)$ are the projective coordinates of the ideal points U_i and u_i, respectively.

We solve $\Psi, \Omega_1, \Omega_2, \Pi_4$ for $f_0, \ldots, f_3$ and plug the obtained expressions in the remaining two equations which yield $\Omega[40]$ (degree 2) and $\Pi[96]$ (degree 4).

Finally, we compute the resultant of $\Omega[40]$ and $\Pi[96]$ with respect to one of the Euler parameters. For e_0 this yields $\Gamma[117\,652]$ (degree 8).

$[3a]$ Computation

In the following, we list the coefficients of $e_1^i e_2^j e_3^k$ of Γ, which are denoted by Γ_{ijk}:

$$\Gamma_{080} = F_1[8]F_2[18]^2, \qquad \Gamma_{800} = (b_2 - b_3)^2(L_1 - g_4)^2 F_3[8],$$

$$\Gamma_{170} = F_2[18]F_4[283], \qquad \Gamma_{710} = (b_2 - b_3)(L_1 - g_4)F_5[170],$$

$$\begin{array}{llll}
\Gamma_{620}[2054], & \Gamma_{602}[1646], & \Gamma_{260}[6126], & \Gamma_{062}[4916], \\
\Gamma_{026}[5950], & \Gamma_{116}[3066], & \Gamma_{530}[4538], & \Gamma_{512}[4512], \\
\Gamma_{152}[6514], & \Gamma_{440}[7134], & \Gamma_{422}[6314], & \Gamma_{242}[7622], \\
\Gamma_{044}[6356], & \Gamma_{314}[6934], & \Gamma_{224}[7096], & \Gamma_{134}[6656], \\
\Gamma_{206}[5950], & \Gamma_{350}[7166], & \Gamma_{404}[5766], & \Gamma_{332}[6982].
\end{array}$$

Based on these 24 equations $\Gamma_{ijk} = 0$ (in 14 unknowns), we were able to proof the following basic result on type II DM self-motions.

$[3b]$ The special cases $(\star)$ and $(\circ)$

We denote the coefficient $e_0^i e_1^j e_2^k e_3^l$ of $\Omega[40]$ by Ω_{ijkl}.

By computing $\Omega_{2000} + \Omega_{0002}$, $\Omega_{2000} - \Omega_{0002}$ and Ω_{1001} it can immediately be seen that Ω does not depend on e_0 and e_3 iff

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = \overline{X}_2 a_2 - \overline{X}_3 a_3 + b_2 - b_3 = \overline{X}_2 b_2 - \overline{X}_3 b_3 - a_2 + a_3 = 0 \quad (\star)$$

By computing $\Omega_{0200} + \Omega_{0020}$, $\Omega_{0200} - \Omega_{0020}$ and Ω_{0110} it can immediately be seen that Ω does not depend on e_1 and e_2 iff

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = \overline{X}_2 a_2 - \overline{X}_3 a_3 - b_2 + b_3 = \overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2 - a_3 = 0 \quad (\circ)$$

Remark: The conditions given in $(\star)$ and $(\circ)$ are equivalent with the conditions given in Eqs. (1-3) of the presented paper. $\diamond$

$[3c]$ Basic result

Theorem 4 (Proof is given in the presented paper and its corresponding technical report [11])
If neither the equations $(\star)$ nor the equations $(\circ)$ are fulfilled, then the corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion, has to have further 3 collinear anchor points in the base or in the platform beside the points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ and $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$.

Due to Lemma 2 of [6] and Theorem 2 we can replace the word "or" in Theorem 4 by the word "and"; i.e.

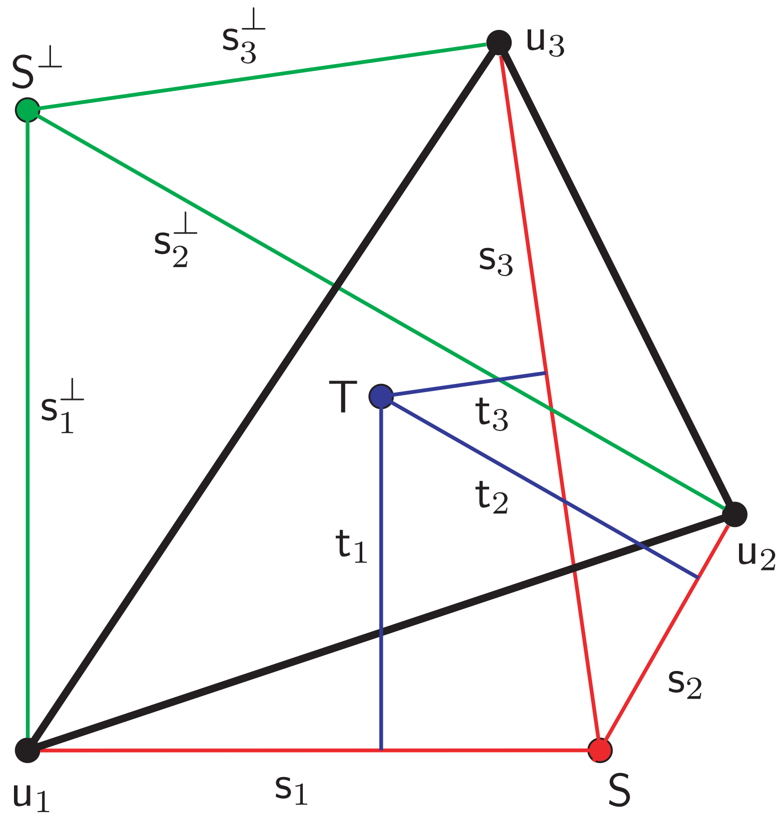
If neither the equations $(\star)$ nor the equations $(\circ)$ are fulfilled, then the corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion, has to have 3 collinear platform points $\mathsf{u}_i, \mathsf{u}_j, \mathsf{u}_k$ and 3 collinear base points $\mathsf{U}_l, \mathsf{U}_m, \mathsf{U}_n$ beside the points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ and $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$ where (i, j, k, l, m, n) consists of all indices from 1 to 6.

$[3d]$ Geometric interpretation of $(\star)$

(I) $L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0$ expresses that the three lines $\mathsf{t}_i \in \Sigma_0$ $(i = 1, 2, 3)$ with homogeneous line coordinates $[L_i : \overline{X}_i : \overline{Y}_i]$ have a common point T ($\Rightarrow$ the three Darboux planes belong to a pencil of planes).

(II) $\overline{X}_2 b_2 - \overline{X}_3 b_3 - a_2 + a_3 = 0$ expresses that the three lines $\mathsf{s}_i := [\mathsf{u}_i, \overline{\mathsf{U}}_i]$ $(i = 1, 2, 3)$ have a common point S.

(III) $\overline{X}_2 a_2 - \overline{X}_3 a_3 + b_2 - b_3 = 0$ expresses that the three lines $\mathsf{s}_i^\perp := [\mathsf{u}_i, \overline{\mathsf{U}}_i^\perp]$ $(i = 1, 2, 3)$ have a common point $S^\perp$.



$[3d]$ Geometric interpretation of $(\star)$ and $(\circ)$

Note that the items (II) and (III) only hold if the coordinate systems are chosen according to Lemma 1 and if these two coordinate systems coincide.

Moreover, the geometric interpretation of $(\circ)$ is equivalent with the one given for $(\star)$, if one rotates the platform about the x-axis with angle π.

The geometric interpretation of $(\star)$ and $(\circ)$ is important because it was recently proved in [L] that $(\star)$ resp. $(\circ)$ are even necessary for a type II DM self-motion:

Theorem 5 (Proof is given in [L])
The corresponding manipulator $\mathrm{u}_1, \ldots, \mathrm{U}_6$ of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion has to fulfill the three conditions of either $(\star)$ or $(\circ)$.

$[3d]$ Line-symmetric Bricard octahedra

Due to [10] every line-symmetric Bricard octahedron $\mathcal{O}$ possess a type II DM self-motion. We denote the vertices of $\mathcal{O}$ by $1_a, 1_b, 2_a, 2_b, 3_a, 3_b$, where i_a and i_b are symmetric with respect to the line l for $i \in \{1, 2, 3\}$. Due to item (I) the following three planes have a common line T_{ijk}:

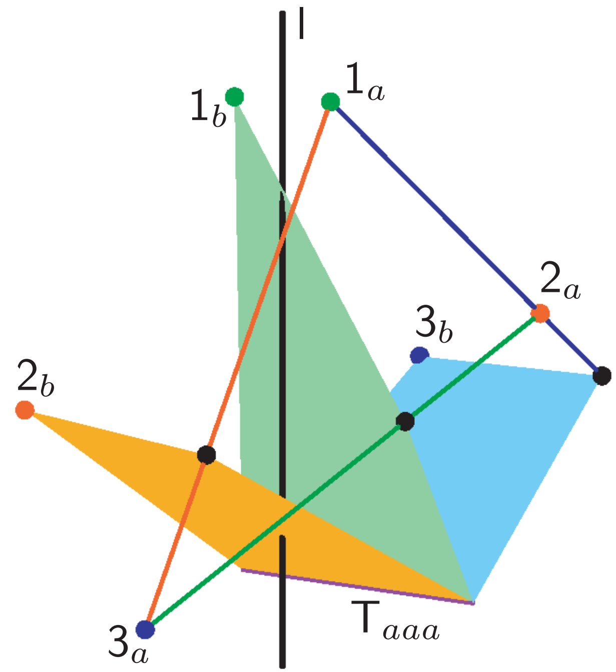
⋆ plane orthogonal to $[1_i, 2_j]$ though $3_{k'}$

⋆ plane orthogonal to $[2_j, 3_k]$ though $1_{i'}$

⋆ plane orthogonal to $[3_k, 1_i]$ though $2_{j'}$

with $i \neq i'$, $j \neq j'$, $k \neq k' \wedge i', j', k' \in \{a, b\}$.

New property: $\mathcal{O}$ possesses the 8 lines T_{ijk}.



[4] References

For [1-12] see the presented paper. The remaining references [A-L] are as follows:

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