On Stewart Gough manipulators with multidimensional self-motions

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On Stewart Gough manipulators with multidimensional self-motions

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Abstract

By means of bond theory, we study Stewart Gough (SG) platforms with n-dimensional self-motions with $n > 2$. It turns out that only architecturally singular manipulators can possess these self-motions. Based on this result, we present a complete list of all SG platforms, which have n-dimensional self-motions. Therefore this paper also solves the famous Borel Bricard problem for n-dimensional motions. We also give some remarks and a new result on SG platforms with 2-dimensional self-motions; nevertheless a full discussion of this case remains open.

Keywords: Self-motion, Stewart Gough platform, Bond Theory, Borel Bricard problem, Kinematotropy

1. Introduction

The geometry of a Stewart Gough (SG) platform is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ with respect to the fixed system and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$ with respect to the moving system (for $i = 1, \ldots, 6$). Each pair $(\mathsf{M}_i, \mathsf{m}_i)$ of corresponding anchor points is connected by a SPS-leg, where only the prismatic joint (P) is active and the spherical joints (S) are passive.

If the geometry of the manipulator is given, as well as the lengths of the six pairwise distinct legs, the SG platform is generically rigid. But, under particular conditions, the manipulator can perform a n-dimensional motion ($n > 0$), which is called self-motion. If $n > 1$ holds, the self-motion is a so-called *multidimensional* one.

Note that self-motions are also solutions to the still unsolved problem posed by the French Academy of Science for the *Prix Vaillant* of the year 1904, which is also known as Borel Bricard problem (cf. [1, 2, 3]) and reads as follows: *"Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."*

It is already known that manipulators, which are singular in every possible configuration, possess self-motions in each pose (over $\mathbb{C}$). These so-called architecturally singular SG platforms are well studied and classified (for the planar case we refer to [4, 5, 6, 7] and for the non-planar case see [8, 9]). In contrast only a few self-motions of non-architecturally singular SG platforms are known. A detailed review of these self-motions was given in [10], which is as complete as possible to the best knowledge of the author. Moreover until now only the following non-architecturally singular SG platform with a *multidimensional* self-motion is known to the author: The platform and the base are *congruent*. This so-called *congruent* SG platform has a 2-dimensional self-motion, if all legs have equal length. It can easily be seen that this self-motion is pure translational.

Example 1. *We consider a special congruent SG platform, which is also known as* Wren *platform (cf. [11]). As in this case, the anchor points are located on a circle (see Fig. 1a), the* Wren *platform is an architecturally singular manipulator (cf. [2, 12, 13]). If all legs have equal length, there also exists a* 1*-dimensional* Schönflies *self-motion (see Fig. 1b), beside the already mentioned* 2*-dimensional translational self-motion (see Fig. 1c). In Fig. 1a, the branching singularity (cf. [14]) of these two self-motions is displayed. Due to* Wohlhart *[11], the manipulator is called kinematotropic, as it can change the dimension of mobility.* ⋄

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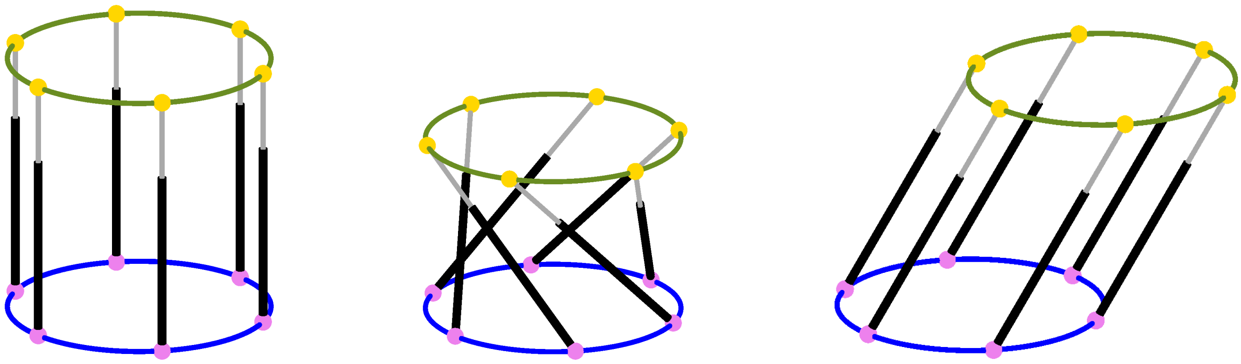

Figure 1: WREN platform: The configuration given in (a) is called a *branching singularity*, as it belongs to the 1-dimensional SCHÖNLFIES self-motion (b) and to the 2-dimensional translational self-motion (c).

Beside the determination of additional examples of non-architecturally singular SG platforms with *multidimensional* self-motions, we also want to close the knowledge gap, whether non-architecturally singular SG platforms with 3-dimensional self-motions exist (cf. footnote 3 of [15]). This still open problem is answered within the article at hand, which is structured as follows:

In Section 2.1, we give a short introduction to the theory of bonds for SG platforms with self-motions. Depending on the dimension β of the bonding surface of the 3-dimensional self-motion after its projection from the STUDY parameter space into the EULER parameter space, we can distinguish four cases: $\beta = -1, 0, 1, 2$ (cf. Section 2.2). In Section 3, we prove that non-architecturally singular SG platforms with 3-dimensional self-motions of type $\beta = 2$ do not exist. In Section 4, we show that this is also the case for $\beta = -1, 0, 1$. Based on the obtained results, we give a complete list of all SG platforms, which have n-dimensional self-motions with $n > 2$, in Section 5. Finally, we close the paper with some remarks and a new result on SG platforms with 2-dimensional self-motions (cf. Section 6).

2. Bond Theory

In Section 2.1, we give a short introduction to the theory of bonds for SG manipulators presented in [16], which was motivated by the bond theory of overconstrained closed linkages with revolute joints given by HEGEDÜS, SCHICHO and SCHRÖCKER in [17] (see also [18]). We start with the direct kinematic problem of parallel manipulators of SG type and proceed with the definition of bonds. Based on these basics, we do some preparatory work in Section 2.2 by giving a classification of 3-dimensional self-motions, which is induced by the bond theory in a natural way.

2.1. Definition of bonds

Due to the result of HUSTY [19], it is advantageous to work with STUDY parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ for solving the forward kinematics. Note that the first four homogeneous coordinates $(e_0 : e_1 : e_2 : e_3)$ are the so-called EULER parameters. Now, all real points of the STUDY parameter space P^7 (7-dimensional projective space), which are located on the so-called STUDY quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the 3-dimensional subspace E of Ψ given by $e_0 = e_1 = e_2 = e_3 = 0$, as its points cannot fulfill the condition $N \neq 0$ with $N = e_0^2 + e_1^2 + e_2^2 + e_3^2$. The translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and the rotation matrix $\mathbf{R} := (r_{ij})$ of the corresponding Euclidean displacement $\mathbf{Rx} + \mathbf{t}$ are given by:

$$t_1 = 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \quad t_2 = 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3), \quad t_3 = 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1),$$

and

$$\mathbf{R} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}, \tag{1}$$

if the normalizing condition $N = 1$ is fulfilled. All points of the complex extension of P^7, which cannot fulfill this normalizing condition, are located on the so-called exceptional cone $N = 0$ with vertex E.

By using the Study parametrization of Euclidean displacements, the condition that the point m_i is located on a sphere centered in M_i with radius R_i, is a quadratic homogeneous equation according to Husty [19]. This so-called sphere condition Λ_i has the following form:

$$\begin{aligned}\Lambda_i:\quad &(a_i^2 + b_i^2 + c_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)N - 2(a_iA_i + b_iB_i + c_iC_i)e_0^2 - 2(a_iA_i - b_iB_i - c_iC_i)e_1^2 \\ &+ 2(a_iA_i - b_iB_i + c_iC_i)e_2^2 + 2(a_iA_i + b_iB_i - c_iC_i)e_3^2 + 4(c_iB_i - b_iC_i)e_0e_1 - 4(c_iA_i - a_iC_i)e_0e_2 \\ &+ 4(b_iA_i - a_iB_i)e_0e_3 - 4(b_iA_i + a_iB_i)e_1e_2 - 4(c_iA_i + a_iC_i)e_1e_3 - 4(c_iB_i + b_iC_i)e_2e_3 \\ &+ 4(a_i - A_i)(e_0f_1 - e_1f_0) + 4(b_i - B_i)(e_0f_2 - e_2f_0) + 4(c_i - C_i)(e_0f_3 - e_3f_0) + 4(a_i + A_i)(e_3f_2 - e_2f_3) \\ &+ 4(b_i + B_i)(e_1f_3 - e_3f_1) + 4(c_i + C_i)(e_2f_1 - e_1f_2) + 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0.\end{aligned} \qquad (2)$$

Now the solution of the direct kinematics over $\mathbb{C}$ can be written as the algebraic variety V of the ideal $\mathcal{I}$ spanned by $\Psi, \Lambda_1, \ldots, \Lambda_6, N = 1$. In general V consists of a discrete set of points with a maximum of 40 elements.

We consider the algebraic motion of the mechanism, which is defined as the set of points on the Study quadric determined by the constraints; i.e. the common points of the seven quadrics $\Psi, \Lambda_1, \ldots, \Lambda_6$. If the manipulator has a n-dimensional self-motion then the algebraic motion also has to be of this dimension. Now the points of the algebraic motion with $N \neq 0$ equal the kinematic image of the algebraic variety V. But we can also consider the points of the algebraic motion, which belong to the exceptional cone $N = 0$. An exact mathematical definition of these so-called bonds can be given as follows (cf. Remark 5 of [16]):

Definition 1. *For a SG manipulator the set $\mathcal{B}$ of bonds is defined as:*

$$\mathcal{B} := ZarClo(V^\star) \ \cap \ \{(e_0 : \ldots : f_3) \in P^7 \mid \Psi, \Lambda_1, \ldots, \Lambda_6, N = 0\},$$

where $V^\star$ denotes the variety V after the removal of all components, which correspond to pure translational motions. Moreover $ZarClo(V^\star)$ is the Zariski closure of $V^\star$, i.e. the zero locus of all algebraic equations that also vanish on $V^\star$.

We have to restrict to non-translational motions for the following reason: A component of V, which corresponds to a pure translational motion, is projected to a single point O (with $N \neq 0$) of the Euler parameter space P^3 by the elimination of $f_0, \ldots, f_3$. Therefore the intersection of O and $N = 0$ equals $\varnothing$. Clearly, the kernel of this projection equals the group of translational motions. Moreover it is important to note that the set of bonds depends on the geometry of the manipulator, and not on the leg lengths (cf. Theorem 1 of [16]). For more details please see [16].

Due to Theorem 2 of [16] a SG platform possesses a pure translational self-motion, if and only if the platform can be rotated about the center $\mathsf{m}_1 = \mathsf{M}_1$ into a pose, where the vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i = 2, \ldots, 6$ fulfill the condition $rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2}, \ldots, \overrightarrow{\mathsf{M}_6\mathsf{m}_6}) \leq 1$. Moreover all 1-dimensional self-motions are circular translations, which can easily be seen by considering a normal projection of the SG manipulator in direction of the parallel vectors $\overrightarrow{\mathsf{M}_i\mathsf{m}_i}$ for $i = 2, \ldots, 6$. If all these five vectors are zero-vectors, which corresponds with the case that the platform and the base are congruent, then we get the already mentioned 2-dimensional translational self-motion.

2.2. *Classification of 3-dimensional self-motions*

We assume that a given SG manipulator has a 3-dimensional self-motion $\mathcal{S}$. As $\mathcal{S}$ corresponds with a 3-dimensional solution of the direct kinematics problem, the corresponding algebraic motion is also 3-dimensional. Due to the fact that this algebraic motion is the kinematic image of a self-motion, it cannot be located within the exceptional cone $N = 0$ (hypersurface). Therefore the bond-set of this self-motion has to be an algebraic variety of dimension 2; i.e. a *bonding surface*.

Now we want to classify $\mathcal{S}$ with respect to the dimension β of the bonding surface after its projection into the Euler parameter space P^3. As we have a bonding surface in the Study parameter space P^7, β can take the values $-1, 0, 1, 2$, where the case $\beta = 2$ is the general one. In order that $\beta = i$ holds for $i = -1, 0, 1$, there has to exist a $(2-i)$-dimensional translational sub-self-motion, which is contained in $\mathcal{S}$, in each pose of $\mathcal{S}$ according to the paragraph below Definition 1. For $i = -1$ this already implies that $\mathcal{S}$ is a 3-dimensional translation.

To make things more clear, we give the following example:

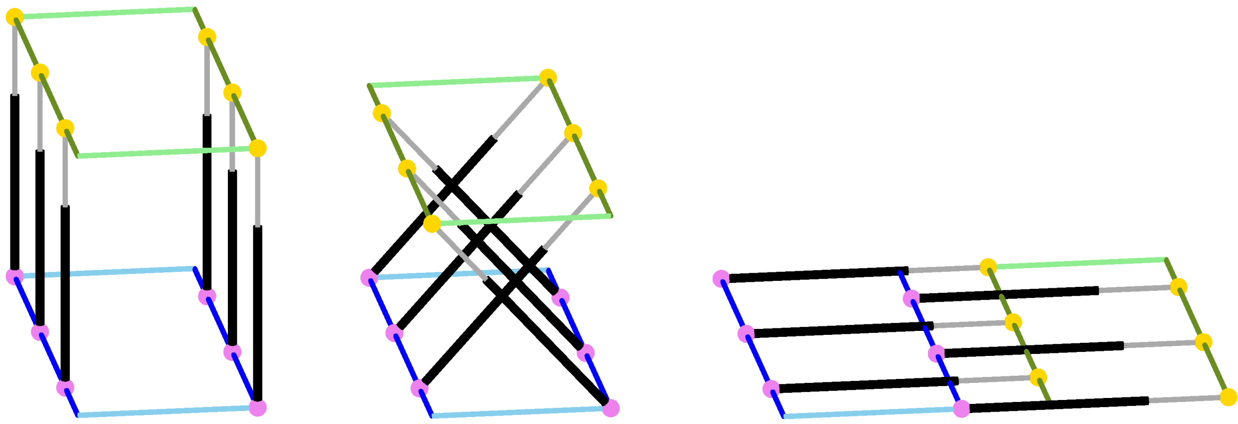

Figure 2: (a) parallelogram mode (b) anti-parallelogram mode (c) bifurcation configuration.

Example 2. *We study a congruent SG platform where the anchor points are located on two parallel lines. As this is also a conic section, the manipulator is again architecturally singular. If all legs have equal lengths, this manipulator has two 2-dimensional self-motions, where the first one (parallelogram mode) is the already known translation (cf. Fig. 2a). Therefore this self-motion is of type $\beta = -1$.*

The second 2-dimensional self-motion (anti-parallelogram mode) has a 1-dimensional translational sub-self-motion (circular translation) in each pose of its self-motion (cf. Fig. 2b).

For this manipulator, we can compute the bond-set according to [16]. The two parallel lines of the base are given by the x-axis of the fixed frame and a x-parallel line through the point $(0, d, 0)^T$ of the fixed frame with $d > 0$. Therefore the base anchor points have coordinates $(g, 0, 0)^T$ and $(h, d, 0)^T$, respectively, with $g, h \in \mathbb{R}$. The corresponding platform anchor points have the same coordinates with respect to the moving frame. Then we can compute the set of bonds $\mathcal{B}$ according to [16], which yields the following four bonds, up to conjugation of coordinates:

$$\mathcal{B} := \{(-2u/d : 2uI/d : 0 : 0 : v : -vI : u : -uI), (2u/d : 2uI/d : 0 : 0 : v : vI : u : -uI),$$
$$(0 : 0 : -2u/d : 2uI/d : u : uI : vI : v), (0 : 0 : 2u/d : 2uI/d : u : uI : -vI : v)\},$$

where I denotes the complex unit. Note that only the first and second bond can belong to the anti-parallelogram self-motion, as the x-axes of the moving and the fixed frame are parallel and equally directed during this self-motion ($\Leftrightarrow e_2 = e_3 = 0$). By restricting us to the first four coordinate entries, we project the first and second bonding curve[1] to the Euler *parameter space P^3, which yields the points $(-1 : I : 0 : 0)$ and $(1 : I : 0 : 0)$. This shows that the anti-parallelogram self-motion is indeed of type $\beta = 0$.*

Moreover it should be noted that this manipulator does not change the dimension of mobility but the type of mobility (cf. [20]) according to the above given classification of self-motions.[2] The flattened position of the manipulator illustrated in Fig. 2c is a bifurcation configuration between the parallelogram mode ($\beta = -1$) and the anti-parallelogram mode ($\beta = 0$). ⋄

3. 3-dimensional self-motions of type $\beta = 2$

A necessary condition for a non-architecturally singular SG platform with a 3-dimensional self-motion is that the 4-legged manipulator resulting from the removal of two legs, which fulfill certain conditions in order to enable the 3-dimensional self-motion, has also a 3-dimensional self-motion.[3]

[1] Note that the ratio $u : v$ can be seen as projective parameter with $(u, v) \neq (0, 0)$.

[2] Note that the Wren platform also changes the type of mobility with respect to this classification in its branching singularity, as the Schönflies self-motion is of type $\beta = 0$ (cf. Example 1 of [16]).

[3] If there is only one leg, which fulfills a certain constraint, then its removal would imply that any arbitrary 5-legged manipulator has two degrees of freedom, which yields a contradiction to the formula of Grübler.

Without loss of generality (w.l.o.g.) we can assume that the removed legs are those with index 5 and 6, respectively. Then the solution of the direct kinematics over $\mathbb{C}$ of this 4-legged manipulator can be written as the algebraic variety V_4 of the ideal $\mathcal{I}_4$ spanned by $\Psi, \Lambda_1, \ldots, \Lambda_4, N = 1$.

Now the bond-set $\mathcal{B}_4$ of the 4-legged manipulator can be defined analogously to Definition 1 by

$$\mathcal{B}_4 := ZarClo(V_4^\star) \ \cap \ \{(e_0 : \ldots : f_3) \in P^7 \mid \Psi, \Lambda_1, \ldots, \Lambda_4, N = 0\},$$

where $V_4^\star$ denotes the variety V_4 after the removal of all components, which correspond to pure translational motions.

As $\mathcal{I}_4$ is a subideal of $\mathcal{I}$, the bond-set $\mathcal{B}$ is a subset of $\mathcal{B}_4$. Based on the four leg constraints $\Lambda_1, \ldots, \Lambda_4$, the bonding surface $\mathcal{B}_4$ can be computed as follows:

We compute the equation $\Delta_{j,i}$, which denotes the difference $\Lambda_j - \Lambda_i$ of two sphere constraints. Note that $\Delta_{j,i}$ is only linear in $f_0, \ldots, f_3$. Now we can solve the equations $\Psi, \Delta_{j,i}, \Delta_{k,i}, \Delta_{l,i}$ with pairwise distinct $i, j, k, l, m \in \{1, \ldots, 4\}$ for $f_0, \ldots, f_3$ and plug the obtained solutions into the remaining equation Λ_i. The numerator of the resulting expression is a polynomial G of degree 8 in the Euler parameters, which can also be seen as an octic surface Γ in the Euler parameter space P^3.

Note that $G = 0$ is the algebraic variety of the elimination-ideal of $\Psi, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ with respect to $f_0, \ldots, f_3$. As the elimination of $f_0, \ldots, f_3$ is done linearly, G does not depend on the choice of $i \in \{1, \ldots, 4\}$. Therefore each point of Γ corresponds to an orientation of the 4-legged manipulator, which can be extended to a solution of the forward kinematics problem (over $\mathbb{C}$). Summed up we can say that the points of Γ with $N \neq 0$ are the projection of V_4 into the Euler parameter space P^3, which is induced by the elimination of $f_0, \ldots, f_3$.

Moreover it should be noted that the linear system $\Psi, \Delta_{j,i}, \Delta_{k,i}, \Delta_{l,i}$ cannot be solved for the f_i's if and only if the 4-legged manipulator has one of the following geometries (cf. pages 512–513 of [21] as well as pages 23–24 of [22]):

1. $\mathsf{m}_1, \ldots, \mathsf{m}_4$ are collinear and $\mathsf{M}_1, \ldots, \mathsf{M}_4$ are collinear,
2. the platform and the base are planar and the anchor points are related within an affinity,
3. the platform and the base are congruent and non-planar.

As these special cases are discussed separately in Section 3.2, we can assume w.l.o.g. that the 4-legged manipulator has non of these three geometries in the general case, which is discussed next.

3.1. General Case

The intersection of Γ and $N = 0$ already yields the projection of the bond surface $\mathcal{B}_4$ into P^3. As $N = 0$ is an irreducible quadratic surface in P^3, the intersection of Γ and $N = 0$ can only be of dimension $\beta = 2$ if:

(A) G is fulfilled identically. Then the intersection equals $N = 0$.

(B) Γ contains $N = 0$; i.e. G is reducible where N factors out. Then the intersection equals again $N = 0$.

Note that both case have to be independent of the choice of $R_1, \ldots, R_4$, as the bonds depend on the geometry of the manipulator and not on the leg lengths (cf. Theorem 1 of [16]).

3.1.1. Ad (A)

If G is fulfilled identically, the 4-legged manipulator has a 3-dimensional self-motion independently of the chosen leg lengths $R_1, \ldots, R_4$. Therefore the 4-legged manipulator has a 3-dimensional self-motion in each pose. By attaching again two arbitrary legs it follows that the resulting SG platform has a 1-dimensional self-motion in each pose and therefore it is singular in each pose ($\Rightarrow$ architecturally singular SG platform). Hence the 4-legged manipulator has to be a *degenerated manipulator* (cf. [23]), which can only be one of the following cases[4] according to Karger (cf. [8, 23]):

(a) $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$ and $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear,

[4] After a possible necessary renumbering of anchor points and exchange of the platform and the base.

(b) $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4$,

(c) $\mathsf{m}_1, \ldots, \mathsf{m}_4$ are collinear, $\mathsf{M}_1, \ldots, \mathsf{M}_4$ are collinear and the condition $CV(\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3, \mathsf{m}_4) = CV(\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4)$ holds, where CV denotes the cross-ratio of points.

Note that only item (a) and (b) can appear in the discussed general case, as item (c) belongs to the special case 1. Due to these considerations, we can assume for the remainder of Section 3.1 that the 4-legged manipulator is not degenerated.

3.1.2. Ad (B)

As the degenerated 4-legged manipulators are excluded we can assume w.l.o.g. that there exist two pairs of anchor points $(\mathsf{M}_i, \mathsf{m}_i)$ and $(\mathsf{M}_j, \mathsf{m}_j)$ with $\mathsf{M}_i \neq \mathsf{M}_j$ and $\mathsf{m}_i \neq \mathsf{m}_j$ for $i \neq j$. Moreover we can assume w.l.o.g. that $i = 1$ and $j = 2$ hold. In addition, we can choose special coordinate systems in the platform and the base in a way that $a_1 = b_1 = c_1 = b_2 = c_2 = c_3 = 0$ and $A_1 = B_1 = C_1 = B_2 = C_2 = C_3 = 0$ hold. Due to $A_2 a_2 \neq 0$, we can eliminate the factor of similarity by setting $A_2 = 1$.

In order that Γ contains $N = 0$, the resultant $P[1955651]$ of G and N with respect to e_3 has to be fulfilled identically, where the number in the brackets gives the number of terms. Fortunately, P is the perfect square of an expression $Q[7589]$, where Q is a homogeneous polynomial of degree 8 in e_0, e_1 and e_2. We denote the coefficient of $e_0^i e_1^j e_2^k$ of Q by Q_{ijk}. Based on this preparatory work we can prove the following lemma.

Lemma 1. *A non-degenerated general* 4*-legged manipulator can only have a* 3*-dimensional self-motion of type* $\beta = 2$*, if three anchor points in the platform or base are collinear.*

Proof. The proof is done by contradiction; i.e. we assume that no three anchor points are collinear and show that no solution exists. To do so, we consider $Q_{116} = -4a_2b_3B_3F_1[6]$ and $Q_{206} - Q_{026} = 4a_2b_3B_3F_2[10]$. As $a_2 = 0$ yields a contradiction and $b_3B_3 = 0$ the collinearity of three anchor points, we remain with $F_1 = 0$ and $F_2 = 0$. We compute the resultant F_{12} of F_1 and F_2 with respect to b_3 which yields:

$$c_4(b_4^2 + c_4^2)(B_4^2 + C_4^2)[(B_3 - B_4)^2 + C_4^2].$$

The second and the third factor can only vanish for $b_4 = c_4 = 0$ and $B_4 = C_4 = 0$, respectively, which already implies three collinear anchor points. Therefore we remain with the following two cases:

1. $(B_3 - B_4)^2 + C_4^2 = 0$: This can only be the case for $B_3 = B_4$ and $C_4 = 0$. Then Q_{116} equals $a_2^2 B_4^4[(b_3 - b_4)^2 + c_4^2]$. As $B_4 = 0$ yields the collinearity of $\mathsf{M}_1, \ldots, \mathsf{M}_4$, the last factor has to vanish, which implies $b_3 = b_4$ and $c_4 = 0$. Then we can compute a_2 from Q_{404} which yields $a_2 = \frac{b_4(A_3-A_4)}{B_4(a_3-a_4)}$. Note that this can be done w.l.o.g. as $B_4(a_3 - a_4) = 0$ implies the collinearity of three anchor points. The computation of $Q_{080} = \frac{(A_3-A_4)^2}{B_4^4(a_3-a_4)^2} F_3[6]^2$ and $Q_{602} = \frac{b_4^2(A_3-A_4)^2}{B_4^4(a_3-a_4)^2} F_4[10]^2$ shows that we remain with the case $F_3 = F_4 = 0$. Therefore we calculate the resultant F_{34} of F_3 and F_4 with respect to A_3 which yields:

$$b_4(a_3 - a_4)(a_3B_4 - a_4B_4 + b_4)(b_4A_4 + a_3B_4 - b_4)(a_4B_4 + b_4A_4).$$

   As $b_4(a_3 - a_4)$ imply the collinearity of three anchor points we remain with the following three cases:

   (a) $b_4 = a_4B_4 - a_3B_4$: Now $F_3 = 0$ and $F_4 = 0$ can only vanish without contradiction (w.c.) for $A_3 = \frac{a_3}{a_3-a_4}$. Moreover Q_{062} implies $A_4 = -\frac{a_4}{a_3-a_4}$. Then Q_{422} cannot vanish w.c.

   (b) $A_4 = \frac{b_4-a_3B_4}{b_4}$ and $(a_3-a_4)B_4+b_4 \neq 0$: Now $F_3 = 0$ and $F_4 = 0$ can only vanish w.c. in one of the following two cases:

      i. $b_4 = (a_3 - a_4)B_4$: Now Q_{062} implies $A_3 = \frac{a_3}{a_3-a_4}$. Then Q_{422} cannot vanish w.c.

      ii. $a_4 = \frac{b_4(1-A_3)}{B_4}$ and $(a_3 - a_4)B_4 - b_4 \neq 0$: Now Q_{062} implies $a_3 = \frac{b_4A_3}{B_4}$. Then Q_{422} cannot vanish w.c.

(c) $A_4 = -\frac{a_4 B_4}{b_4}$ and $(a_3B_4 - a_4B_4 + b_4)(b_4A_4 + a_3B_4 - b_4) \neq 0$: Now $F_3 = 0$ and $F_4 = 0$ can only vanish w.c. for $A_3 = -\frac{a_3B_4}{b_4}$. Moreover Q_{062} implies $b_4 = -(a_3 + a_4)B_4$. Then Q_{422} cannot vanish w.c.

2. $c_4 = 0$: Now we get $F_1 = b_4(b_4 - b_3)B_3C_4$ and $F_2 = b_4(b_4 - b_3)(C_4^2 + B_4^2 - B_3B_4)$. As $b_4 = 0$ implies the collinearity of $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_4$ we remain with the following two cases:

   (a) $b_3 = b_4$: Now Q_{206} equals $b_4[(B_3 - B_4)^2 + C_4^2]$, which can only vanish for $B_3 = B_4$ and $C_4 = 0$, but this case was already discussed in item 1.

   (b) $b_3 \neq b_4$: Now F_1 and F_2 can only vanish w.c. for $C_4 = 0$ and $B_3 = B_4$. Then Q_{206} cannot vanish w.c. □

Lemma 2. *A non-degenerated general* 4*-legged manipulator has a* 3*-dimensional self-motion of type* $\beta = 2$ *if and only if one of the following conditions is fulfilled (under consideration of footnote 4):*

I. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$,

II. $\mathsf{m}_1 = \mathsf{m}_2$ *and* $\mathsf{M}_3 = \mathsf{M}_4$.

In both cases the 3*-dimensional self-motion equals the complete spherical motion group.*

Proof. The proof of the sufficiency can easily be done by direct computations as follows. For both designs I and II we compute Γ, where N can be factored out. The remaining polynomial is denoted by Υ. Therefore the conditions are sufficient for the factorization. Moreover we have to show that the designs are also sufficient for the existence of a 3-dimensional self-motion. This is the case, if there exist leg lengths $R_1, \ldots, R_4$ in a way that Υ is fulfilled identically. It can easily be verified that the resulting system of equations has the following solution for:

$$\text{design I:} \quad R_1 = \overline{\mathsf{M}_1\mathsf{M}_4}, \quad R_2 = \overline{\mathsf{M}_2\mathsf{M}_4}, \quad R_3 = \overline{\mathsf{M}_3\mathsf{M}_4}, \quad R_4 = \overline{\mathsf{m}_1\mathsf{m}_4}, \tag{3}$$

$$\text{design II:} \quad R_1 = \overline{\mathsf{M}_1\mathsf{M}_4}, \quad R_2 = \overline{\mathsf{M}_2\mathsf{M}_4}, \quad R_3 = \overline{\mathsf{m}_1\mathsf{m}_3}, \quad R_4 = \overline{\mathsf{m}_1\mathsf{m}_4}. \tag{4}$$

This proves the sufficiency. Moreover the conditions of Eq. (3) and Eq. (4) show that m_1 coincides with M_4 during the self-motion. As a consequence, the 3-dimensional self-motion equals the complete spherical motion group.

Therefore we only remain with the proof of the necessity of the conditions given for design I and II, which is done as follows: Due to Lemma 1 we can assume w.l.o.g. that $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3$ are collinear. Moreover we can assume that there exist two pairs of anchor points $(\mathsf{M}_i, \mathsf{m}_i)$ and $(\mathsf{M}_j, \mathsf{m}_j)$ with $\mathsf{M}_i \neq \mathsf{M}_j$ and $\mathsf{m}_i \neq \mathsf{m}_j$ for distinct $i, j \in \{1, 2, 3\}$, as the non-existence already implies the conditions of design I (under consideration of footnote 4). W.l.o.g. we can set $i = 1$ and $j = 2$. In addition we can choose special coordinate systems in the platform and the base in a way that $a_1 = b_1 = c_1 = b_2 = c_2 = b_3 = c_3 = c_4 = 0$ and $A_1 = B_1 = C_1 = B_2 = C_2 = C_3 = 0$ hold. Due to $A_2a_2 \neq 0$, we can eliminate the factor of similarity by setting $A_2 = 1$.

Now the expression $Q[7589]$ simplifies to $(e_0^2 + e_1^2)Q^\star[1498]$, where $Q^\star$ is a homogeneous polynomial of degree 6 in e_0, e_1 and e_2. We denote the coefficient of $e_0^ie_1^je_2^k$ of $Q^\star$ by $Q^\star_{ijk}$. Based on this preparatory work, the proof of the necessity is split into the following two parts:

1. **We assume that no four anchor points are aligned.** We distinguish the following two cases:

   (a) $B_3 \neq 0$: Now $Q^\star_{006}$ can only vanish w.c. for $B_4 = C_4 = 0$ ($\Rightarrow \mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_4$ are collinear). Then $Q^\star_{015}$ implies $a_3 = a_2(1 - A_4)$. As a consequence $Q^\star_{312}$ can only vanish w.c. in one of the following cases:

      i. $A_4 = 0$: As this implies $\mathsf{M}_1 = \mathsf{M}_4$ and $\mathsf{m}_2 = \mathsf{m}_3$, we get design II.

      ii. $A_4 = 1$: As this implies $\mathsf{M}_2 = \mathsf{M}_4$ and $\mathsf{m}_1 = \mathsf{m}_3$, we get design II.

      iii. $c_4 = 0$ and $A_4(A_4 - 1) \neq 0$: Now $Q^\star_{042} - Q^\star_{402}$ cannot vanish w.c.

   (b) $B_3 = 0$: As in this case $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear, we can assume w.l.o.g. that $C_4 = 0$ holds. Now $Q^\star_{312}$ implies $a_3 = A_3a_2$. Then $Q^\star_{402} - Q^\star_{042}$ can only vanish w.c. for:

      i. $A_3 = 0$: The first leg and the third leg coincide.

      ii. $A_3 = 1$: The second leg and the third leg coincide.

2. **We assume that four anchor points are aligned.** W.l.o.g. we can state that these are the four platform anchor points, which additionally implies $b_4 = 0$. As a consequence $Q^{\star}[1498]$ simplifies to $(e_0^2 + e_1^2)Q^{\star\star}[447]$, where $Q^{\star\star}$ is a homogeneous polynomial of degree 4 in e_0, e_1 and e_2.

   $Q^{\star\star}_{004} = a_2 B_3(a_3 - a_4)(B_4^2 + C_4^2)$ can only vanish w.c. in one of the following three cases:

   (a) $B_3 = 0$: As in this case $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear, we can assume w.l.o.g. that $C_4 = 0$ holds. Moreover, we can assume $B_4 \neq 0$, as otherwise we get the special case 1. Now $Q^{\star\star}_{202}$ can only vanish for $a_3(a_4 - a_2) + a_2 A_3(a_3 - a_4) = 0$: We have to distinguish two cases:

      i. $a_3 - a_4 \neq 0$: Now we can solve this condition for A_3. Then $Q^{\star\star}_{400}$ can only vanish for:
         - $a_3 = 0$: The first leg and the third leg coincide.
         - $a_2 = a_3$: The second leg and the third leg coincide.
         - $a_4 = 0$: As $\mathsf{m}_1 = \mathsf{m}_4$ and $\mathsf{M}_2 = \mathsf{M}_3$ holds, we get a special case of design II.
         - $a_2 = a_4$: As $\mathsf{m}_2 = \mathsf{m}_4$ and $\mathsf{M}_1 = \mathsf{M}_3$ holds, we get a special case of design II.

      ii. $a_3 = a_4$: Now the condition can only vanish for:
         - $a_4 = 0$: As $\mathsf{m}_1 = \mathsf{m}_3 = \mathsf{m}_4$ holds, we get a special case of design I.
         - $a_2 = a_4$: As $\mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4$ holds, we get a special case of design I.

   (b) $B_4 = C_4 = 0$ and $B_3 \neq 0$: Now $Q^{\star\star}_{202}$ can only vanish w.c. for $a_4(a_3 - a_2) + a_2 A_4(a_4 - a_3) = 0$. A comparison of this condition with the one of item 2a shows that they are identical if the indices 3 and 4 are exchanged. Therefore this case yields the analogous results with respect to this exchange of indices.

   (c) $a_3 = a_4$, $B_3 \neq 0$ and we can assume that $B_4 = C_4 = 0$ does not hold: Now $Q^{\star\star}_{202}$ can only vanish w.c. for:

      i. $a_4 = 0$: As $\mathsf{m}_1 = \mathsf{m}_3 = \mathsf{m}_4$ holds, we get design I.

      ii. $a_2 = a_4$: As $\mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4$ holds, we get design I. □

Remark 1. It should be noted that every SG manipulator, which possesses one of the 4-legged manipulators of Lemma 2 as subassembly, already has at least a 1-dimensional spherical self-motion. ⋄

3.2. Special Cases

In the following three lemmata the remaining special cases are discussed. In order to improve the readability of the article, the proofs of these lemmata are given in the Appendix.

Lemma 3. *A non-degenerated* 4*-legged manipulator, which belongs to the special case 1, cannot have a* 3*-dimensional self-motion of type* $\beta = 2$.

Lemma 4. *A non-degenerated* 4*-legged manipulator, which belongs to the special case 2, cannot have a* 3*-dimensional self-motion of type* $\beta = 2$.

Lemma 5. *A* 4*-legged manipulator, which belongs to the special case 3, cannot have a* 3*-dimensional self-motion of type* $\beta = 2$.

The results of Section 3 are summed up within the next theorem:

Theorem 1. *Non-architecturally singular SG platforms with* 3*-dimensional self-motions of type* $\beta = 2$ *do not exist.*

Proof. The proof of this theorem is based on the results obtained in Lemma 1–5. Now we have to add two further legs (legs 5 and 6) to the only non-degenerated 4-legged manipulators with a 3-dimensional self-motion (design I and II of Lemma 2) without restricting the dimension of this self-motion. A necessary condition for this is that every 4-legged manipulator of the resulting SG platform equals design I or II, respectively.

As a consequence, we can only attach the fifth and sixth leg to design I under the side condition that $\mathsf{M}_4 = \mathsf{M}_5 = \mathsf{M}_6$ holds. But this yields an architecturally singular SG manipulator (cf. item 2 of Theorem 3 of [8]).

Analogous considerations for design II show that we can only attach the fifth and sixth leg if $\mathsf{m}_1 = \mathsf{m}_i$ and $\mathsf{M}_4 = \mathsf{M}_j$ hold with distinct $i, j \in \{5, 6\}$. This yields again the architecturally singular manipulator given in item 2 of Theorem 3 of [8]. □

4. 3-dimensional self-motions of type $\beta = -1, 0, 1$

We start this section by studying 3-dimensional self-motions of type $\beta = -1$:

Theorem 2. *SG platforms (architecturally singular or not) with 3-dimensional self-motions of type $\beta = -1$ do not exist.*

Proof. Due to the last paragraph of Section 2.1 it is clear that 3-dimensional self-motions of type $\beta = -1$ cannot exist. But this can also easily be seen as follows: We assume that a SG manipulator configuration exists, where the manipulator can perform a 3-dimensional self-motion of type $\beta = -1$. Therefore the manipulator can be moved out of this configuration by any translation of arbitrary direction. By choosing e.g. the direction of the P-joint of the i-th leg, we see that the corresponding sphere condition Λ_i is violated, which already yields the contradiction. □

We proceed with the 3-dimensional self-motions of type $\beta = 1$:

Theorem 3. *Non-architecturally singular SG platforms with 3-dimensional self-motions of type $\beta = 1$ do not exist.*

Proof. For a 3-dimensional self-motion $\mathcal{S}$ of type $\beta = 1$, there has to exist a 1-dimensional translational sub-self-motion in each pose of $\mathcal{S}$. According to the last paragraph of Section 2.1, a necessary condition for the existence of a SG manipulator with $\mathcal{S}$ of type $\beta = 1$ is that there exists a 2-parametric set of platform orientations with $\mathsf{m}_1 = \mathsf{M}_1$ and $rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2}, \ldots, \overrightarrow{\mathsf{M}_6\mathsf{m}_6}) \leq 1$.

A necessity for this circumstance is that for $\mathsf{m}_1 = \mathsf{M}_1$ the vectors $\overrightarrow{\mathsf{M}_2\mathsf{m}_2}$ and $\overrightarrow{\mathsf{M}_3\mathsf{m}_3}$ are linearly dependent. In the following we use this necessary condition for the proof of Theorem 3:

As the legs are not allowed to coincide, we can assume w.l.o.g. that $\mathsf{M}_1 \neq \mathsf{M}_2$ holds. We can choose special coordinate systems in the platform and the base in a way that $a_1 = b_1 = c_1 = b_2 = c_2 = c_3 = 0$ and $A_1 = B_1 = C_1 = B_2 = C_2 = C_3 = 0$ hold. Due to $A_2 \neq 0$ we can eliminate the factor of similarity by setting $A_2 = 1$.

Now we compute the images of m_2 and m_3 under the spherical motion with center $\mathsf{m}_1 = \mathsf{M}_1$ by the multiplication of the rotation matrix $\mathbf{R}$ of Eq. (1) with $\mathbf{m}_2$ and $\mathbf{m}_3$, respectively. Therefore the above mentioned condition of linear dependency can be written as:

$$(\mathbf{R}\mathbf{m}_2 - N\mathbf{M}_2) \times (\mathbf{R}\mathbf{m}_3 - N\mathbf{M}_3) = \mathbf{o}, \tag{5}$$

where $\mathbf{o}$ is the zero-vector. This implies three conditions, which are denoted by S_i, where $i = 1, 2, 3$ denotes the corresponding line of Eq. (5). Now these three conditions have to have a common factor. A necessary condition for this is that the resultant T_{ij} of S_i and S_j with respect to e_0 is fulfilled identically for distinct $i, j \in \{1, 2, 3\}$. T_{12} factors into $4a_2b_3e_3T[10]$ with

$$T[10] := (1 - a_2)(B_3 + b_3)e_1^2 + (1 + a_2)(B_3 - b_3)e_2^2 + 2(a_2A_3 - a_3)e_1e_2.$$

Therefore we have to do the following discussion of cases:

1. $a_2b_3 \neq 0$: Therefore $T[10]$ has to be fulfilled identically. It can easily be seen that this can only be the case for either $a_2 = 1$, $a_3 = A_3$, $b_3 = B_3$ or $a_2 = -1$,$a_3 = -A_3$, $b_3 = -B_3$. The geometric meaning of both cases is that the planar figures $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3$ and $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are congruent. We distinguish the following cases:
   (a) SG manipulator is planar: In this case the platform and the base are congruent. It is well known (cf. [13, 24]) that this class of manipulators can only have non-translational self-motions, if they are architecturally singular. This already closes the planar case.
   (b) SG manipulator is non-planar: In this case we have to distinguish between two possibilities:
      i. Platform and base are congruent: For this class of manipulators it is known (cf. [25]) that they cannot have 1-dimensional translational self-motions, but only 2-dimensional ones (if all legs have equal length). Therefore this case can only result in 4-dimensional self-motion of type $\beta = 1$ and we are done. Note that this is a special case of the one discussed in the next theorem (cf. Theorem 4).

ii. Platform and base are reflection-congruent: Any reflection-congruent SG platform possesses a 2-parametric set of platform orientations with $\mathsf{m}_1 = \mathsf{M}_1$ and $rk(\overrightarrow{\mathsf{M}_2\mathsf{m}_2}, \ldots, \overrightarrow{\mathsf{M}_6\mathsf{m}_6}) = 1$. This can easily be seen by reflecting the base with respect to the 2-parametric set of planes through M_1 into the platform (cf. [26]).
In the following we show by means of computation that a reflection-congruent SG manipulator cannot have a self-motion of type $\beta = 1$. W.l.o.g. we can assume that the first four anchor points span a tetrahedron. Moreover we can choose special coordinate systems in the platform and the base in a way that we get:

$$\mathbf{m}_1 = \mathbf{M}_1 = (0,0,0)^T, \qquad \mathbf{m}_2 = \mathbf{M}_2 = (a_2,0,0)^T, \qquad \mathbf{m}_3 = \mathbf{M}_3 = (a_3,b_3,0)^T,$$
$$\mathbf{M}_4 = (a_4,b_4,c_4)^T, \qquad \mathbf{m}_4 = (a_4,b_4,-c_4)^T,$$

with $b_3c_4 \neq 0$. In addition we can eliminate the factor of similarity by setting $a_2 = 1$. For this setup, it can easily be seen that the 2-parametric set of platform orientations, which cause 1-dimensional translational sub-self-motions, is determined by $e_3 = 0$. As a consequence, we can assume $e_0e_1e_2 \neq 0$, as otherwise we only get at most a 2-dimensional self-motion of type $\beta = 0$.
Therefore we can solve $\Psi, \Delta_{2,1}$ for f_0, f_3 w.l.o.g. and plug the obtained solutions into $\Delta_{3,1}$ and $\Delta_{4,1}$. The numerators of the resulting expressions are denoted by G_3 and G_4, respectively, which are both homogeneous cubic polynomials in e_0, e_1, e_2. Note that G_3 and G_4 do not depend on f_1 and f_2. These two Study parameters only appear in Λ_1, but this equation is not of interest for the further computation of bonds.
We eliminate e_0 from G_i by calculating the resultant H_i of G_i and N with respect to e_0 for $i = 3,4$. Now H_3 can only vanish w.c. for either $e_1 = \frac{a_3}{b_3}e_2$ or $e_1 = \frac{a_3-1}{b_3}e_2$. In both cases H_4 has to be fulfilled identically. The resulting condition can in both cases be solved for a_4 w.l.o.g. and it can be seen that none of the obtained solutions can be real for $b_3c_4 \neq 0$, which finishes this case.

2. $a_2 = 0$: As now m_1 and m_2 coincide, we can assume w.l.o.g. that $b_3 = 0$ holds. Now S_1 is already fulfilled identically and we remain with:

$$S_2 = 2a_3(e_1e_3 - e_0e_2), \qquad S_3 = B_3N - 2a_3(e_1e_2 + e_0e_3).$$

It can easily be seen that these two equations cannot have a common factor and therefore they can only be fulfilled identically, which is the case for $a_3 = B_3 = 0$. But this implies $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$ and $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ collinear, which is an architecturally singular design (cf. item (a) of Section 3.1.1).

3. $b_3 = 0$ and $a_2 \neq 0$: In this case we consider T_{13} and T_{23} which factor into $4a_2^2B_3^2(e_2^2 + e_3^2)T[6]$ and $4(a_2A_3 - a_3)^2(e_2^2 + e_3^2)T[6]$, respectively, with $T[6] = B_3(1-a_2)e_1^2 + B_3(1+a_2)e_2^2 + 2(a_2A_3 - a_3)e_1e_2$. Therefore we remain with two possibilities:

   (a) $a_3 = a_2A_3$, $B_3 = 0$: In this case $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3$ are collinear and $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear and the corresponding points are related by a similarity transformation. If we extend this solution to all six pairs of anchor points, we get a trivial architecturally singularity, as all anchor points are located on a line (cf. item 1 of Theorem 3 of [8]).

   (b) It remains to check whether T can be fulfilled identically. It can easily be seen that this happens if and only if $B_3 = 0$ and $a_3 = a_2A_3$ hold, thus we end up with the case already discussed in the last item. □

Therefore we remain with the 3-dimensional self-motions of type $\beta = 0$, which are studied next:

Theorem 4. *Non-architecturally singular SG platforms with 3-dimensional self-motions of type $\beta = 0$ do not exist.*

Proof. Analogous considerations as in the proof of Theorem 3 show that a necessary condition for the existence of a SG manipulator with a 3-dimensional self-motion of type $\beta = 0$ is that there exists a 1-parametric set of platform orientations with $\mathsf{m}_i = \mathsf{M}_i$ for $i = 1,2,3$. Clearly, this can only be the case if all three anchor points are located on a line. Therefore this condition implies a congruent SG platform, where all anchor points are collinear. This is a trivial architecturally singular manipulator (cf. item 1 of Theorem 3 of [8]), which finishes the proof of Theorem 4. □

5. SG manipulators with n-dimensional self-motions ($n > 2$)

The answer to the question asked in footnote 3 of [15], whether non-architecturally singular SG platforms with 3-dimensional self-motions exist, is included within the next theorem.

Theorem 5. *Non-architecturally singular SG platforms, which possess a n-dimensional self-motion with $n = 3, 4$, do not exist. SG platforms (architecturally singular or not) with higher-dimensional self-motions than 4 do not exist.*

Proof. Due to the Theorems 1 – 4, this is true for $n = 3$. Therefore we focus on the case $n = 4$, for which Theorem 5 is proven by contradiction. If we replace a leg of the non-architecturally singular SG platform, which fulfills a condition in order to enable the 4-dimensional self-motion, by an arbitrary leg (not causing an architecturally singular design), we end up with a non-architecturally singular manipulator with a 3-dimensional self-motion; a contradiction.

Therefore we only remain with the proof of the last sentence of Theorem 5. If we couple the platform and the base only by one leg with non-zero length, the motion is a 5-dimensional one.[5] A second leg with non-zero length, which differs from the first one, already restricts the motion to a 4-dimensional one, which finishes the proof. □

Due to this theorem, only architecturally singular manipulators can have n-dimensional self-motions with $n = 3, 4$. As preparatory work for a complete list of these manipulators, which is given in Theorem 6 and 7, we have to prove the following lemma:

Lemma 6. *A 4-legged manipulator can only have a 4-dimensional self-motion, if it is the following degenerated one (under consideration of footnote 4): All base anchor points are collinear and the four platform anchor points collapse into one point.*

Proof. We assume that there exists a 4-legged manipulator with a 4-dimensional self-motion. If we replace one of the four legs, which fulfills a certain constraint in order to enable the 4-dimensional self-motion, by an arbitrary leg, we end up with a 4-legged manipulator with a 3-dimensional self-motion, where one leg is free of any constraints. Therefore it can only be either design I of Lemma 2 or item (a) of Section 3.1.1. As the latter is a special case of design I we can restrict ourselves to this case.

Now the question arises, how design I can be specified to obtain even a 4-dimensional self-motion. In order to find an answer, we first consider the 3-legged manipulator with $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$. Trivially the 3-dimensional spherical self-motion with center m_1 can only be extended by one further dimension (circular translation), if $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are collinear. This 4-dimensional self-motion is not restricted by the attachment of a fourth leg, if and only if the fourth leg belongs to the pencil of lines spanned by the first three legs. This results in the conditions given in Lemma 6. □

Based on this lemma we can prove the following theorem:

Theorem 6. *If a SG platform has a 4-dimensional self-motion, it has to be the following architecturally singular design (under consideration of footnote 4): All base anchor points are collinear and the six platform anchor points collapse into one point (cf. Fig. 3a).*

Proof. As each 4-legged manipulator of the SG platform has to have a 4-dimensional self-motion, the result is implied by Lemma 6. □

Theorem 7. *If a SG platform has a 3-dimensional self-motion, it has to be one of the following architecturally singular designs (under consideration of footnote 4):*

1. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$ *and* $\mathsf{M}_4 = \mathsf{M}_5 = \mathsf{M}_6$ *(cf. Fig. 3b).*
2. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4$ *and* $\mathsf{M}_5 = \mathsf{M}_6$ *(cf. Fig. 3c).*
3. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4 = \mathsf{m}_5$ *(cf. Fig. 3d).*

[5] A leg with zero length already restricts the mobility to three dimensions.

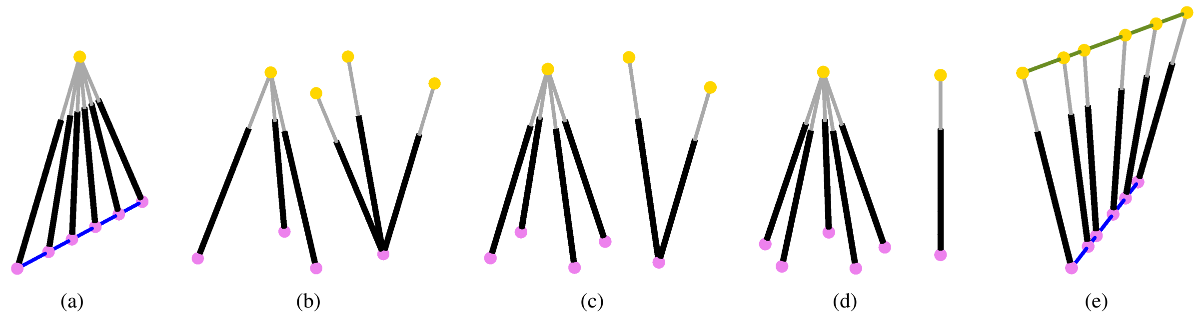

Figure 3: SG platform with a 4-dimensional self-motion (a) and a 3-dimensional self-motion (b-e). In each graphic, the legs are displayed in ascending order from left to right.

4. *All base anchor points are collinear and all platform anchor points are collinear. Moreover corresponding anchor points are related by a regular projectivity* κ: $\mathsf{M}_i \mapsto \mathsf{m}_i$ *for* $i = 1, \ldots, 6$ *(cf. Fig. 3e).*

Proof. The designs 1, 2 and 3 possess at least a 3-dimensional spherical self-motion if m_1 coincides with M_6. Design 4 is highly redundant as the six legs belong to a regulus of lines and therefore one can remove any three legs without changing the direct kinematics and singularity surface. As a consequence the manipulator has trivially a 3-dimensional self-motion.

Therefore these conditions are sufficient for the existence of 3-dimensional self-motions. In the following we show that they are necessary as well. To do so we distinguish the following cases with respect to the rank of the manipulator's Jacobian **J** in a generic configuration.

Due to Theorem 5 we only have to consider the cases $1 < rk(\mathbf{J}) < 6$, where the case $rk(\mathbf{J}) = 2$ yields a 4-dimensional self-motion, which was already discussed in Theorem 6. Therefore we proceed with the case $rk(\mathbf{J}) = 3$, which trivially implies a 3-dimensional self-motion:

- $rk(\mathbf{J}) = 3$: It follows directly from the items (a–c) of Section 3.1.1 that these SG platforms have either the design of item 4 or one of the following four designs (under consideration of footnote 4):

  i. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4 = \mathsf{m}_5$ and $\mathsf{M}_1, \ldots, \mathsf{M}_5$ collinear (cf. Fig. 4a).

  ii. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4 = \mathsf{m}_5 = \mathsf{m}_6$ (cf. Fig. 4b).

  iii. $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are collinear, $\mathsf{M}_1, \ldots, \mathsf{M}_6$ are collinear, $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$ and $\mathsf{M}_4 = \mathsf{M}_5 = \mathsf{M}_6$ (cf. Fig. 4c).

  iv. $\mathsf{m}_1, \ldots, \mathsf{m}_6$ are collinear, $\mathsf{M}_1, \ldots, \mathsf{M}_6$ are collinear, $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4$ and $\mathsf{M}_5 = \mathsf{M}_6$ (cf. Fig. 4d).

  Note that item (iii) and (iv) are special cases of item 4 (regulus splits up into two planes). Moreover item 1, 2 and 3 are necessary conditions for the listed four designs.

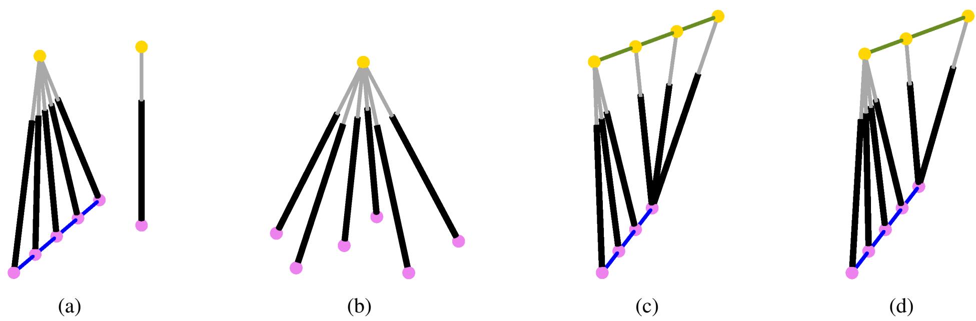

Figure 4: SG platforms with $rk(\mathbf{J}) = 3$, which possess a 3-dimensional self-motion. The legs are displayed in ascending order from left to right.

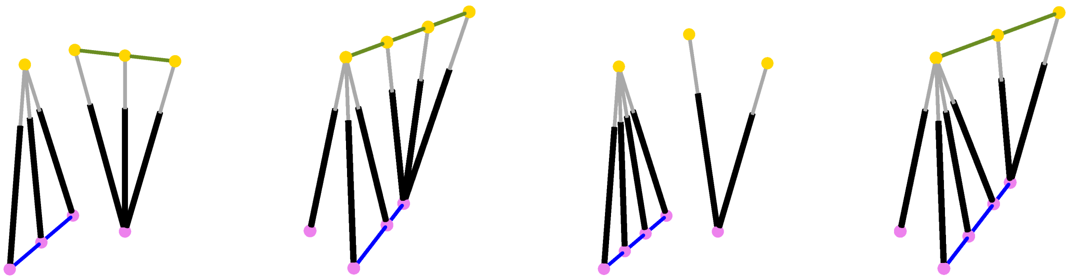

Figure 5: SG platforms with $rk(\mathbf{J}) = 4$, which possess a 3-dimensional self-motion. The legs are displayed in ascending order from left to right.

If we have an architecturally singular manipulator with $rk(\mathbf{J}) > 3$, it has to contain a non-degenerated 4-legged manipulator with a 3-dimensional self-motion. Due to our results obtained in the proofs given in the Sections 3 and 4, there only exist the two designs I and II of Lemma 2 with this property.

Therefore we have to add two further legs (legs 5 and 6) to these two designs without restricting the 3-dimensional self-motion. A necessary condition for this is that every 4-legged manipulator of the resulting SG platform is either a degenerated one (items (a–c) of Section 3.1.1) or equals design I and II, respectively.

- $rk(\mathbf{J}) = 4$: If we check the possible combinatorial cases with respect to the condition $rk(\mathbf{J}) = 4$, it is not difficult to figure out that we get the following cases (under consideration of footnote 4) beside the design given in item 3:

  i. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3, \mathsf{M}_4 = \mathsf{M}_5 = \mathsf{M}_6, \mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6$ collinear and $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ collinear (cf. Fig. 5a).
  ii. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3, \mathsf{M}_4 = \mathsf{M}_5 = \mathsf{M}_6, \mathsf{m}_1, \ldots, \mathsf{m}_6$ collinear and $\mathsf{M}_2, \ldots, \mathsf{M}_6$ collinear (cf. Fig. 5b).
  iii. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4, \mathsf{M}_5 = \mathsf{M}_6$ and $\mathsf{M}_1, \ldots, \mathsf{M}_4$ collinear (cf. Fig. 5c).
  iv. $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3 = \mathsf{m}_4, \mathsf{M}_5 = \mathsf{M}_6, \mathsf{m}_1, \ldots, \mathsf{m}_6$ collinear and $\mathsf{M}_2, \ldots, \mathsf{M}_6$ collinear (cf. Fig. 5d).

  Note that item 1 and 2 are necessary conditions for the listed four designs.

- $rk(\mathbf{J}) = 5$: If we check the possible combinatorial cases with respect to the condition $rk(\mathbf{J}) = 5$, it is not difficult to figure out that we get the two cases given in item 1 and 2 of Theorem 7. □

Remark 2. In the author's opinion, the following interesting property of the design, given in item 4 of Theorem 7, should be noted: In the general case the 3-dimensional self-motion is of type $\beta = 2$. If κ is a similarity (cf. proof of Theorem 3), we get a type $\beta = 1$ self-motion. Moreover if κ is the congruence and all legs have equal length, we even get a 3-dimensional self-motion of type $\beta = 0$ (cf. proof of Theorem 4). Finally it should be mentioned that the manipulator design with the 4-dimensional self-motion, which is always of type $\beta = 2$, can also be obtained from item 4 of Theorem 7 by assuming that κ is singular. ⋄

6. Conclusion and remarks on SG platforms with 2-dimensional self-motions

Within this article we gave a complete list of all SG platforms, which have n-dimensional self-motions with $n > 2$ (cf. Theorems 6 and 7). All these manipulators are architecturally singular (cf. Theorem 5).

Therefore only the study of SG platforms with 2-dimensional self-motions remains open within the field of multidimensional self-motions. Based on Theorem 3 of [8], it is not difficult to give a list of all architecturally singular manipulators with $rk(\mathbf{J}) = 4$. This is left to the reader. The more challenging problem, which is still unsolved, is the determination of all architecturally singular designs with $rk(\mathbf{J}) = 5$ possessing 2-dimensional self-motions. This list is of interest as from each entry non-architecturally singular SG platforms with 1-dimensional self-motions can be constructed in the same way as done for SG manipulators with so-called type II DM self-motions (cf. [27]). This topic is dedicated to future research.

Finally, we want to give a new result on non-architecturally singular SG platforms with 2-dimensional self-motions. Beside the pure translational self-motion (⇒ type $\beta = -1$) of the congruent SG platform, there exists a further trivial example, which was not mentioned before in the literature, to the best knowledge of the author. This design, implied by the results obtained in Section 3, can be given as follows (under consideration of footnote 4): For $\mathsf{m}_1 = \mathsf{m}_2 = \mathsf{m}_3$ and $\mathsf{M}_4 = \mathsf{M}_5$ there exists a 2-dimensional self-motion if m_1 coincides with M_4. This self-motion is a pure spherical one (⇒ type $\beta = 1$) with center $\mathsf{m}_1 = \mathsf{M}_4$.

It remains open, whether further non-architecturally singular SG platforms with 2-dimensional self-motions exist (especially a design with a type $\beta = 0$ self-motion) beside the two already known ones. This topic is dedicated to future research as well.

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References

[1] Borel, E.: *Mémoire sur les déplacements à trajectoires sphériques.* Mémoire présenteés par divers savants à l'Académie des Sciences de l'Institut National de France **33**(1) 1–128 (1908)

[2] Bricard, R.: *Mémoire sur les déplacements à trajectoires sphériques.* Journal de École Polytechnique **11** 1–96 (1906)

[3] Husty, M.: *E. Borel's and R. Bricard's Papers on Displacements with Spherical Paths and their Relevance to Self-Motions of Parallel Manipulators.* Int. Symp. on History of Machines and Mechanisms (M. Ceccarelli ed.), 163–172, Kluwer (2000)

[4] Karger, A.: *Architecture singular planar parallel manipulators.* Mechanism and Machine Theory **38**(11) 1149–1164 (2003)

[5] Nawratil, G.: *On the degenerated cases of architecturally singular planar parallel manipulators.* Journal for Geometry and Graphics **12**(2) 141–149 (2008)

[6] Röschel, O., Mick, S.: *Characterisation of architecturally shaky platforms.* Advances in Robot Kinematics: Analysis and Control (J. Lenarcic, M.L. Husty eds.), 465–474, Kluwer (1998)

[7] Wohlhart, K.: *From higher degrees of shakiness to mobility.* Mechanism and Machine Theory **45**(3) 467–476 (2010)

[8] Karger, A.: *Architecturally singular non-planar parallel manipulators.* Mechanism and Machine Theory **43**(3) 335–346 (2008)

[9] Nawratil, G.: *A new approach to the classification of architecturally singular parallel manipulators.* Computational Kinematics (A. Kecskemethy, A. Müller eds.), 349–358, Springer (2009)

[10] Nawratil, G.: *Review and recent results on Stewart Gough platforms with self-motions.* Applied Mechanics and Materials **162** 151–160 (2012)

[11] Wohlhart, K.: *Kinematotropic linkages.* Recent Advances in Robot Kinematics (J. Lenarcic, V. Parenti-Castelli eds.), 359–368, Kluwer (1996)

[12] Chasles, M.: *Sur les six droites qui peuvent étre les directions de six forces en équilibre.* Comptes Rendus des Séances de l'Académie des Sciences **52** 1094–1104 (1861)

[13] Karger, A.: *Singularities and self-motions of a special type of platforms.* Advances in Robot Kinematics: Theory and Applications (J. Lenarcic, F. Thomas eds.), 155-164, Springer (2002)

[14] Gogu, G.: *Branching singularities in kinematotropic parallel mechanisms.* Computational Kinematics (A. Kecskemethy, A. Müller eds.), 341–348, Springer (2009)

[15] Husty, M.L., Karger, A.: *Self motions of Stewart-Gough platforms: an overview.* Proc. of the workshop on fundamental issues and future research directions for parallel mechanisms and manipulators (C.M. Gosselin, I. Ebert-Uphoff eds.), 131–141 (2002)

[16] Nawratil, G.: *Introducing the theory of bonds for Stewart Gough platforms with self-motions.* ASME Journal of Mechanisms and Robotics, in press (2013)

[17] Hegedüs, G., Schicho, J., Schröcker, H-P.: *Bond Theory and Closed 5R Linkages.* Latest Advances in Robot Kinematics (J. Lenarcic, M. Husty eds.), 221–228, Springer (2012)

[18] Hegedüs, G., Schicho, J., Schröcker, H-P.: *The theory of bonds: A new method for the analysis of linkages.* Mechanism and Machine Theory **70** 407–424 (2013)

[19] Husty, M.L.: *An algorithm for solving the direct kinematics of general Stewart-Gough platforms.* Mechanism and Machine Theory **31**(4) 365–380 (1996)

[20] Fanghella, P., Galletti, C., Giannotti, E.: *Parallel robots that change their group of motion.* Advances in Robot Kinematics: Mechanisms and Motion (J. Lencarcic, B. Roth eds.), 49–56, Springer (2006)

[21] Husty, M., Karger, A., Sachs, H., Steinhilper, W.: *Kinematik und Robotik.* Springer (1997)

[22] Nawratil, G.: *Zur Geometrie von Stewart Gough Plattformen.* Master thesis, Vienna University of Technology (2004)

[23] Karger, A.: *Architecture singular parallel manipulators.* Advances in Robot Kinematics: Analysis and Control (J. Lenarcic, M.L. Husty eds.), 445–454, Kluwer (1998)

[24] Nawratil, G.: *Self-motions of planar projective Stewart Gough platforms.* Latest Advances in Robot Kinematics (J. Lenarcic, M. Husty eds.), 27–34, Springer (2012)

[25] Nawratil, G.: *Congruent Stewart Gough platforms with non-translational self-motions.* Proc. of the 16th International Conference on Geometry and Graphics, submitted

[26] Nawratil, G.: *On equiform Stewart Gough platforms with self-motions.* Journal for Geometry and Graphics, submitted

[27] Nawratil, G.: *Types of self-motions of planar Stewart Gough platforms.* Meccanica **48**(5) 1177–1190 (2013)

Appendix

Proof of Lemma 3:

W.l.o.g. we can assume that m_1 (resp. M_1) equals the origin of the moving (resp. fixed) system and that the remaining platform (resp. base) anchor points are located on its x-axis. Therefore this implies $a_1 = A_1 = b_i = B_i = c_i = C_i = 0$ for $i = 1, \ldots, 4$. As the legs are not allowed to coincide, we can eliminate the factor of similarity by setting $a_2 = 1$.

Now we can solve the equations $\Psi, \Delta_{2,1}, \Delta_{3,1}$ for f_0, f_1, f_2 if the mapping $\mathsf{m}_j \mapsto \mathsf{M}_j$ for $j = 1, 2, 3$ is no similarity. If this cannot be achieved by a renumbering of the legs, we already have a special case of the degenerated manipulator given in item (c) of Section 3.1.1.

After plugging the obtained solutions for f_0, f_1, f_2 into the equation Λ_1, we can compute f_3 from it[6], but this is not of interest. If we substitute the expressions for f_0, f_1, f_2 into $\Delta_{4,1}$ we get in the numerator a homogeneous polynomial $G[120]$ of degree 2 in the Euler parameters. Therefore Γ is in this case a quadric in P^3.

Again the intersection of Γ and $N = 0$ can only be of dimension $\beta = 2$ if either case (A) or case (B) of Section 3.1 holds. The argumentation given in Section 3.1.1 for case (A) is still true, where special cases of (a) and (b) of Section 3.1.1 can appear beside item (c) of Section 3.1.1. Therefore we can assume for the following discussion of case (B), that the manipulator is non-degenerated.

In order that Γ contains $N = 0$, the resultant P of G and N with respect to e_3 has to be fulfilled identically. P factors into $(e_0^2 + e_1^2)^2$ and

$$[a_3a_4(A_2A_3 - A_2A_4) + a_3(A_2A_4 - A_3A_4) + a_4(A_3A_4 - A_2A_3)]^2,$$

which equals the squared condition $CV(\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3, \mathsf{m}_4) = CV(\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4)$ of item (c) of Section 3.1.1. This finishes the proof of Lemma 3. □

Therefore we can assume for the discussion of the following proof of Lemma 4 that the four platform anchor points and the corresponding base anchor points are not both collinear.

Proof of Lemma 4:

Now the platform and the base are planar ($\Rightarrow C_i = c_i = 0$ for $i = 1, \ldots, 4$) and corresponding anchor points are related within an affinity. W.l.o.g. we can assume that m_1 (resp. M_1) equals the origin of the moving (resp. fixed) frame and that m_2 (resp. M_2) is located on its x-axis. Therefore this implies $a_1 = b_1 = b_2 = 0$ and $A_1 = B_1 = B_2 = 0$. As the legs are not allowed to coincide, we can eliminate the factor of similarity by setting $a_2 = 1$.

This already implies that the affinity is of the form $(A_i, B_i)^T = \mathbf{A}(a_i, b_i)^T$ for $i = 1, \ldots, 4$, where the 2×2 transformation matrix $\mathbf{A}$ is given by:

$$\mathbf{A} := \begin{pmatrix} u & v \\ 0 & w \end{pmatrix}.$$

Now we can solve the equations $\Psi, \Delta_{2,1}, \Delta_{3,1}$ for f_0, f_1, f_2 if the first three anchor points are not collinear. This can always be achieved by a renumbering of the legs.

After plugging the obtained solutions for f_0, f_1, f_2 into the equation Λ_1, we can compute f_3 from it (cf. footnote 6), but this is not of interest. If we substitute the expressions for f_0, f_1, f_2 into $\Delta_{4,1}$ we get in the numerator a homogeneous polynomial $G[140]$ of degree 2 in the Euler parameters. Therefore Γ is again a quadric in P^3.

[6] If Λ_1 does not depend on f_3, we can only get a 3-dimensional self-motion of type $\beta = 1$, as one translational parameter remains free.

Again the intersection of Γ and $N = 0$ can only be of dimension $\beta = 2$ if either case (A) or case (B) of Section 3.1 holds. The argumentation given in Section 3.1.1 for case (A) is still true, where only the cases (a) and (b) of Section 3.1.1 can appear. In these cases the affinity is singular; i.e. $rk(\mathbf{A}) < 2$. Therefore we can assume for the following discussion of case (B), that the manipulator is non-degenerated.

In order that Γ contains $N = 0$, the resultant $P[244]$ of G and N with respect to e_3 has to be fulfilled identically. Note that P is a homogeneous polynomial of degree 4 in e_0, e_1, e_2. It is not difficult to verify by a discussion of cases that P can only vanish if two legs coincide or in the degenerated cases (a) and (b) of Section 3.1.1. This finishes the proof of Lemma 4. □

Proof of Lemma 5:

Now the platform and the base are congruent and non-planar. W.l.o.g. we can assume that m_1 (resp. M_1) equals the origin of the moving (resp. fixed) frame and that m_2 (resp. M_2) is located on its x-axis and that m_3 (resp. M_3) belongs to the xy-plane of the moving (resp. fixed) frame. Therefore this implies $a_1 = b_1 = b_2 = c_1 = c_2 = c_3 = 0$ and $A_1 = B_1 = B_2 = C_1 = C_2 = C_3 = 0$. Moreover we have $a_i = A_i$, $b_i = B_i$ and $c_i = C_i$ for the remaining coordinates. In addition we can assume w.l.o.g. that $c_4 \neq 0$ holds as otherwise the SG manipulator is planar. As the legs are not allowed to coincide, we can eliminate the factor of similarity by setting $a_2 = 1$.

Now we can solve the equations $\Psi, \Delta_{2,1}, \Delta_{3,1}$ for f_0, f_1, f_2 if the first three anchor points are not collinear. This is always the case, as otherwise the SG manipulator is planar.

After plugging the obtained solutions for f_0, f_1, f_2 into the equation Λ_1, we can compute f_3 from it (cf. footnote 6), but this is not of interest. If we substitute the expressions for f_0, f_1, f_2 into $\Delta_{4,1}$ we get in the numerator a homogeneous polynomial $G[83]$ of degree 3 in the Euler parameters. Therefore Γ is in this case a cubic surface in P^3.

Again the intersection of Γ and $N = 0$ can only be of dimension $\beta = 2$ if either case (A) or case (B) of Section 3.1 holds. The argumentation given in Section 3.1.1 for case (A) is still true, but non of the three degenerated cases of Section 3.1.1 can appear.

Therefore we remain with the discussion of case (B), which can be done as follows: In order that Γ contains $N = 0$, the resultant $P[342]$ of G and N with respect to e_3 has to be fulfilled identically. Note that P is a homogeneous polynomial of degree 6 in e_0, e_1, e_2. It is not difficult to verify by a discussion of cases that P can only vanish if two legs coincide. This finishes the proof of Lemma 5. □