Planar Stewart Gough platforms with quadratic singularity surface

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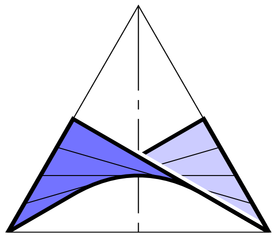


Overview

1. Stewart Gough platform (SGP)

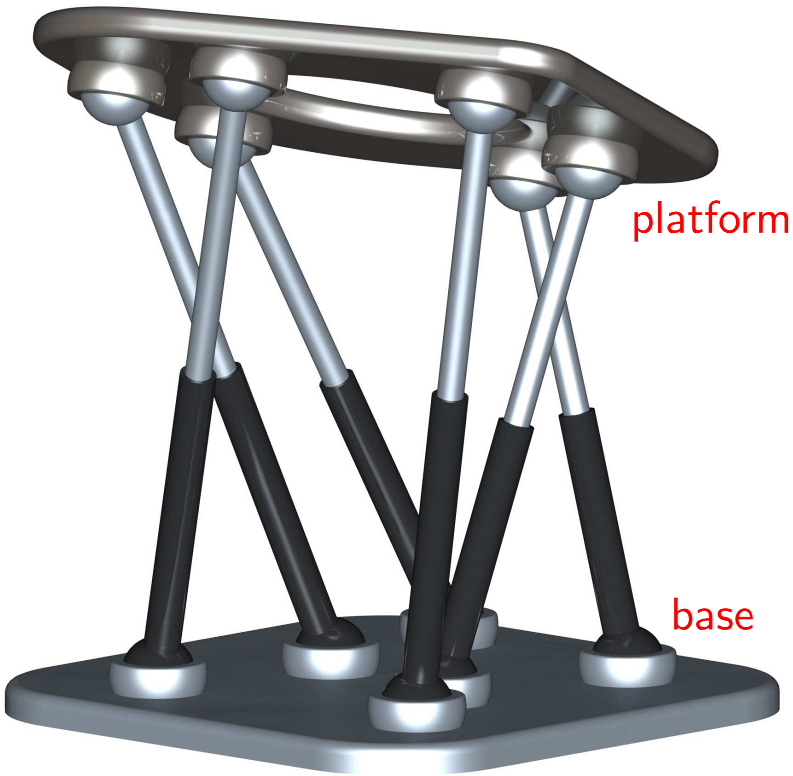
The geometry of a (planar) SGP is given by:

- six (coplanar) base anchor points M_i
- six (coplanar) platform points m_i

M_i and m_i are connected with a S<u>P</u>S leg.

Theorem. MERLET [9]
A SGP is *singular* (*infinitesimal flexible*, *shaky*) if and only if the carrier lines of the six S<u>P</u>S legs belong to a linear line complex.

A SGP is called *architecturally singular* if it is singular in any possible configuration.



1. Review

Theorem 1. Nawratil [10,11]
A planar SGP possesses a quadratic singularity surface in the space of translations for any orientation of the platform if and only if $rk(\mathbf{N}) = 4$ holds with

$$\mathbf{N}^T := \begin{pmatrix} 1 & a_1 & b_1 & A_1 & B_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_6 & b_6 & A_6 & B_6 \end{pmatrix}, \qquad (1)$$

where $\mathbf{M}_i = (A_i, B_i, 0)^T$ (resp. $\mathbf{m}_i = (a_i, b_i, 0)^T$) are the coordinates of M_i (resp. m_i) with respect to the fixed frame (resp. moving frame).

Theorem 2. Karger [8]
$rk(\mathbf{N}) = 3 \iff$ base and platform are affinely equivalent.

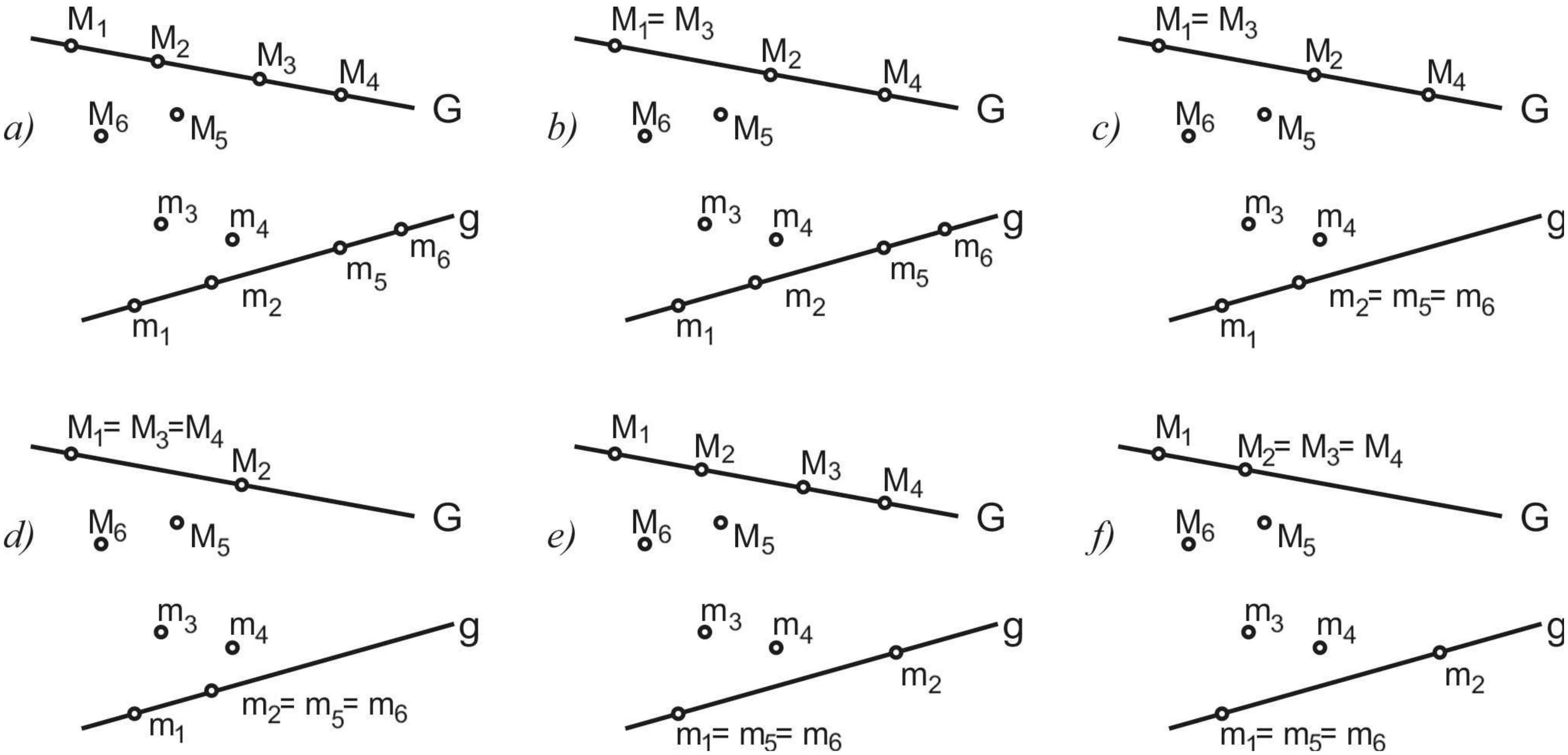
Moreover Karger [8] noted that $rk(\mathbf{N}) < 3$ implies architecturally singular designs and that for $rk(\mathbf{N}) = 4$ *"no special properties are known so far"*.

2. Existence theorem

Theorem 3.
Every non-architecturally singular SGP has triples of anchor points $(\mathsf{M}_i, \mathsf{M}_j, \mathsf{M}_k)$ and $(\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_k)$ with $i, j, k \in \{1, \ldots, 6\}$ in a way that the triangles $\triangle(\mathsf{M}_i\mathsf{M}_j\mathsf{M}_k)$ and $\triangle(\mathsf{m}_i\mathsf{m}_j\mathsf{m}_k)$ are not degenerated; i.e. they do not collapse into a line or even a point.

Proof: The proof is split into three Lemmata. For details please see Lemma 1-3 of the presented paper.

2. Existence theorem



Lemma 3 involves a very lengthy discussion of cases, where some of them are illustrated above.

3. Geometric interpretation of rk(N)=4

W.l.o.g. we can choose the fixed and moving frame in a way that the first base and platform anchor point are located in their origins; i.e. $a_1 = b_1 = A_1 = B_1 = 0$. $rk(\mathbf{N}) = 4 \Leftrightarrow rk(\mathbf{n}) = 3$ with

$$\mathbf{n}^T := \begin{pmatrix} a_2 & b_2 & A_2 & B_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_6 & b_6 & A_6 & B_6 \end{pmatrix}. \tag{2}$$

Lemma 4.
The rank of $\mathbf{n}$ is invariant under regular affinities of the platform and the base.

Proof: We apply planar affine transformations τ_0 to the base and τ to the platform, respectively,

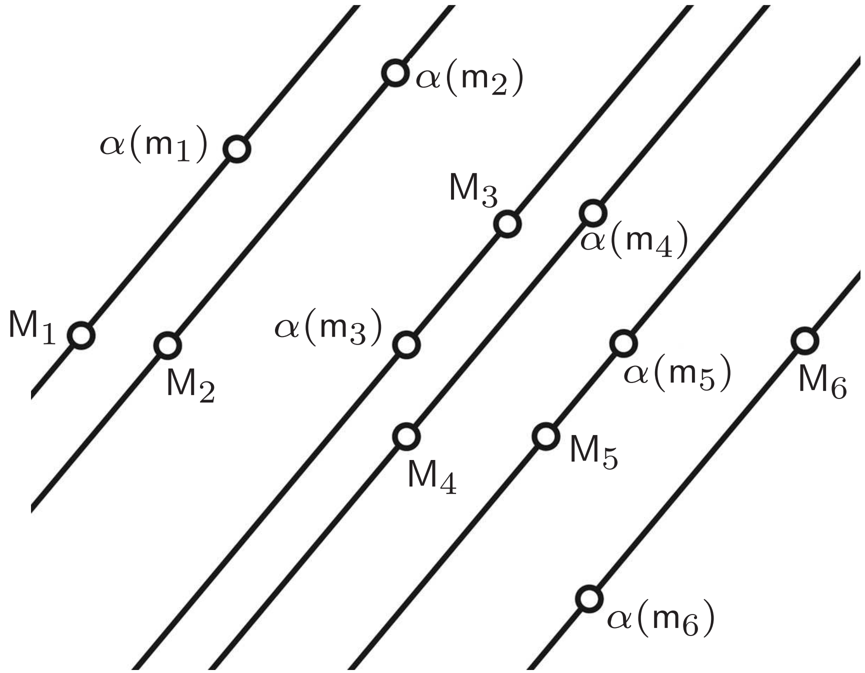
$$\tau_0 : \mathbf{M}_i \mapsto \mathbf{T}\mathbf{M}_i \quad \text{and} \quad \tau : \mathbf{m}_i \mapsto \mathbf{t}\mathbf{m}_i,$$

where $\mathbf{T}$ and $\mathbf{t}$ with $\det(\mathbf{T}) \neq 0 \neq \det(\mathbf{t})$ are the matrices of these transformations.

3. Geometric interpretation of rk(N)=4

We build the analogous matrix as given in Eq. (2) with respect to the coordinates of $\tau_0(\mathsf{M}_i)$ and $\tau(\mathsf{m}_i)$. This 5×4 matrix is denoted by $\widetilde{\mathbf{n}}^T$. Then the determinant of the 4×4 submatrix $\widetilde{\mathbf{n}}_i$ of $\widetilde{\mathbf{n}}$, which is obtained by removing the ith column of $\widetilde{\mathbf{n}}$, factors into $\det(\mathbf{T})\det(\mathbf{t})\det(\mathbf{n}_i)$. $\square$

Theorem 4.
$rk(\mathbf{N}) = 4 \iff$ There exists a regular affinity α from the planar platform to the planar base of the non-architecturally singular SGP in a way that $\alpha(\mathsf{m}_i)$ and M_i are located on lines of a parallel line pencil with vertex P at infinity.



3. Geometric interpretation of rk(N)=4

Proof: Due to the existence Theorem and the last lemma we can assume w.l.o.g.:

$$\mathbf{M}_1 = \mathbf{m}_1 = (0,0,0)^T, \quad \mathbf{M}_2 = \mathbf{m}_2 = (1,0,0)^T, \quad \mathbf{M}_3 = \mathbf{m}_3 = (0,1,0)^T.$$

"$\Rightarrow$" By basic operations on columns of the matrix $\mathbf{n}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & A_4 - a_4 & A_5 - a_5 & A_6 - a_6 \\ 0 & 0 & B_4 - b_4 & B_5 - b_5 & B_6 - b_6 \end{pmatrix},$$

which implies that $(A_i - a_i)(B_j - b_j) = (A_j - a_j)(B_i - b_i)$ has to hold for pairwise distinct $i, j \in \{4, 5, 6\}$, which already proves "$\Rightarrow$".

3. Geometric interpretation of rk(N)=4

"$\Leftarrow$" Under our assumptions the coordinates of the anchor points can be written as:

$$\mathbf{M}_1 = \mathbf{m}_1 = (0,0,0)^T, \qquad \mathbf{M}_4 = \mathbf{m}_4 + \xi_4\mathbf{p}, \tag{3}$$

$$\mathbf{M}_2 = \mathbf{m}_2 = (1,0,0)^T, \qquad \mathbf{M}_5 = \mathbf{m}_5 + \xi_5\mathbf{p}, \tag{4}$$

$$\mathbf{M}_3 = \mathbf{m}_3 = (0,1,0)^T, \qquad \mathbf{M}_6 = \mathbf{m}_6 + \xi_6\mathbf{p}, \tag{5}$$

where $\mathbf{p} = (p_1, p_2, 0)^T$ is the direction of the parallel line pencil. By applying analogous column operations as in "$\Rightarrow$" we see that $rk(\mathbf{n}) = 3$ holds. $\square$

Corollary 1.
$rk(\mathbf{N}) = 4 \Leftrightarrow$ There exists a configuration of the planar non-architecturally singular SGP where all six legs are not coplanar but are contained in a parallel plane pencil.

4. Rational parametrization of the singularity locus

Coordinatization of planar SGPs with $rk(\mathbf{N}) = 4$:
Take the coordinates of Eqs. (3-5) and apply planar affine transformations τ_0 and τ, where one can assume w.l.o.g. that $T_{21} = t_{21} = 0$ and $T_{11} > 0 < t_{11}$ holds. If one wants to eliminate the factor of similarity one can set e.g. $t_{11} = 1$.

Based on this coordinatization we compute the Plücker coordinates of the carrier lines l_i of the six legs by $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{m}'_i - \mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$, where $\mathbf{m}'_i$ is the location-vector of m_i with respect to the fixed system; i.e. $\mathbf{m}'_i = N^{-1}\mathbf{R}\mathbf{m}_i + \mathbf{s}$ with

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix},$$

$$N = e_0^2 + e_1^2 + e_2^2 + e_3^2 \quad \text{and} \quad \mathbf{s} = (s_1, s_2, s_3)^T.$$

4. Rational parametrization of the singularity locus

The lines $\mathsf{l}_1, \ldots, \mathsf{l}_6$ belong to a linear line complex if and only if $S = 0$ holds with

$$S := \begin{vmatrix} \mathbf{l}_1 & \ldots & \mathbf{l}_6 \\ \widehat{\mathbf{l}}_1 & \ldots & \widehat{\mathbf{l}}_6 \end{vmatrix}.$$

Therefore the singularity locus Σ, which is quadratic in s_1, s_2, s_3 according to Theorem 1, is given by $S = 0$ (can be computed explicitly by e.g. MAPLE).

For the parametrization of the singularity locus we follow the idea of COSTE AND MOUSSA [3]. It can be seen by direct computations that the ideal point W in direction $(r_{23} : -r_{13} : 0)$ is located on the projective closure of Σ.

Remark: W is the ideal point of the intersection line of platform and base. $\diamond$

4. Rational parametrization of the singularity locus

Now the parallel line bundle $\mathcal{B}$ through W given by:

$$\mathcal{B} : \begin{pmatrix} r_{13}u \\ r_{23}u \\ v \end{pmatrix} + w \begin{pmatrix} r_{23} \\ -r_{13} \\ 0 \end{pmatrix} \tag{6}$$

can be used for the rational parametrization: We plug these coordinates of $\mathcal{B}$ into $S = 0$ and obtain an equation of the form $p + qw = 0$. If we insert $w = -p/q$ back into Eq. (6) we get the desired rational parametrization of $\Sigma(u, v, e_0 : \ldots : e_3)$, which is well defined for all orientations with $q \neq 0$.

Remark: Note that one can obtain a rational parametrization by using a line bundle through any (rational) point of the quadric; for instance the origin instead of the ideal point W. The advantage of using W is that we even get a graph-representation of the singularity surface. ◇

5. Direct kinematics

The condition that the point m_i is located on a sphere with center M_i and radius d_i is a quadratic condition $K_i = 0$ (e.g. HUSTY [5]) in $e_0, \ldots, e_3, s_1, s_2, s_3$. Then one considers the linear combination:

$$Q := \kappa_1 K_1 + \kappa_2 K_2 + \kappa_3 K_3 + \kappa_4 K_4 + \kappa_5 K_5 + \kappa_6 K_6.$$

According to KARGER [8] there exists a $(5 - rk(\mathbf{N}))$-dimensional solution set for $(\kappa_1 : \ldots : \kappa_6) \neq (0 : \ldots : 0)$ in a way that Q is free of translation parameters (but still quadratic in the Euler parameters).

Therefore $rk(\mathbf{N}) = 4$ implies the existence of two linear independent linear combinations Q_1 and Q_2. Moreover one can solve the system of equations $K_2 - K_1 = K_3 - K_1 = K_4 - K_1 = 0$, which is linear in s_1, s_2, s_3 for these unknowns. Inserting the resulting expressions into K_1 implies a condition O of degree 8 in the Euler parameters.

5. Direct kinematics

Therefore the direct kinematics reduces to the intersection of an octic surface $O = 0$ and two quadrics $Q_1 = Q_2 = 0$ in the Euler parameter space, which shows the following result considering Bezout's theorem.

Theorem 5.
A planar SGP with $rk(\mathbf{N}) = 4$ has not more than 32 solutions (over $\mathbb{C}$) for the direct kinematics problem.

An example verifying these upper bound of 32 (over $\mathbb{C}$) is given in Aigner [1].

Open problems in this context:

- **?** Example of planar SGPs with $rk(\mathbf{N}) = 4$ possessing 32 real solutions.
- **?** Characterization of planar SGPs with $rk(\mathbf{N}) = 4$ having self-motions.

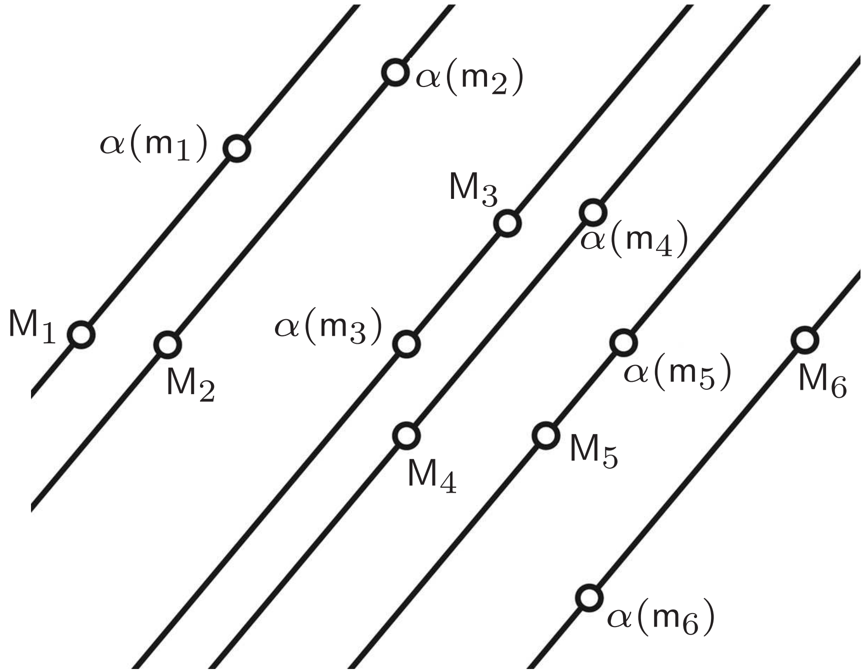
6. Leg-replacement

We consider the set $\mathcal{L}$ of legs, which can be added to a SGP without changing:

- neither the direct kinematics (cf. Husty et al [6]),
- nor the set of singular configurations (cf. Borras et al [2]).

From these papers it is known that for a planar SGP a one-parametric set $\mathcal{L}$ exists, where the platform (resp. base) anchor points are located on a planar cubic curve c (resp. C) on the platform (resp. base).

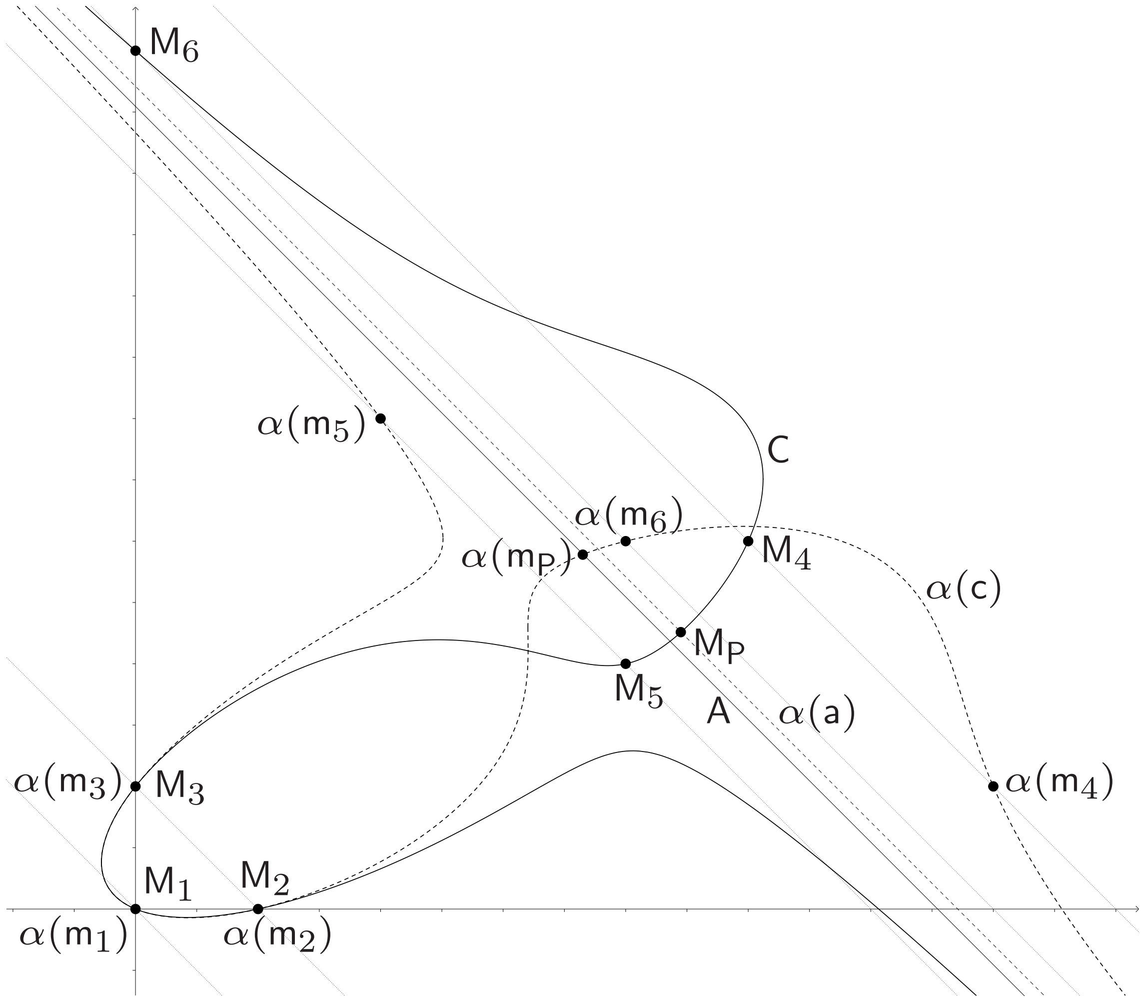
It can be shown (cf. Aigner [1]) by direct computation that P is located on C and α^{-1}(P) on c with P and α of Theorem 4.



6. Leg-replacement

As the leg-replacement is singular-invariant, $\alpha(\mathrm{m}) \in \alpha(\mathrm{c})$ and the corresponding base point $\mathrm{M} \in \mathrm{C}$ have to be located on a line through P.

The corresponding platform point $\alpha(\mathrm{m_P}) \in \alpha(\mathrm{c})$ (resp. base point $\mathrm{M_P} \in \mathrm{C}$) of $\mathrm{P} \in \mathrm{C}$ (resp. $\mathrm{P} \in \alpha(\mathrm{c})$) is located on the asymptote A of C (resp. asymptote $\alpha(\mathrm{a})$ of $\alpha(\mathrm{c})$) through P.



7. Conclusion and references

This detailed study of planar SGPs with $rk(\mathbf{N}) = 4$ gives:

- a geometric interpretation of the rank condition,
- a rational parametrization of the singularity locus,
- an upper bound for the solution of the direct kinematics problem,
- special properties of the anchor point loci for singular-invariant leg-replacement.

In addition an existence theorem for non-architecturally singular SGPs is presented.

All references refer to the list of publications given in the presented paper:

Aigner, B., Nawratil, G.: Planar Stewart Gough platforms with quadratic singularity surface. New Trends in Mechanism and Machine Science: Theory and Industrial Applications (P. Wenger, P. Flores eds.) pages 93–102, Springer (2016) ISBN: 978-3-319-44155-9