Journal for Geometry and Graphics
Volume VOL (YEAR), No. NO, 1–12.

Necessary conditions for type II DM self-motions of planar Stewart Gough platforms

Georg Nawratil
Institute of Discrete Mathematics and Geometry, Vienna University of Technology
Wiedner Hauptstrasse 8-10/104, Vienna, A-1040, Austria
nawratil@geometrie.tuwien.ac.at

Abstract. Due to a previous publication of the author, it is already known that one-parametric self-motions of Stewart Gough platforms with planar base and planar platform can be classified into two so-called Darboux Mannheim (DM) types (I and II). Moreover, the author also presented a method for computing the set of equations yielding a type II DM self-motion explicitly. Based on these equations we prove in this article the necessity of three conditions for obtaining a type II DM self-motion. Finally, we give a geometric interpretation of these conditions, which also identifies a property of line-symmetric Bricard octahedra, which was not known until now, to the best knowledge of the author.

Key Words: Self-motion, Stewart Gough platform, Borel Bricard problem, Bricard octahedra

1. Introduction

The geometry of a Stewart Gough (SG) platform with planar base and planar platform (which is also known as planar SG platform) is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, 0)^T$ with respect to the fixed system Σ_0 and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, 0)^T$ with respect to the moving system Σ (cf. Fig. 1). By using Study parameters $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$ for the parametrization of Euclidean displacements, the coordinates $\mathbf{m}_i'$ of the platform anchor points with respect to Σ_0 can be written as $K\mathbf{m}_i' = \mathbf{R}\,\mathbf{m}_i + (t_1, t_2, t_3)^T$ with

$$t_1 = 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \qquad t_2 = 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3),$$
$$t_3 = 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1), \qquad K = e_0^2 + e_1^2 + e_2^2 + e_3^2 \neq 0 \quad \text{and}$$
$$\mathbf{R} = (r_{ij}) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}.$$

Now all points of the real 7-dimensional space $P_{\mathbb{R}}^7$, which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the three-dimensional subspace $e_0 = \ldots = e_3 = 0$ of Ψ, as its points cannot fulfill the normalizing condition $K = 1$.

ISSN 1433-8157/$ 2.50 © YEAR Heldermann Verlag, Berlin

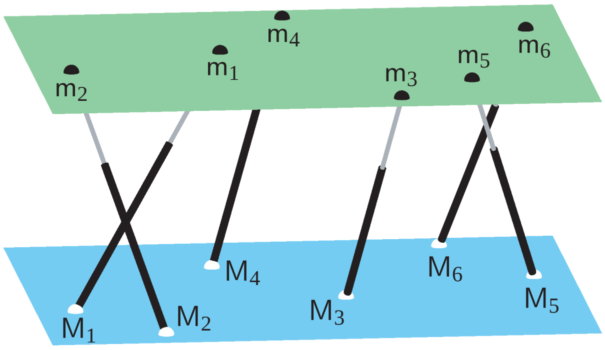

Figure 1: Sketch of a planar SG platform $m_1, \ldots, M_6$.

If the geometry of the manipulator is given as well as the six leg lengths, then the SG platform is generically rigid, but under particular conditions, the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion. Note that such motions are also solutions to the still unsolved problem posed 1904 by the French Academy of Science for the Prix Vaillant, which is also known as Borel Bricard problem (cf. [1, 4, 6, 10]) and reads as follows:

"Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."

Especially those rigid-body motions are of interest, where more than five points possess spherical trajectories. In this context, we only want to mention the well-known theorem of Duporcq [5], which can be formulated in the following way:

"If five points of a plane P move on five fixed spheres whose centers lie on a fixed plane P', then there exist on P a sixth point which also describes such a sphere."

1.1. Types of self-motions

In this and the next subsection we sketch the results and ideas of the central work [15] in this context.

It is already known, that manipulators which are singular in every possible configuration, possess self-motions in each pose (over $\mathbb{C}$). As these so-called architecturally singular SG platforms are well studied and classified (for the planar case we refer to [8, 12, 20, 21] and for the non-planar case note [9, 13]), we are only interested in non-architecturally singular SG platforms with self-motions. Until now only few self-motions of this type are known, as their computation is a very complicated task. To the best knowledge of the author, a complete and detailed review of these self-motions was given in [18].

Due to the publications [7, 11], it is known that the set $\mathscr{L}$ of additional legs, which can be attached to a given planar SG platform $m_1, \ldots, M_6$ without restricting the forward kinematics, is determined by a linear system of equations given in Eq. (30) of [11]. As the solvability condition of this system is equivalent to the criterion given in Eq. (12) of [2], also the singularity surface of the manipulator does not change by adding legs of $\mathscr{L}$. Moreover, it was shown in [11], that in the general case $\mathscr{L}$ is one-parametric and that the base anchor points M_i as well as the corresponding platform anchor points m_i of $\mathscr{L}$ are located on planar cubic curves C and c, respectively.

Assumption 1 *We assume that there exist such cubics c and C (which can also be reducible) in the Euclidean domain of the platform and the base, respectively.*

Now, we consider the complex projective extension $P^3_{\mathbb{C}}$ of the Euclidean 3-space E^3, i.e.

$$a_i = \frac{x_i}{w_i},\ b_i = \frac{y_i}{w_i},\ A_i = \frac{X_i}{W_i},\ B_i = \frac{Y_i}{W_i}, \qquad (1)$$

and replacing the coordinates $(a_i, b_i, 0)^T$ of m_i and $(A_i, B_i, 0)^T$ of M_i by homogeneous coordinates $(w_i : x_i : y_i : 0)^T$ and $(W_i : X_i : Y_i : 0)^T$, respectively. Note that ideal points of the platform (base) are characterized by $w_i = 0$ ($W_i = 0$). Therefore, we denote in the remainder of this article the coordinates of anchor points, which are ideal points, by x_i, y_i and X_i, Y_i, respectively. For finite anchor points we use the coordinates a_i, b_i and A_i, B_i, respectively.

The correspondence between the points of C and c in $P^3_{\mathbb{C}}$, which is determined by the geometry of the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$, can be computed according to [7, 11] or [2] under consideration of Eq. (1). As this correspondence has not to be a bijection, a point $\in P^3_{\mathbb{C}}$ of c (resp. C) is in general mapped to a non-empty set of points $\in P^3_{\mathbb{C}}$ of C (resp. c). We denote this set by the term *corresponding location* and indicate this fact by the usage of bracelets $\{\}$. Moreover, it should be noted that the corresponding location of a real point contains real points as well.

In $P^3_{\mathbb{C}}$ the cubic C has three ideal points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$, where at least one of these points (e.g. U_1) is real. The remaining points U_2 and U_3 are real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{u}_1\}, \{\mathsf{u}_2\}, \{\mathsf{u}_3\}$ of c ($\Rightarrow \{\mathsf{u}_1\}$ contains real points). We denote the ideal points of c by $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$, where again one (e.g. u_4) has to be real. The remaining points u_5 and u_6 are again real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{U}_4\}, \{\mathsf{U}_5\}, \{\mathsf{U}_6\}$ of C ($\Rightarrow \{\mathsf{U}_4\}$ contains real points).

Assumption 2 *For guaranteeing a general case, we assume that each of the corresponding locations* $\{\mathsf{u}_1\}$, $\{\mathsf{u}_2\}$, $\{\mathsf{u}_3\}, \{\mathsf{U}_4\}$, $\{\mathsf{U}_5\}$, $\{\mathsf{U}_6\}$ *consists of a single point. Moreover, we assume that no four collinear platform anchor points* u_j *or base anchor points* U_j *(*$j = 1, \ldots, 6$*) exist.*

Now the basic idea can simply be expressed by attaching the special "legs"[1] $\overline{\mathsf{u}_i\mathsf{U}_i} \in \mathscr{L}$ with $i = 1, \ldots, 6$ to the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$, which have the following kinematic interpretation (cf. [15]): The attachment of the "leg" $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i \in \{1, 2, 3\}$ corresponds with the so-called Darboux constraint, that the platform anchor point u_i moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i. Moreover, the attachment of the "leg" $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i \in \{4, 5, 6\}$ corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to u_i slides through U_i. Note that this Mannheim condition is the inverse of the Darboux condition.

By removing the originally six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ with $i = 1, \ldots, 6$ we remain with the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$, which is uniquely determined due to Assumption 1 and 2. Moreover, under consideration of Assumption 1 and 2, the following statement holds (cf. [15]):

Theorem 1 *The manipulator* $\mathsf{u}_1, \ldots, \mathsf{U}_6$ *is redundant and therefore architecturally singular. Moreover, all anchor points of the platform* $\mathsf{u}_1, \ldots, \mathsf{u}_6$ *and as well of the base* $\mathsf{U}_1, \ldots, \mathsf{U}_6$ *are distinct.*

It was also proven in [15] that there only exist type I and type II Darboux Mannheim (DM) self-motions, where the definition of types reads as follows:

Definition 1 *Assume* $\mathscr{M}$ *is a one-parametric self-motion of a non-architecturally singular SG platform* $\mathsf{m}_1, \ldots, \mathsf{M}_6$. *Then* $\mathscr{M}$ *is of the type n DM if the corresponding architecturally singular manipulator* $\mathsf{u}_1, \ldots, \mathsf{U}_6$ *has an n-parametric self-motion* $\mathscr{U}$ *(which includes* $\mathscr{M}$*). Note that the numbering of types is done with Roman numerals; i.e.* $n = I, II, \ldots$

[1] We have to quote the word legs in this context, as it is impossible to attach physical legs with infinite length to the platform.

1.2. Type II DM self-motions

In the remainder of the article we only study type II DM self-motions. The author [15] was already able to compute the set of equations yielding a type II DM self-motion explicitly. This symbolic computation, which is repeated in subsection 2.5, is based on the analytical versions of the Darboux and Mannheim constraints, which are given next:

Darboux constraint: The constraint that the platform anchor point u_i ($i=1,2,3$) moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i can be written as (cf. [15])

$$\Omega_i:\ \overline{X}_i(a_ir_{11}+b_ir_{12}+t_1)+\overline{Y}_i(a_ir_{21}+b_ir_{22}+t_2)+L_iK=0,$$

with $X_i,Y_i,a_i,b_i,L_i\in\mathbb{C}$. This is a homogeneous quadratic equation in the Study parameters $e_0,\dots,f_3$, where $\overline{X}_i$ and $\overline{Y}_i$ denote the conjugate complex of X_i and Y_i, respectively.

Mannheim constraint: The constraint that the plane orthogonal to u_i ($i=4,5,6$) through the platform point $(g_i,h_i,0)$ slides through the point U_i of the fixed system can be written as (cf. [15])

$$\begin{aligned}\Pi_i:\overline{x}_i[A_ir_{11}+B_ir_{21}-g_iK-2(e_0f_1-e_1f_0-e_2f_3+e_3f_2)]+\\ \overline{y}_i[A_ir_{12}+B_ir_{22}-h_iK-2(e_0f_2+e_1f_3-e_2f_0-e_3f_1)]=0,\end{aligned}$$

with $x_i,y_i,A_i,B_i,g_i,h_i\in\mathbb{C}$. This is again a homogeneous quadratic equation in the Study parameters $e_0,\dots,f_3$, where $\overline{x}_i$ and $\overline{y}_i$ denote the conjugate complex of x_i and y_i.

The content of the following lemma was also proven in [15]:

Lemma 1 *Without loss of generality (w.l.o.g.) we can assume that the algebraic variety of the two-parametric self-motion of the manipulator* $\mathsf{u}_1,\dots,\mathsf{U}_6$ *is spanned by* $\Psi,\Omega_1,\Omega_2,\Omega_3,\Pi_4,\Pi_5$. *Moreover, we can choose following special coordinate systems in* Σ_0 *and* Σ *w.l.o.g.:* $X_1=Y_2=Y_3=x_4=y_5=1$ *and* $a_1=b_1=y_4=A_4=B_4=Y_1=h_4=g_5=0$.

An important step in direction of a complete classification of type II DM self-motions was done by the following basic result, which was proven in [14]:

Theorem 2 *If the corresponding manipulator* $\mathsf{u}_1,\dots,\mathsf{U}_6$ *of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion does not fulfill neither the three equations*

$$\begin{aligned}L_1(\overline{X}_2-\overline{X}_3)-L_2+L_3=0,\quad & a_2(\overline{X}_2-\overline{X}_3)+\overline{X}_3(\overline{X}_2b_2-\overline{X}_3b_3)+b_2-b_3=0,\\ & a_3(\overline{X}_2-\overline{X}_3)+\overline{X}_2(\overline{X}_2b_2-\overline{X}_3b_3)+b_2-b_3=0,\end{aligned}\qquad(2)$$

nor the three equations

$$\begin{aligned}L_1(\overline{X}_2-\overline{X}_3)-L_2+L_3=0,\quad & a_2(\overline{X}_2-\overline{X}_3)-\overline{X}_3(\overline{X}_2b_2-\overline{X}_3b_3)-b_2+b_3=0,\\ & a_3(\overline{X}_2-\overline{X}_3)-\overline{X}_2(\overline{X}_2b_2-\overline{X}_3b_3)-b_2+b_3=0,\end{aligned}\qquad(3)$$

then it has to have further three collinear anchor points in the base or in the platform beside the points $\mathsf{U}_1,\mathsf{U}_2,\mathsf{U}_3$ *and* $\mathsf{u}_4,\mathsf{u}_5,\mathsf{u}_6$.

Based on this theorem we prove the following much stronger result within this article:

Theorem 3 *The corresponding manipulator* $\mathsf{u}_1,\dots,\mathsf{U}_6$ *of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion has to fulfill the three conditions either of Eq. (2) or Eq. (3).*

2. Preparatory work for the proof of Theorem 3

For the proof of Theorem 3 we have to show that there exists no corresponding manipulator $\mathsf{u}_1,\ldots,\mathsf{U}_6$ of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion, which does not fulfill either the three conditions of Eq. (2) or Eq. (3).

Due to Theorem 2 and due to Lemma 2 of [8] we can even restrict ourselves to manipulators $\mathsf{u}_1,\ldots,\mathsf{U}_6$, which have three collinear platform points $\mathsf{u}_i,\mathsf{u}_j,\mathsf{u}_k$ and three collinear base points $\mathsf{U}_l,\mathsf{U}_m,\mathsf{U}_n$ beside the points $\mathsf{U}_1,\mathsf{U}_2,\mathsf{U}_3$ and $\mathsf{u}_4,\mathsf{u}_5,\mathsf{u}_6$ where (i,j,k,l,m,n) consists of all indices from 1 to 6.

As we have different types of anchor points (real, complex, finite, infinite), we have to distinguish the following four cases of three collinear points (beside the triples $\mathsf{U}_1,\mathsf{U}_2,\mathsf{U}_3$ and $\mathsf{u}_4,\mathsf{u}_5,\mathsf{u}_6$):

A. $\mathsf{U}_1,\mathsf{U}_4,\mathsf{U}_5$ collinear ($\Leftrightarrow$ $\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_6$ collinear): As u_5 and u_6 are both real or conjugate complex, this case is equivalent to $\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_5$ collinear ($\Leftrightarrow$ $\mathsf{U}_1,\mathsf{U}_4,\mathsf{U}_6$ collinear).
   Moreover, by exchanging the platform and the base the above two cases are also equivalent to $\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_4$ collinear ($\Leftrightarrow$ $\mathsf{U}_3,\mathsf{U}_5,\mathsf{U}_6$ collinear) and $\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_4$ collinear ($\Leftrightarrow$ $\mathsf{U}_2,\mathsf{U}_5,\mathsf{U}_6$ collinear), respectively.

B. $\mathsf{U}_2,\mathsf{U}_4,\mathsf{U}_5$ collinear ($\Leftrightarrow$ $\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_6$ collinear): As u_5 and u_6 are both real or conjugate complex, this case is equivalent to $\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_5$ collinear ($\Leftrightarrow$ $\mathsf{U}_2,\mathsf{U}_4,\mathsf{U}_6$ collinear).
   Moreover, as U_2 and U_3 are both real or conjugate complex, these cases are also equivalent to $\mathsf{U}_3,\mathsf{U}_4,\mathsf{U}_5$ collinear ($\Leftrightarrow$ $\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_6$ collinear) and $\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_5$ collinear ($\Leftrightarrow$ $\mathsf{U}_3,\mathsf{U}_4,\mathsf{U}_6$ collinear), respectively.

C. $\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_4$ collinear ($\Leftrightarrow$ $\mathsf{U}_1,\mathsf{U}_5,\mathsf{U}_6$ collinear)

D. $\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_3$ collinear ($\Leftrightarrow$ $\mathsf{U}_4,\mathsf{U}_5,\mathsf{U}_6$ collinear)

In the following we discuss these four types A–D in more detail:

2.1. Collinearity of type A

$\mathsf{U}_1,\mathsf{U}_4,\mathsf{U}_5$ are collinear for $B_5=0$. As due to Assumption 2 no four platform anchor points u_i or base anchor points U_i are allowed to be collinear, we can stop the discussion of type A if:

- $\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_4$ collinear ($\Leftrightarrow b_2-b_3=0$),
- $\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_3$ collinear ($\Leftrightarrow a_2b_3-a_3b_2=0$),
- $\mathsf{u}_2,\mathsf{u}_3,\mathsf{u}_5$ collinear ($\Leftrightarrow x_5(b_2-b_3)-a_2+a_3=0$),

because then the points $\mathsf{U}_1,\mathsf{U}_4,\mathsf{U}_5,\mathsf{U}_6$ are collinear due to Lemma 2 of [8], which yields a contradiction. Due to Theorem 1 also $A_5(X_2-X_3)\neq 0$ has to hold, as otherwise the base anchor points are not pairwise distinct. Finally, we can assume $X_2\neq 0$ w.l.o.g., because both points U_2 and U_3 do not belong to the triple of collinear points.

2.2. Collinearity of type B

$\mathsf{U}_2,\mathsf{U}_4,\mathsf{U}_5$ are collinear for $A_5=X_2B_5$. Now we can stop the discussion of case B if:

- $\mathsf{u}_1,\mathsf{u}_2,\mathsf{u}_3$ collinear ($\Leftrightarrow a_2b_3-a_3b_2=0$),
- $\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_4$ collinear ($\Leftrightarrow b_3=0$),
- $\mathsf{u}_1,\mathsf{u}_3,\mathsf{u}_5$ collinear ($\Leftrightarrow a_3-x_5b_3=0$),

because then the points $\mathsf{U}_2,\mathsf{U}_4,\mathsf{U}_5,\mathsf{U}_6$ are collinear, a contradiction. Due to Theorem 1 also $B_5(X_2-X_3)\neq 0$ has to hold, as otherwise the base anchor points are not pairwise distinct. Moreover, we can stop the discussion of case B, if U_2 is real ($\Leftrightarrow X_2\in\mathbb{R}$, especially $X_2=0$) because then this case is equivalent to case A.

2.3. Collinearity of type C

u_2, u_3, u_4 are collinear for $b_2 = b_3$. We can stop the discussion of case C if U_1, U_4, U_5 are collinear ($\Leftrightarrow B_5 = 0$), because then the points u_2, u_3, u_4, u_6 are collinear, a contradiction. Moreover $b_2 \neq 0$ has to hold because otherwise u_1, u_2, u_3, u_4 are collinear, a contradiction. Due to Theorem 1 also $(a_2 - a_3)(X_2 - X_3) \neq 0$ has to hold, as $u_2 = u_3$ resp. $U_2 = U_3$ yield a contradiction. In addition, we can assume $X_2 \neq 0$ w.l.o.g., because the corresponding points of U_2 and U_3 belong to the triple of collinear points.

We can also assume that U_2, U_4, U_5 are not collinear ($\Leftrightarrow A_5 - X_2 B_5 \neq 0$), because this case was already discussed in case B.

2.4. Collinearity of type D

u_1, u_2, u_3 are collinear for $a_2 b_3 - a_3 b_2 = 0$. Now we can stop the discussion of case D if:

- U_1, U_4, U_5 collinear ($\Leftrightarrow B_5 = 0$),
- U_2, U_4, U_5 collinear ($\Leftrightarrow A_5 - X_2 B_5 = 0$),
- U_3, U_4, U_5 collinear ($\Leftrightarrow A_5 - X_3 B_5 = 0$),

because then the points u_1, u_2, u_3, u_6 are collinear, a contradiction. Moreover, we can assume $b_2 b_3 \neq 0$ because otherwise u_1, u_2, u_3, u_4 are collinear ($\Rightarrow a_2 = a_3 b_2 / b_3$). Clearly, also the points u_1, u_2, u_3, u_5 are not allowed to be collinear which implies $a_3 - x_5 b_3 \neq 0$. Moreover we can assume $b_2 \neq b_3$ because otherwise we get $u_2 = u_3$, a contradiction. Due to Theorem 1 also $(X_2 - X_3) \neq 0$ has to hold, as $U_2 = U_3$ yields a contradiction. In addition, we can assume $X_2 \neq 0$ w.l.o.g., because the corresponding points of U_2 and U_3 belong to the triple of collinear points.

2.5. Preparatory computations

In the following we describe how the set $\mathscr{E}$ of equations yielding a type II DM self-motion can be computed explicitly (cf. section 3.2 of [15]). Note that the proof for the general case of Theorem 3 (cf. section 3) is based on this set $\mathscr{E}$.

We solve the linear system of equations $\Psi, \Omega_1, \Omega_2, \Pi_4$ for $f_0, \ldots, f_3$ and plug the obtained expressions in the remaining two equations.[2] This yields in general two homogeneous polynomials $\Omega[40]$ and $\Pi[96]$ in the Euler parameters of degree 2 and 4, respectively. The number in the square brackets gives the number of terms.

Finally, we compute the resultant of Ω and Π with respect to one of the Euler parameters. Here we choose[3] e_0. This yields a homogeneous polynomial $\Gamma[117\,652]$ of degree 8 in e_1, e_2, e_3. In the following we denote the coefficients of e_1^i, e_2^j, e_3^k of Γ by Γ_{ijk}. We get a set $\mathscr{E}$ of 24 equations $\Gamma_{ijk} = 0$ in the 14 unknowns (a_2, b_2, a_3, b_3, A_5, B_5, X_2, X_3, x_5, L_1, L_2, L_3, g_4, h_5).

Moreover, we denote the coefficients of $e_0^i e_1^j, e_2^k, e_3^l$ of Ω and Π by Ω_{ijkl} and Π_{ijkl}, respectively.

Finally, it should be said that all symbolic computations were done with MAPLE 14 on a high-capacity computer.[4]

---

[2] For $e_0 e_2 - e_1 e_3 \neq 0$ this can be done w.l.o.g., as this factor belongs to the denominator of f_i.

[3] Therefore we are looking for a common factor of Ω and Π, which depends on e_0.

[4] CPU: Intel(R) Core(TM)2 Quad CPU Q6600 @ 2.40 GHz, RAM: 8 GB, Hard disk: 2x250 GB, Graphic: nVidia 7x00GT or 8x00GT, Operating system: Linux x64 (Kernel 2.6.18-53)

3. Proving the general case of Theorem 3

For the general case we have to assume $\Omega_{2000}\Pi_{3000} \neq 0$, as only those solutions of $\mathscr{E}$ correspond to type II self-motions, which do not cause a vanishing of the coefficient of the highest power of e_0 in Ω or Π. In the following we prove this general case for all types A–D of collinearity. For each type the proof is done by contradiction, i.e. we stop the discussion for the cases listed in the respective subsections (subsection 2.1–2.4) or if the three conditions of Eq. (2) or Eq. (3) are fulfilled.

3.1. Collinearity of type A

Γ_{800} can only vanish without contradiction (w.c.) for $L_1 = g_4$ or for $F_A[8] = 0$.

3.1.1. $F_A = 0$

We can express L_1 from $F_A = 0$. Now we distinguish two cases:

1. $L_1 \neq g_4$: Then $\Gamma_{710} = 0$ implies $a_2 = a_3 - \overline{X}_2 b_2 + \overline{X}_3 b_3$. Now Γ_{620} cannot vanish w.c..
2. $L_1 = g_4$: We can compute h_5 from the only non-contradicting (non-c.) factor of Γ_{602}. Now Γ_{530} can only vanish w.c. for:
   a. $L_3 = \overline{X}_3(L_2 - b_2)/\overline{X}_2 + \overline{X}_3(a_2 - a_3) + b_3$: We can express A_5 from the only non-c. factor of Γ_{422}. Again we distinguish two cases:
      i. $\overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2 - a_3 \neq 0$: Now Γ_{350} has only one non-c. factor, which can be solved for L_2. Then $\Gamma_{314} = 0$ implies $b_3 = 0$. Now we get $x_5 = -X_3$ from $\Gamma_{206} = 0$. Then Γ_{080} can only vanish w.c. for:
         - ⋆ $X_3 = 0$: Now $\Gamma_{026} = 0$ yields the contradiction.
         - ⋆ $b_2 = \overline{X}_2 a_2 - \overline{X}_3 a_3$, $X_3 \neq 0$: Γ_{026} cannot vanish w.c..
      ii. $a_3 = \overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2$: Then $\Gamma_{260} = 0$ implies $L_2 = 2\overline{X}_2^2 b_2 + \overline{X}_2 a_2 + b_2$. Moreover, we can solve the only non-c. factor of Γ_{242} for $\overline{x}_5$.
         - ⋆ Assuming $\overline{X}_2 b_3 - \overline{X}_3 b_2 \neq 0$: Under this assumption we can compute a_2 from the only non-c. factor of Γ_{080}. Now $\Gamma_{224} = 0$ yields the contradiction.
         - ⋆ $b_3 = \overline{X}_3 b_2/\overline{X}_2$: Then Γ_{080} can only vanish w.c. for $X_3 = 0$ or $X_2 = -X_3$. In both cases $\Gamma_{026} = 0$ yields the contradiction.
   b. $a_2 = \overline{X}_3 b_3 - \overline{X}_2 b_2 + a_3$, $\overline{X}_2\overline{X}_3(a_2 - a_3) + \overline{X}_2(b_3 - L_3) - \overline{X}_3(b_2 - L_2) \neq 0$: Now $\Gamma_{440} = 0$ yields the contradiction.

3.1.2. $F_A \neq 0$

Now $L_1 = g_4$ has to hold. Then Γ_{080} factors into $G_A[8]H_A[16]^2$.

1. $G_A[8] = 0$: We can express L_1 from $G_A[8] = 0$. Now Γ_{170} can only vanish w.c. for:
   a. $a_2 = \overline{X}_3 b_3 - \overline{X}_2 b_2 + a_3$: We can solve the only non-c. factor of Γ_{620} for h_5. Now we can express L_3 from the only non-c. factor of Γ_{602}.
      i. $x_5 = 0$: We distinguish two cases:
         - ⋆ $\overline{X}_2 b_3 - \overline{X}_3 b_2 \neq 0$: Now we can express a_3 from the only non-c. factor of Γ_{260}. Then we can compute A_5 from the only non-c. factor of Γ_{440}. Now Γ_{404} cannot vanish w.c..
         - ⋆ $b_3 = \overline{X}_3 b_2/\overline{X}_2$: Now Γ_{260} can only vanish w.c. for $X_3 = 0$. Then $\Gamma_{440} = 0$ implies $A_5 = -a_3$. Now we can solve the only non-c. factor of Γ_{422} for L_2. Finally, $\Gamma_{026} = 0$ yields the contradiction.

ii. $x_5 \neq 0$: Under this assumption we can compute A_5 from the only non-c. factor of Γ_{260}. Then we can express L_2 from the only non-c. factor of Γ_{062}. Now the resultant of the only non-c. factors of Γ_{404} and Γ_{440} with respect to $\overline{X}_3$ can only vanish w.c. for:

- ⋆ $b_3 = 0$: Now Γ_{404} implies $x_5 = X_3$. Finally, $\Gamma_{026} = 0$ yields the contradiction.
- ⋆ $x_5 = X_3$, $b_3 \neq 0$: Now $\Gamma_{440} = 0$ implies $a_3 = \overline{X}_2 b_3$ and $\Gamma_{404} = 0$ yields the contradiction.
- ⋆ $a_3 = -\overline{X}_2 b_3$, $b_3(x_5 - X_3) \neq 0$: Now $\Gamma_{404} = 0$ implies $b_2 = -b_3$ and $\Gamma_{440} = 0$ yields the contradiction.

b. $V_A[16] = 0$, $\overline{X}_3 b_3 - \overline{X}_2 b_2 - a_2 + a_3 \neq 0$:

i. $x_5 = 0$: Now we can solve $V_A = 0$ for L_3. Then we can compute b_3 from the only non-c. factor of Γ_{620}. Now Γ_{602} implies $h_5 = 0$. Then the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $X_3 = 0$. Now $\Gamma_{440} = 0$ implies $a_3 = -A_5$. From the only non-c. factor of $\Gamma_{422} = 0$ we express L_2. Then $\Gamma_{026} = 0$ yields the contradiction.

ii. $x_5 \neq 0$: Under this assumption we can compute A_5 from $V_A[16] = 0$. Then can solve the only non-c. factor of Γ_{620} for h_5. Now we can express L_3 from the only non-c. factor of Γ_{602}. Moreover, we can solve the only non-c. factor of Γ_{062} for L_2. Now the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Then $\Gamma_{440} = 0$ implies $x_5 = X_3$ and $\Gamma_{422} = 0$ yields the contradiction.

2. $H_A[16] = 0$, $G_A[8] \neq 0$: We distinguish two cases:

a. $\overline{X}_2 a_2 - \overline{X}_3 a_3 \neq 0$: Under this assumption we can compute h_5 from $H_A[16] = 0$.

i. $x_5 = 0$: We can solve the only non-c. factor of Γ_{620} for b_3. Then we express L_3 from the only non-c. factor of Γ_{602}. Then the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $X_3 = 0$. Now $\Gamma_{440} = 0$ implies $a_3 = -A_5$ and from $\Gamma_{422} = 0$ we get $L_1 = -2A_5$. Then $\Gamma_{026} = 0$ yields the contradiction.

ii. $x_5 \neq 0$: Now we can compute A_5 from the only non-c. factor of Γ_{620}. Moreover, we can compute L_3 from the only non-c. factor of Γ_{602}. Now the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Then $\Gamma_{440} = 0$ implies $x_5 = X_3$. Now $\Gamma_{422} = 0$ implies $L_1 = 2a_3$. Finally, $\Gamma_{242} = 0$ yields the contradiction.

b. $a_2 = \overline{X}_3 a_3 / \overline{X}_2$: Now H_A can only vanish w.c. for $A_5 \overline{x}_5 + \overline{X}_3 a_3 = 0$.

i. $x_5 = 0$: Now $H_A = 0$ implies $X_3 = 0$. Then we can express h_5 from the only non-c. factor of Γ_{620}. Moreover, we can compute L_3 from the only non-c. factor of Γ_{602}. Then the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Now $\Gamma_{440} = 0$ implies $a_3 = -A_5$ and from $\Gamma_{422} = 0$ we get $L_1 = -2A_5$. Then $\Gamma_{026} = 0$ yields the contradiction.

ii. $x_5 \neq 0$: Under this assumption we can solve the last equation for A_5. Now we can express h_5 from the only non-c. factor of Γ_{620}. Then we can compute L_3 from the only non-c. factor of Γ_{602}. Now the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Then $\Gamma_{440} = 0$ implies $x_5 = X_3$. Now $\Gamma_{422} = 0$ implies $L_1 = 2a_3$. Finally, $\Gamma_{242} = 0$ yields the contradiction.

3.2. Collinearity of type B–D

For the collinearity of type B, C and D the case study can be done in an analogous way, which is given in full detail in the corresponding technical report [16].

4. Proving the special cases of Theorem 3

For the proof of the special cases $\Omega_{2000}\Pi_{3000} = 0$ and $e_0e_2 - e_1e_3 = 0$ (cf. footnote 2) we also refer to section 4 and 5, respectively, of the corresponding technical report [16], as these case studies exceed the number of pages for an usual journal article. Nevertheless, we encourage the interested reader to have a look at [16], as the presented discussion is not trivial.

Note that the discussion of special cases given in [16] finishes the proof of Theorem 3. □

5. Addendum

At the time of writing the convolute of papers [14, 15, 16, 17] and the article at hand, the author was under the assumption that Duporcq's theorem (cf. section 1) is correct (under consideration of the projective closure). However, recent studies on Duporcq's theorem (cf. [19]) showed, that this is not the case, which also has the following minor effect on the problem under consideration:

Due to the new result obtained in [19], it can also occur that $(\mathsf{u}_i, \mathsf{U}_i) = (\mathsf{u}_j, \mathsf{U}_j)$ holds for $i \neq j$ with[5] either $i, j \in \{1,2,3\}$ or $i, j \in \{4,5,6\}$, which contradicts the second part of Theorem 1. After a perhaps necessary renumbering of indices and an exchange of the platform and the base, one can assume w.l.o.g., that the sixth "leg" (cf. footnote 1) coincides with the fifth one.

Still, the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is redundant as two "legs" coincide, but only the points $\mathsf{u}_1, \ldots, \mathsf{u}_5$ as well as $\mathsf{U}_1, \ldots, \mathsf{U}_5$ are distinct (cf. Theorem 1). Nevertheless, Lemma 1 is also true for this case and therefore Theorem 2 holds for this exceptional case as well.[6]

However, Lemma 2 of [8], which was used in section 2, still holds with the exception, that the collinearity of $\mathsf{u}_i, \mathsf{u}_5, \mathsf{u}_6$ (resp. $\mathsf{U}_i, \mathsf{U}_5, \mathsf{U}_6$) no longer implies the collinearity of $\mathsf{U}_j, \mathsf{U}_k, \mathsf{U}_l$ (resp. $\mathsf{u}_j, \mathsf{u}_k, \mathsf{u}_l$) for pairwise distinct $i, j, k, l \in \{1, \ldots, 4\}$. This only effects the proof of Theorem 3 in the way that additionally the collinearity of $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4$ has to be checked, as this case is no longer equivalent with the collinearity of type A. This can be done analogously to the outlined procedure, which shows that Theorem 3 remains valid for the special case.

Moreover, as the determination of all planar SG platforms with a type II DM self-motion is based on Theorem 3, also the results within [17] take this exceptional case under consideration. Therefore, this special case does not cause any additional type II DM self-motions.

This closes the small gap, opened by the new result on Duporcq's theorem (cf. [19]).

6. Geometric interpretation of the necessary conditions

As noted in [14], the equations Eq. (2) and Eq. (3) arise from the condition that Ω of subsection 2.5 does not depend on e_0 and e_3 or e_1 and e_2, respectively. By computing $\Omega_{2000} + \Omega_{0002}$, $\Omega_{2000} - \Omega_{0002}$ and Ω_{1001} it can immediately be seen that the conditions of Eq. (2) can also be written as:

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0, \quad \overline{X}_2 a_2 - \overline{X}_3 a_3 + b_2 - b_3 = 0, \quad \overline{X}_2 b_2 - \overline{X}_3 b_3 - a_2 + a_3 = 0. \qquad (4)$$

By computing $\Omega_{0200} + \Omega_{0020}$, $\Omega_{0200} - \Omega_{0020}$ and Ω_{0110} it can immediately be seen that Eq. (3) can be rewritten as:

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0, \quad \overline{X}_2 a_2 - \overline{X}_3 a_3 - b_2 + b_3 = 0, \quad \overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2 - a_3 = 0. \qquad (5)$$

In the following we give the geometric interpretation of Eq. (4), which is sketched in Fig. 2a:

[5] If $(\mathsf{u}_i, \mathsf{U}_i) = (\mathsf{u}_j, \mathsf{U}_j)$ holds for $i \in \{1,2,3\}$ and $j \in \{4,5,6\}$, we would end up with four collinear points.

[6] To be totally correct, we also have to prove Theorem 2 for the special case $x_5 = 0$ (as $x_5 \neq 0$ cannot be assumed any longer w.l.o.g.), which can be done analogously to [14].

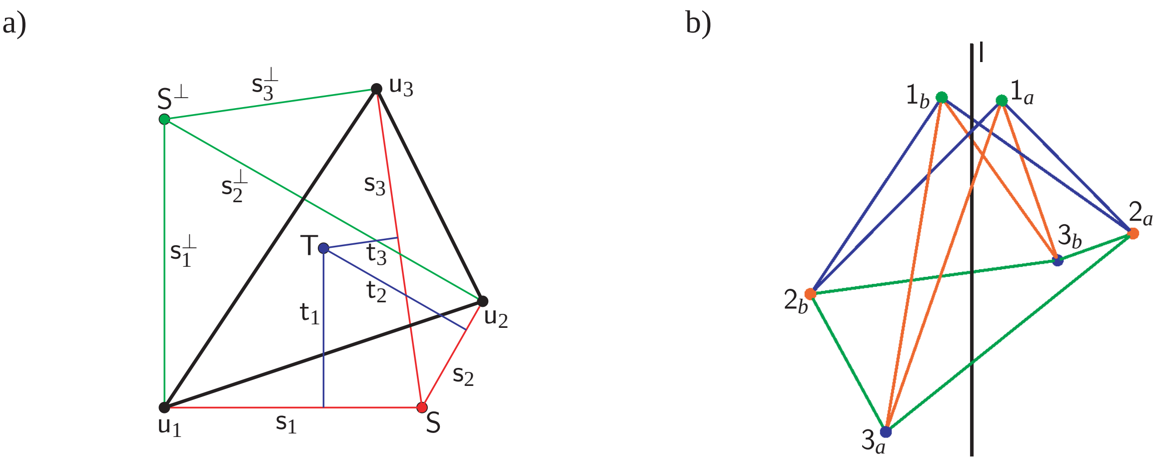

Figure 2: a) Sketch of the geometric interpretation of the necessary conditions. b) Axonometric view of a line-symmetric Bricard octahedron: $1_a = (1,0,0)$, $2_a = (5,3,-6)$, $3_a = (-2,-7,-9)$ and the line of symmetry l is the z-axis.

I. $L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0$ expresses that the three lines $\mathsf{t}_i \in \Sigma_0$ $(i = 1,2,3)$ with homogeneous line coordinates $[L_i : \overline{X}_i : \overline{Y}_i]$ have a common point T ($\Rightarrow$ the three Darboux planes belong to a pencil of planes).

II. $\overline{X}_2 b_2 - \overline{X}_3 b_3 - a_2 + a_3 = 0$ expresses that the three lines $\mathsf{s}_i := [\mathsf{u}_i, \overline{\mathsf{U}}_i]$ $(i = 1,2,3)$ with $\overline{\mathsf{U}}_i = (0 : \overline{X}_i : \overline{Y}_i)$ have a common point S.

III. $\overline{X}_2 a_2 - \overline{X}_3 a_3 + b_2 - b_3 = 0$ expresses that the three lines $\mathsf{s}_i^\perp := [\mathsf{u}_i, \overline{\mathsf{U}}_i^\perp]$ $(i = 1,2,3)$ with $\overline{\mathsf{U}}_i^\perp = (0 : -\overline{Y}_i : \overline{X}_i)$ have a common point $\mathsf{S}^\perp$.

Note that the items II and III only hold if the coordinate systems of the platform and base are chosen according to Lemma 1 and if these two coordinate systems coincide.

The geometric interpretation of Eq. (5) is equivalent with the one given above, if one rotates the platform about the x-axis with angle π. Therefore the two triples of necessary conditions are connected by this rotation, which is represented in the Euler parameter space by the transformation (cf. [10]): $(e_0, e_1, e_2, e_3) \mapsto (-e_1, e_0, -e_3, e_2)$.

Remark 1 *It is interesting to note, that the given necessary conditions only arise from the three Darboux constraints. A purely geometric proof of the necessity of these conditions for a type II DM self-motion of a general planar SG platform seems to be a complicated task.* ◇

6.1. Line-symmetric Bricard octahedra

We denote the vertices of the line-symmetric Bricard octahedron [3] by 1_a, 1_b, 2_a, 2_b, 3_a, 3_b, where v_a and v_b are symmetric with respect to the line l for $\mathsf{v} \in \{1,2,3\}$ (see Fig. 2b). Moreover, ε_{ijk} denotes the face spanned by $1_i, 2_j, 3_k$ with $i,j,k \in \{a,b\}$. Under consideration of this notation we can formulate the following theorem, which is illustrated in Fig. 3:

Theorem 4 *Every line-symmetric Bricard octahedron has the property that the following three planes, orthogonal to ε_{ijk}, have a common line T_{ijk}:*

- *plane orthogonal to $[1_i, 2_j]$ though $3_{k'}$ where $k \neq k' \in \{a,b\}$,*
- *plane orthogonal to $[2_j, 3_k]$ though $1_{i'}$ where $i \neq i' \in \{a,b\}$,*
- *plane orthogonal to $[3_k, 1_i]$ though $2_{j'}$ where $j \neq j' \in \{a,b\}$.*

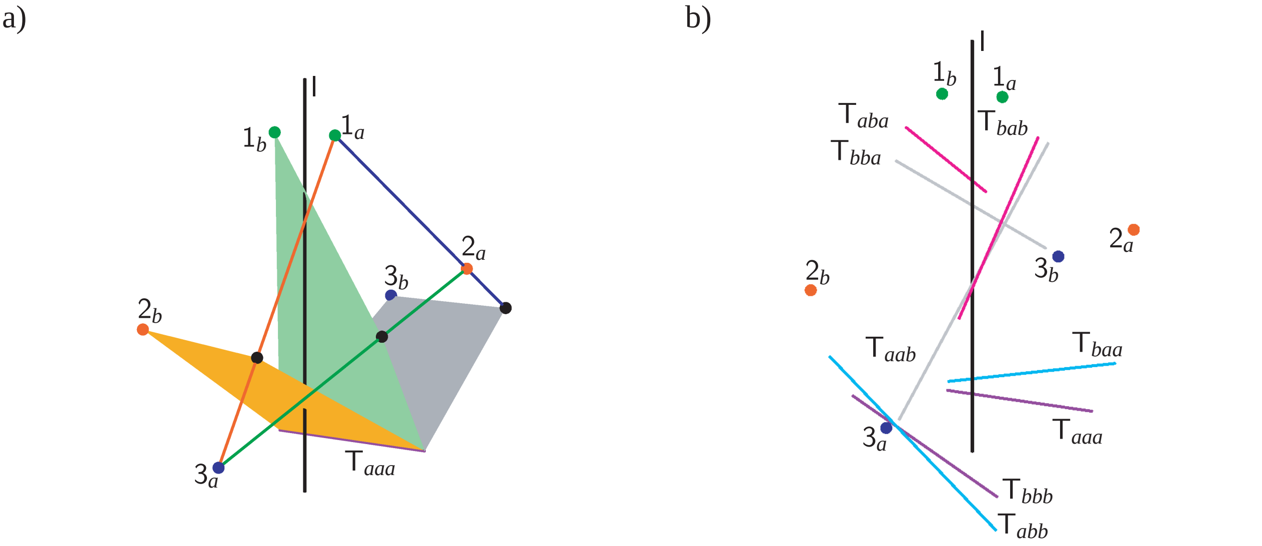

Figure 3: a) Illustration of Theorem 4 on the basis of the octahedron given in Fig. 2b for $i = j = k = a$, where T_{aaa} is printed between ε_{aaa} and ε_{bbb}. b) All eight possible axes T_{ijk} of this octahedron are drawn between ε_{ijk} and $\varepsilon_{i'j'k'}$.

Proof: It was already proven by the author in Corollary 1 of [15] that the continuous flexion of a line-symmetric Bricard octahedron is a type II DM self-motion. Then the theorem follows immediately by item I. □

7. Conclusion

In this article we have proven the necessity of three conditions for obtaining a type II DM self-motion of a general planar SG platform (cf. Theorem 3). Moreover, we also gave a geometric interpretation of these conditions (cf. section 6), which identified a property of line-symmetric Bricard octahedra, which was not known until now, to the best knowledge of the author (cf. Theorem 4).

Finally, it should be noted that Theorem 3 is the key for the determination of all planar SG platforms with a type II DM self-motion (cf. [17]).

Acknowledgement The work of the author is supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

References

[1] Borel, E.: *Mémoire sur les déplacements à trajectoires sphériques*, Mémoire présenteés par divers savants à l'Académie des Sciences de l'Institut National de France **33**(1) 1–128 (1908).

[2] Borras, J., Thomas, F., Torras, C.: *Singularity-invariant leg rearrangements in doubly-planar Stewart-Gough platforms*, In Proc. of Robotics Science and Systems, Zaragoza, Spain (2010).

[3] Bricard, R.: *Mémoire sur la théorie de l'octaèdre articulé*, Journal de Mathématiques pures et appliquées, Liouville **3** 113–148 (1897).

[4] Bricard, R.: *Mémoire sur les déplacements à trajectoires sphériques*, Journal de École Polytechnique **11** 1–96 (1906).

[5] Duporcq, E.: *Sur la correspondance quadratique et rationnelle de deux figures planes et sur un déplacement remarquable*, Comptes Rendus des Séances de l'Académie des Sciences **126** 1405–1406 (1898).

[6] Husty, M.: *E. Borel's and R. Bricard's Papers on Displacements with Spherical Paths and their Relevance to Self-Motions of Parallel Manipulators*, Int. Symp. on History of Machines and Mechanisms (M. Ceccarelli ed.), 163–172, Kluwer (2000).

[7] Husty, M., Mielczarek, S., Hiller, M.: *A redundant spatial Stewart-Gough platform with a maximal forward kinematics solution set*, Advances in Robot Kinematics: Theory and Applications (J. Lenarcic, F. Thomas eds.), 147–154, Kluwer (2002).

[8] Karger, A.: *Architecture singular planar parallel manipulators*, Mechanism and Machine Theory **38**(11) 1149–1164 (2003).

[9] Karger, A.: *Architecturally singular non-planar parallel manipulators*, Mechanism and Machine Theory **43**(3) 335–346 (2008).

[10] Karger, A.: *Parallel manipulators and Borel-Bricard problem*, Computer Aided Geometric Design, Special Issue: Advances in Applied Geometry (B. Jüttler, M. Lavicka, O. Röschel eds.) **27**(8) 669–680 (2010).

[11] Mielczarek, S., Husty, M.L., Hiller, M.: *Designing a redundant Stewart-Gough platform with a maximal forward kinematics solution set*, In Proc. of the International Symposion of Multibody Simulation and Mechatronics (MUSME), Mexico City, Mexico (2002).

[12] Nawratil, G.: *On the degenerated cases of architecturally singular planar parallel manipulators*, Journal for Geometry and Graphics **12**(2) 141–149 (2008).

[13] Nawratil, G.: *A new approach to the classification of architecturally singular parallel manipulators*, Computational Kinematics (A. Kecskemethy, A. Müller eds.), 349–358, Springer (2009).

[14] Nawratil, G.: *Basic result on type II DM self-motions of planar Stewart Gough platforms*, Mechanisms, Transmissions, Applications (E.Chr. Lovasz, B. Corves eds.), 235–244, Springer (2011).

[15] Nawratil, G.: *Types of self-motions of planar Stewart Gough platforms*, Meccanica, DOI: 10.1007/s11012-012-9659-6, SpringerLink (2012).

[16] Nawratil, G.: *Necessary conditions for type II DM self-motions of planar Stewart Gough platforms*, Technical Report No. 219, Geometry Preprint Series, TU Vienna (2011).

[17] Nawratil, G.: *Planar Stewart Gough platforms with a type II DM self-motion*, Journal of Geometry **102**(1) 149–169 (2011).

[18] Nawratil, G.: *Review and recent results on Stewart Gough platforms with self-motions*, Applied Mechanics and Materials **162** 151–160 (2012).

[19] Nawratil, G.: *Correcting Duporcq's theorem*, Technical Report No. 224, Geometry Preprint Series, TU Vienna (2012).

[20] Röschel, O., Mick, S.: *Characterisation of architecturally shaky platforms*, Advances in Robot Kinematics: Analysis and Control (J. Lenarcic, M.L. Husty eds.), 465–474, Kluwer (1998).

[21] Wohlhart, K.: *From higher degrees of shakiness to mobility*, Mechanism and Machine Theory **45**(3) 467–476 (2010).

Received