Necessary conditions for type II DM self-motions of planar Stewart Gough platforms

Georg Nawratil

Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8-10/104, Vienna, A-1040, Austria

Abstract

Due to previous publications of the author, it is already known that one-parametric self-motions of general planar Stewart Gough platforms can be classified into two so-called Darboux Mannheim (DM) types (I and II). Moreover, the author also presented a method for computing the set of equations yielding a type II DM self-motion explicitly. Based on these equations we prove in this article the necessity of three conditions for obtaining a type II DM self-motion. Finally, we give a geometric interpretation of these conditions, which also identifies a property of line-symmetric Bricard octahedra, which was not known until now, to the best knowledge of the author.

Keywords: Self-motion, Stewart Gough platform, Borel Bricard problem, Bricard octahedra

1. Introduction

The geometry of a planar Stewart Gough (SG) platform is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, 0)^T$ with respect to the fixed system Σ_0 and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, 0)^T$ with respect to the moving system Σ. By using Study parameters $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$ for the parametrization of Euclidean displacements, the coordinates $\mathbf{m}_i'$ of the platform anchor points with respect to Σ_0 can be written as $K\mathbf{m}_i' = \mathbf{R}\,\mathbf{m}_i + (t_1, t_2, t_3)^T$ with

$$t_1 = 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \qquad t_2 = 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3),$$
$$t_3 = 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1), \qquad K = e_0^2 + e_1^2 + e_2^2 + e_3^2 \neq 0 \quad \text{and}$$
$$\mathbf{R} = (r_{ij}) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}.$$

Now all points of the real 7-dimensional space $P^7_{\mathbb{R}}$, which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the subspace $e_0 = \ldots = e_3 = 0$ of Ψ, as these points cannot fulfill the normalizing condition $K = 1$.

If the geometry of the manipulator is given as well as the six leg lengths, then the SG platform is in general rigid, but it can even happen that the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion. Note that such motions are also solutions to the famous Borel Bricard problem (cf. [1, 4, 5, 10, 19]). This still unsolved problem was posed 1904 by the French Academy of Science for the Prix Vaillant and reads as follows: "*Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.*"

Email address: `nawratil@geometrie.tuwien.ac.at` (Georg Nawratil)
URL: `http://www.geometrie.tuwien.ac.at/nawratil` (Georg Nawratil)

1.1. Types of self-motions

In this and the next subsection we give a very short review of the results and ideas stated in [16], where more details and also some concrete examples can be found.

It is already known, that manipulators which are singular in every possible configuration, possess self-motions in each pose. These manipulators are so-called architecturally singular SG platforms [11] and they are well studied: For the characterization of architecturally singular planar SG platforms we refer to [8, 13, 18, 20]. For the non-planar case we refer to [9, 14]. Therefore, we are only interested in the computation of self-motions of non-architecturally singular SG platforms. Until now only few self-motions of this type are known, as their computation is a very complicated task. A detailed review of these self-motions was given by the author in [16] (see also [6]).

Moreover, it is known that if a planar SG platform with anchor points $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is not architecturally singular, then at least a one-parametric set $\mathcal{L}$ of legs exists, which can be attached to the given manipulator without restricting the forward kinematics [7, 12]. The underlying linear system of equations is given in Eq. (30) of [12]. As the solvability condition of this system is equivalent to the criterion given in Eq. (12) of [2], also the singularity surface of the manipulator does not change by adding legs of $\mathcal{L}$. Moreover, it was shown that in general the base anchor points M_i as well as the corresponding platform anchor points m_i of $\mathcal{L}$ are located on planar cubic curves C and c, respectively.

Assumption 1. *We assume that there exist such cubics* c *and* C *(which can also be reducible) in the Euclidean domain of the platform and the base, respectively.*

Now, we consider the complex projective extension $P^3_{\mathbb{C}}$ of the Euclidean 3-space E^3, i.e.

$$a_i = \frac{x_i}{w_i}, \quad b_i = \frac{y_i}{w_i}, \quad A_i = \frac{X_i}{W_i}, \quad B_i = \frac{Y_i}{W_i}. \tag{1}$$

Note that ideal points are characterized by $w_i = 0$ and $W_i = 0$, respectively. Therefore we denote in the remainder of this article the coordinates of anchor points, which are ideal points, by x_i, y_i and X_i, Y_i, respectively. For all other anchor points we use the coordinates a_i, b_i and A_i, B_i, respectively.

The correspondence between the points of C and c in $P^3_{\mathbb{C}}$, which is determined by the geometry of the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$, can be computed according to [7, 12] or [2] under consideration of Eq. (1). As this correspondence has not to be a bijection, a point $\in P^3_{\mathbb{C}}$ of c resp. C is in general mapped to a non-empty set of points $\in P^3_{\mathbb{C}}$ of C resp. c. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{\,\}$.

In $P^3_{\mathbb{C}}$ the cubic C has three ideal points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$, where at least one of these points (e.g. U_1) is real. The remaining points U_2 and U_3 are real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{u}_1\}, \{\mathsf{u}_2\}, \{\mathsf{u}_3\}$ of c ($\Rightarrow \{\mathsf{u}_1\}$ contains real points). We denote the ideal points of c by $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$, where again one (e.g. u_4) has to be real. The remaining points u_5 and u_6 are again real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{U}_4\}, \{\mathsf{U}_5\}, \{\mathsf{U}_6\}$ of C ($\Rightarrow \{\mathsf{U}_4\}$ contains real points).

Assumption 2. *For guaranteeing a general case, we assume that each of the corresponding locations* $\{\mathsf{u}_1\}$, $\{\mathsf{u}_2\}$, $\{\mathsf{u}_3\}$, $\{\mathsf{U}_4\}$, $\{\mathsf{U}_5\}$, $\{\mathsf{U}_6\}$ *consists of a single point. Moreover, we assume that no 4 collinear platform anchor points* u_j *or base anchor points* U_j *(*$j = 1, \ldots, 6$*) exist.*

Now the basic idea can simply be expressed by attaching the special legs $\overline{\mathsf{u}_i\mathsf{U}_i} \in \mathcal{L}$ with $i = 1, \ldots, 6$ to the manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$. The attachment of the special leg $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i \in \{1, 2, 3\}$ corresponds with the so-called Darboux constraint, that the platform anchor point u_i moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i. Moreover, the attachment of the special leg $\overline{\mathsf{u}_i\mathsf{U}_i}$ for $i \in \{4, 5, 6\}$ corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to u_i slides through the point U_i.

By removing the originally six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ with $i = 1, \ldots, 6$ we remain with the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$, which is uniquely determined due to Assumption 1 and 2. Moreover, under consideration of Assumption 1 and 2, the following statement holds (cf. [16]):

Theorem 1. *The manipulator* $\mathsf{u}_1, \ldots, \mathsf{U}_6$ *is redundant and therefore architecturally singular. Moreover, all anchor points of the platform* $\mathsf{u}_1, \ldots, \mathsf{u}_6$ *and as well of the base* $\mathsf{U}_1, \ldots, \mathsf{U}_6$ *are distinct.*

It was also proven in [16] that there only exist type I and type II Darboux Mannheim (DM) self-motions, where the definition of types reads as follows:

Definition 1. *Assume $\mathcal{M}$ is a one-parametric self-motion of a non-architecturally singular SG platform $\mathsf{m}_1, \ldots, \mathsf{M}_6$. Then $\mathcal{M}$ is of the type n DM if the corresponding architecturally singular manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ has an n-parametric self-motion $\mathcal{U}$ (which includes $\mathcal{M}$).*

1.2. Type II DM self-motions

In the remainder of the article we focus on type II DM self-motions. The author [16] was already able to compute the set of equations yielding a type II DM self-motion explicitly (cf. subsection 2.5). This was only possible by the usage of the analytical versions of the Darboux and Mannheim constraints, which are repeated next:

Darboux constraint:. The constraint that the platform anchor point u_i ($i = 1, 2, 3$) moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i can be written as (cf. [16])

$$\Omega_i: \ \overline{X}_i(a_i r_{11} + b_i r_{12} + t_1) + \overline{Y}_i(a_i r_{21} + b_i r_{22} + t_2) + L_i K = 0,$$

with $X_i, Y_i, a_i, b_i, L_i \in \mathbb{C}$. This is a homogeneous quadratic equation in the Study parameters where $\overline{X}_i$ and $\overline{Y}_i$ denote the conjugate complex of X_i and Y_i, respectively.

Mannheim constraint:. The constraint that the plane orthogonal to u_i ($i = 4, 5, 6$) through the platform point $(g_i, h_i, 0)$ slides through the point U_i of the fixed system can be written as (cf. [16])

$$\begin{aligned}\Pi_i: \ &\overline{x}_i[A_i r_{11} + B_i r_{21} - g_i K - 2(e_0 f_1 - e_1 f_0 - e_2 f_3 + e_3 f_2)] + \\ &\overline{y}_i[A_i r_{12} + B_i r_{22} - h_i K - 2(e_0 f_2 + e_1 f_3 - e_2 f_0 - e_3 f_1)] = 0,\end{aligned}$$

with $x_i, y_i, A_i, B_i, g_i, h_i \in \mathbb{C}$. This is again a homogeneous quadratic equation in the Study parameters where $\overline{x}_i$ and $\overline{y}_i$ denote the conjugate complex of x_i and y_i.

The content of the following lemma was also proven in [16]:

Lemma 1. *Without loss of generality (w.l.o.g.) we can assume that the algebraic variety of the two-parametric self-motion of the manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ is spanned by $\Psi, \Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5$. Moreover, we can choose following special coordinate systems in Σ_0 and Σ w.l.o.g.: $X_1 = Y_2 = Y_3 = x_4 = y_5 = 1$ and $a_1 = b_1 = y_4 = A_4 = B_4 = Y_1 = h_4 = g_5 = 0$.*

An important step in direction of a complete classification of type II DM self-motions was done by the following basic result, which was proven in [15]:

Theorem 2. *If neither the three equations*

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0, \quad a_2(\overline{X}_2 - \overline{X}_3) + \overline{X}_3(\overline{X}_2 b_2 - \overline{X}_3 b_3) + b_2 - b_3 = 0, \quad a_3(\overline{X}_2 - \overline{X}_3) + \overline{X}_2(\overline{X}_2 b_2 - \overline{X}_3 b_3) + b_2 - b_3 = 0, \tag{2}$$

nor the three equations

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0, \quad a_2(\overline{X}_2 - \overline{X}_3) - \overline{X}_3(\overline{X}_2 b_2 - \overline{X}_3 b_3) - b_2 + b_3 = 0, \quad a_3(\overline{X}_2 - \overline{X}_3) - \overline{X}_2(\overline{X}_2 b_2 - \overline{X}_3 b_3) - b_2 + b_3 = 0, \tag{3}$$

are fulfilled, then the corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion, has to have further three collinear anchor points in the base or in the platform beside the points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ and $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$.

Based on this theorem we prove the following much stronger result within this article:

Theorem 3. *The corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion has to fulfill the three conditions either of Eq. (2) or Eq. (3).*

2. Preparatory work for the proof of Theorem 3

For the proof of Theorem 3 we have to show that there exists no corresponding manipulator $\mathsf{u}_1, \ldots, \mathsf{U}_6$ of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion, which does not fulfill either the three conditions of Eq. (2) or Eq. (3).

Due to Theorem 2 and due to Lemma 2 of [8] we can even restrict ourselves to manipulators $\mathsf{u}_1, \ldots, \mathsf{U}_6$, which have three collinear platform points $\mathsf{u}_i, \mathsf{u}_j, \mathsf{u}_k$ and three collinear base points $\mathsf{U}_l, \mathsf{U}_m, \mathsf{U}_n$ beside the points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ and $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$ where (i, j, k, l, m, n) consists of all indices from 1 to 6.

As we have different types of anchor points (real, complex, finite, infinite), we have to distinguish the following four cases of three collinear points (beside the triples $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ and $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$):

A. $\mathsf{U}_1, \mathsf{U}_4, \mathsf{U}_5$ collinear ($\Leftrightarrow \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_6$ collinear): As u_5 and u_6 are both real or conjugate complex, this case is equivalent to $\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_5$ collinear ($\Leftrightarrow \mathsf{U}_1, \mathsf{U}_4, \mathsf{U}_6$ collinear).

   Moreover, by exchanging the platform and the base the above two cases are also equivalent to $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_4$ collinear ($\Leftrightarrow \mathsf{U}_3, \mathsf{U}_5, \mathsf{U}_6$ collinear) and $\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_4$ collinear ($\Leftrightarrow \mathsf{U}_2, \mathsf{U}_5, \mathsf{U}_6$ collinear), respectively.

B. $\mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_5$ collinear ($\Leftrightarrow \mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_6$ collinear): As u_5 and u_6 are both real or conjugate complex, this case is equivalent to $\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_5$ collinear ($\Leftrightarrow \mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_6$ collinear).

   Moreover, as U_2 and U_3 are both real or conjugate complex, these cases are also equivalent to $\mathsf{U}_3, \mathsf{U}_4, \mathsf{U}_5$ collinear ($\Leftrightarrow \mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_6$ collinear) and $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_5$ collinear ($\Leftrightarrow \mathsf{U}_3, \mathsf{U}_4, \mathsf{U}_6$ collinear), respectively.

C. $\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4$ collinear ($\Leftrightarrow \mathsf{U}_1, \mathsf{U}_5, \mathsf{U}_6$ collinear)

D. $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3$ collinear ($\Leftrightarrow \mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_6$ collinear)

In the following we discuss these four types A–D in more detail:

2.1. *Collinearity of type A*

$\mathsf{U}_1, \mathsf{U}_4, \mathsf{U}_5$ are collinear for $B_5 = 0$. As due to Assumption 2 no four platform anchor points u_i or base anchor points U_i are allowed to be collinear, we can stop the discussion of type A if:

- $\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4$ collinear ($\Leftrightarrow b_2 - b_3 = 0$),
- $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3$ collinear ($\Leftrightarrow a_2b_3 - a_3b_2 = 0$),
- $\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_5$ collinear ($\Leftrightarrow x_5(b_2 - b_3) - a_2 + a_3 = 0$),

because then the points $\mathsf{U}_1, \mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_6$ are collinear due to Lemma 2 of [8], which yields a contradiction. Due to Theorem 1 also $A_5(X_2 - X_3) \neq 0$ has to hold, as otherwise the base anchor points are not pairwise distinct. Finally, we can assume $X_2 \neq 0$ w.l.o.g., because both points U_2 and U_3 do not belong to the triple of collinear points.

2.2. *Collinearity of type B*

$\mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_5$ are collinear for $A_5 = X_2B_5$. Now we can stop the discussion of case B if:

- $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3$ collinear ($\Leftrightarrow a_2b_3 - a_3b_2 = 0$),
- $\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_4$ collinear ($\Leftrightarrow b_3 = 0$),
- $\mathsf{u}_1, \mathsf{u}_3, \mathsf{u}_5$ collinear ($\Leftrightarrow a_3 - x_5b_3 = 0$),

because then the points $\mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_5, \mathsf{U}_6$ are collinear, a contradiction. Due to Theorem 1 also $B_5(X_2 - X_3) \neq 0$ has to hold, as otherwise the base anchor points are not pairwise distinct. Moreover, we can stop the discussion of case B, if U_2 is real ($\Leftrightarrow X_2 \in \mathbb{R}$, especially $X_2 = 0$) because then this case is equivalent to case A.

2.3. Collinearity of type C

$\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4$ are collinear for $b_2 = b_3$. We can stop the discussion of case C if $\mathsf{U}_1, \mathsf{U}_4, \mathsf{U}_5$ are collinear ($\Leftrightarrow B_5 = 0$), because then the points $\mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4, \mathsf{u}_6$ are collinear, a contradiction. Moreover $b_2 \neq 0$ has to hold because otherwise $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4$ are collinear, a contradiction. Due to Theorem 1 also $(a_2 - a_3)(X_2 - X_3) \neq 0$ has to hold, as $\mathsf{u}_2 = \mathsf{u}_3$ resp. $\mathsf{U}_2 = \mathsf{U}_3$ yield a contradiction. In addition, we can assume $X_2 \neq 0$ w.l.o.g., because the corresponding points of U_2 and U_3 belong to the triple of collinear points.

We can also assume that $\mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_5$ are not collinear ($\Leftrightarrow A_5 - X_2 B_5 \neq 0$), because this case was already discussed in case B.

2.4. Collinearity of type D

$\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3$ are collinear for $a_2 b_3 - a_3 b_2 = 0$. Now we can stop the discussion of case D if:

- $\mathsf{U}_1, \mathsf{U}_4, \mathsf{U}_5$ collinear ($\Leftrightarrow B_5 = 0$),
- $\mathsf{U}_2, \mathsf{U}_4, \mathsf{U}_5$ collinear ($\Leftrightarrow A_5 - X_2 B_5 = 0$),
- $\mathsf{U}_3, \mathsf{U}_4, \mathsf{U}_5$ collinear ($\Leftrightarrow A_5 - X_3 B_5 = 0$),

because then the points $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_6$ are collinear, a contradiction. Moreover, we can assume $b_2 b_3 \neq 0$ because otherwise $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_4$ are collinear ($\Rightarrow a_2 = a_3 b_2 / b_3$). Clearly, also the points $\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \mathsf{u}_5$ are not allowed to be collinear which implies $a_3 - x_5 b_3 \neq 0$. Moreover we can assume $b_2 \neq b_3$ because otherwise we get $\mathsf{u}_2 = \mathsf{u}_3$, a contradiction. Due to Theorem 1 also $(X_2 - X_3) \neq 0$ has to hold, as $\mathsf{U}_2 = \mathsf{U}_3$ yields a contradiction. In addition, we can assume $X_2 \neq 0$ w.l.o.g., because the corresponding points of U_2 and U_3 belong to the triple of collinear points.

2.5. Preparatory computations

In the following we describe how the set $\mathcal{E}$ of equations yielding a type II DM self-motion can be computed explicitly. Note that the proof for the general case of Theorem 3 (cf. section 3) is based on this set $\mathcal{E}$.

We solve the linear system of equations $\Psi, \Omega_1, \Omega_2, \Pi_4$ for $f_0, \ldots, f_3$ and plug the obtained expressions in the remaining two equations.[1] This yields in general two homogeneous polynomials $\Omega[40]$ and $\Pi[96]$ in the Euler parameters of degree 2 and 4, respectively. The number in the square brackets gives the number of terms.

Finally, we compute the resultant of Ω and Π with respect to one of the Euler parameters. Here we choose[2] e_0. This yields a homogeneous polynomial $\Gamma[117\,652]$ of degree 8 in e_1, e_2, e_3. In the following we denote the coefficients of e_1^i, e_2^j, e_3^k of Γ by Γ_{ijk}. We get a set $\mathcal{E}$ of 24 equations $\Gamma_{ijk} = 0$ in the 14 unknowns $(a_2, b_2, a_3, b_3, A_5, B_5, X_2, X_3, x_5, L_1, L_2, L_3, g_4, h_5)$.

Moreover, we denote the coefficients of $e_0^i e_1^j, e_2^k, e_3^l$ of Ω and Π by Ω_{ijkl} and Π_{ijkl}, respectively.

Finally, it should be said that all symbolic computations were done with Maple 14 on a high-capacity computer.[3]

3. Proving the general case of Theorem 3

For the general case we have to assume $\Omega_{2000}\Pi_{3000} \neq 0$, as only those solutions of $\mathcal{E}$ correspond to type II self-motions, which do not cause a vanishing of the coefficient of the highest power of e_0 in Ω and Π, respectively. In the following we prove this general case for the four different types A–D of collinearity.

For each type the proof is done by contradiction, i.e. we stop the discussion for the cases listed in the respective subsections (subsection 2.1–2.4) or if the three conditions of Eq. (2) or Eq. (3) are fulfilled.

3.1. For the collinearity of type A

Γ_{800} can only vanish without contradiction (w.c.) for $L_1 = g_4$ or for $F_A[8] = 0$.

[1] For $e_0 e_2 - e_1 e_3 \neq 0$ this can be done w.l.o.g., as this factor belongs to the denominator of f_i.
[2] Therefore we are looking for a common factor of Ω and Π, which depends on e_0.
[3] CPU: Intel(R) Core(TM)2 Quad CPU Q6600 @ 2.40 GHz, RAM: 8 GB, Hard disk: 2x250 GB, Graphic: nVidia 7x00GT or 8x00GT, Operating system: Linux x64 (Kernel 2.6.18-53)

3.1.1. $F_A = 0$

We can express L_1 from $F_A = 0$. Now we distinguish two cases:

1. $L_1 \neq g_4$: Then $\Gamma_{710} = 0$ implies $a_2 = a_3 - \overline{X}_2 b_2 + \overline{X}_3 b_3$. Now Γ_{620} cannot vanish w.c..
2. $L_1 = g_4$: We can compute h_5 from the only non-contradicting (non-c.) factor of Γ_{602}. Now Γ_{530} can only vanish w.c. for:
   a. $L_3 = \overline{X}_3(L_2 - b_2)/\overline{X}_2 + \overline{X}_3(a_2 - a_3) + b_3$: We can express A_5 from the only non-c. factor of Γ_{422}. Again we distinguish two cases:
      i. $\overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2 - a_3 \neq 0$: Now Γ_{350} has only one non-c. factor, which can be solved for L_2. Then $\Gamma_{314} = 0$ implies $b_3 = 0$. Now we get $x_5 = -X_3$ from $\Gamma_{206} = 0$. Then Γ_{080} can only vanish w.c. for:
         - ⋆ $X_3 = 0$: Now $\Gamma_{026} = 0$ yields the contradiction.
         - ⋆ $b_2 = \overline{X}_2 a_2 - \overline{X}_3 a_3$, $X_3 \neq 0$: Γ_{026} cannot vanish w.c..
      ii. $a_3 = \overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2$: Then $\Gamma_{260} = 0$ implies $L_2 = 2\overline{X}_2^2 b_2 + \overline{X}_2 a_2 + b_2$. Moreover, we can solve the only non-c. factor of Γ_{242} for $\overline{x}_5$.
         - ⋆ Assuming $\overline{X}_2 b_3 - \overline{X}_3 b_2 \neq 0$: Under this assumption we can compute a_2 from the only non-c. factor of Γ_{080}. Now $\Gamma_{224} = 0$ yields the contradiction.
         - ⋆ $b_3 = \overline{X}_3 b_2/\overline{X}_2$: Then Γ_{080} can only vanish w.c. for $X_3 = 0$ or $X_2 = -X_3$. In both cases $\Gamma_{026} = 0$ yields the contradiction.
   b. $a_2 = \overline{X}_3 b_3 - \overline{X}_2 b_2 + a_3$, $\overline{X}_2\overline{X}_3(a_2 - a_3) + \overline{X}_2(b_3 - L_3) - \overline{X}_3(b_2 - L_2) \neq 0$: Now $\Gamma_{440} = 0$ yields the contradiction.

3.1.2. $F_A \neq 0$

Now $L_1 = g_4$ has to hold. Then Γ_{080} factors into $G_A[8]H_A[16]^2$.

1. $G_A[8] = 0$: We can express L_1 from $G_A[8] = 0$. Now Γ_{170} can only vanish w.c. for:
   a. $a_2 = \overline{X}_3 b_3 - \overline{X}_2 b_2 + a_3$: We can solve the only non-c. factor of Γ_{620} for h_5. Now we can express L_3 from the only non-c. factor of Γ_{602}.
      i. $x_5 \neq 0$: Under this assumption we can compute A_5 from the only non-c. factor of Γ_{260}. Then we can express L_2 from the only non-c. factor of Γ_{062}. Now the resultant of the only non-c. factors of Γ_{404} and Γ_{440} with respect to $\overline{X}_3$ can only vanish w.c. for:
         - ⋆ $b_3 = 0$: Now Γ_{404} implies $x_5 = X_3$. Finally, $\Gamma_{026} = 0$ yields the contradiction.
         - ⋆ $x_5 = X_3$, $b_3 \neq 0$: Now $\Gamma_{440} = 0$ implies $a_3 = \overline{X}_2 b_3$ and $\Gamma_{404} = 0$ yields the contradiction.
         - ⋆ $a_3 = -\overline{X}_2 b_3$, $b_3(x_5 - X_3) \neq 0$: Now $\Gamma_{404} = 0$ implies $b_2 = -b_3$ and $\Gamma_{440} = 0$ yields the contradiction.
      ii. $x_5 = 0$: We distinguish two cases:
         - ⋆ $\overline{X}_2 b_3 - \overline{X}_3 b_2 \neq 0$: Under this assumption we can express a_3 from the only non-c. factor of Γ_{260}. Then we can compute A_5 from the only non-c. factor of Γ_{440}. Now Γ_{404} cannot vanish w.c..
         - ⋆ $b_3 = \overline{X}_3 b_2/\overline{X}_2$: Now Γ_{260} can only vanish w.c. for $X_3 = 0$. Then $\Gamma_{440} = 0$ implies $A_5 = -a_3$. Now we can solve the only non-c. factor of Γ_{422} for L_2. Finally, $\Gamma_{026} = 0$ yields the contradiction.
   b. $V_A[16] = 0$, $\overline{X}_3 b_3 - \overline{X}_2 b_2 - a_2 + a_3 \neq 0$:
      i. $x_5 \neq 0$: Under this assumption we can compute A_5 from $V_A[16] = 0$. Then can solve the only non-c. factor of Γ_{620} for h_5. Now we can express L_3 from the only non-c. factor of Γ_{602}. Moreover, we can solve the only non-c. factor of Γ_{062} for L_2. Now the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Then $\Gamma_{440} = 0$ implies $x_5 = X_3$ and $\Gamma_{422} = 0$ yields the contradiction.
      ii. $x_5 = 0$: Now we can solve $V_A = 0$ for L_3. Then we can compute b_3 from the only non-c. factor of Γ_{620}. Now Γ_{602} implies $h_5 = 0$. Then the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $X_3 = 0$. Now $\Gamma_{440} = 0$ implies $a_3 = -A_5$. From the only non-c. factor of $\Gamma_{422} = 0$ we express L_2. Then $\Gamma_{026} = 0$ yields the contradiction.

2. $H_A[16] = 0$, $G_A[8] \neq 0$: We distinguish two cases:

   a. $\overline{X}_2 a_2 - \overline{X}_3 a_3 \neq 0$: Under this assumption we can compute h_5 from $H_A[16] = 0$.

      i. $x_5 \neq 0$: Under this assumption we can compute A_5 from the only non-c. factor of Γ_{620}. Moreover, we can compute L_3 from the only non-c. factor of Γ_{602}. Now the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Then $\Gamma_{440} = 0$ implies $x_5 = X_3$. Now $\Gamma_{422} = 0$ implies $L_1 = 2a_3$. Finally, $\Gamma_{242} = 0$ yields the contradiction.

      ii. $x_5 = 0$: We can solve the only non-c. factor of Γ_{620} for b_3. Then we express L_3 from the only non-c. factor of Γ_{602}. Then the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $X_3 = 0$. Now $\Gamma_{440} = 0$ implies $a_3 = -A_5$ and from $\Gamma_{422} = 0$ we get $L_1 = -2A_5$. Then $\Gamma_{026} = 0$ yields the contradiction.

   b. $a_2 = \overline{X}_3 a_3 / \overline{X}_2$: Now H_A can only vanish w.c. for $A_5 \overline{x}_5 + \overline{X}_3 a_3 = 0$.

      i. $x_5 \neq 0$: Under this assumption we can solve the last equation for A_5. Now we can express h_5 from the only non-c. factor of Γ_{620}. Then we can compute L_3 from the only non-c. factor of Γ_{602}. Now the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Then $\Gamma_{440} = 0$ implies $x_5 = X_3$. Now $\Gamma_{422} = 0$ implies $L_1 = 2a_3$. Finally, $\Gamma_{242} = 0$ yields the contradiction.

      ii. $x_5 = 0$: Now $H_A = 0$ implies $X_3 = 0$. Then we can express h_5 from the only non-c. factor of Γ_{620}. Moreover, we can compute L_3 from the only non-c. factor of Γ_{602}. Then the difference of the only non-c. factors of Γ_{440} and Γ_{404} can only vanish w.c. for $b_3 = 0$. Now $\Gamma_{440} = 0$ implies $a_3 = -A_5$ and from $\Gamma_{422} = 0$ we get $L_1 = -2A_5$. Then $\Gamma_{026} = 0$ yields the contradiction.

3.2. For the collinearity of type B

Γ_{800} can only vanish w.c. for $b_2 = b_3$, $L_1 = g_4$ or $F_B[8] = 0$.

3.2.1. $b_2 = b_3$

From the only non-c. factor of Γ_{620} we can compute B_5. Then we can express g_4 from the only non-c. factor of Γ_{602}. Now we compute h_5 from the difference of the only non-c. factors of Γ_{440} and Γ_{404}. Then we can express $\overline{x}_5$ from the only non-c. factor of Γ_{440}. Now we can compute L_3 from the only non-c. factor of Γ_{422}.

1. $X_3 \neq 0$: Then we can solve the only non-c. factor of Γ_{260} for b_2. Finally, $\Gamma_{206} = 0$ yields the contradiction.

2. $X_3 = 0$: Now the only non-c. factor of Γ_{260} can be solved for a_3. Then we can compute L_2 from the only non-c. factor of Γ_{224}. Finally, $\Gamma_{242} = 0$ yields the contradiction.

3.2.2. $F_B = 0$, $b_2 \neq b_3$

We can express L_1 from $F_B = 0$. As for $L_1 \neq g_4$, $\Gamma_{710} = 0$ yields $a_3 = a_2 - \overline{X}_3 b_3 + \overline{X}_2 b_2$ and $\Gamma_{620} = 0$ the contradiction, we can assume $L_1 = g_4$: Now we can compute h_5 from the only non-c. factor of Γ_{602}. Then Γ_{530} can only vanish w.c. for:

1. $L_3 = \overline{X}_3(L_2 - b_2)/\overline{X}_2 + \overline{X}_3(a_2 - a_3) + b_3$: We can express $\overline{x}_5$ from the only non-c. factor of Γ_{422}.

   a. $\overline{X}_2 b_2 - \overline{X}_3 b_3 + a_2 - a_3 \neq 0$: Now Γ_{350} has only one non-c. factor, which can be solved for L_2. Then we can express a_2 from the only non-c. factor of $\Gamma_{314} = 0$.

      i. $X_2(B_5^2 - b_2 b_3) + \overline{X}_2 B_5(b_3 - B_5) - a_3(b_2 - B_5) \neq 0$: Under this assumption we can compute $\overline{X}_3$ from the only non-c. factor of Γ_{206}. Then $\Gamma_{314} = 0$ yields the contradiction.

      ii. $X_2(B_5^2 - b_2 b_3) + \overline{X}_2 B_5(b_3 - B_5) - a_3(b_2 - B_5) = 0$: As for $B_5 = b_2$ this equation yields a contradiction, we can assume $B_5 \neq b_2$. Now we can solve this equation for a_3. Then $\Gamma_{206} = 0$ implies $b_2 = 0$ and $\Gamma_{134} = 0$ yields the contradiction.

   b. $a_2 = \overline{X}_3 b_3 - \overline{X}_2 b_2 + a_3$: Then we can solve the only non-c. factor of Γ_{260} for L_2.

      i. $X_2^2(b_3 - b_2) + a_3(\overline{X}_3 - \overline{X}_2) - \overline{X}_2(\overline{X}_3 b_3 - \overline{X}_2 b_2) \neq 0$: Under this assumption we can express B_5 from the only non-c. factor of Γ_{242}. Now Γ_{116} can only vanish w.c. for:

- ⋆ $a_3 = -b_3X_2$: As for $b_3 = -X_2b_2/\overline{X}_2$ the condition Γ_{080} cannot vanish w.c. we can assume $\overline{X}_2b_3 + X_2b_2 \neq 0$. Under this assumption we can solve the only non-c. factor of Γ_{080} for $\overline{X}_3$.

  (α) $X_2 - \overline{X}_2 - 2\overline{X}_2^3 \neq 0$: Under this assumption we can compute b_2 from the only non-c. factor of Γ_{062}. Then it is not difficult to see, that Γ_{026} and Γ_{044} cannot vanish w.c..

  (β) $X_2 - \overline{X}_2 - 2\overline{X}_2^3 = 0$: In this case we set $\overline{X}_2 = m + in$ with $m, n \in \mathbb{R}$. Then the resultant of the equation of item (β) and the only non-c. factor of Γ_{062} with respect to n cannot vanish w.c..
- ⋆ $b_2 = (\overline{X}_3b_3 + a_3)/(\overline{X}_2 - X_2)$, $a_3 + b_3X_2 \neq 0$: Then $\Gamma_{080} = 0$ implies $b_3 = -\overline{X}_3a_3/(\overline{X}_2X_2)$. Now we can solve the only non-c. factor of Γ_{062} for $\overline{X}_3$. Finally, $\Gamma_{044} = 0$ yields the contradiction.
- ⋆ $a_3 = \overline{X}_2(\overline{X}_3b_3 - \overline{X}_2b_2)/(\overline{X}_3 - \overline{X}_2)$, $(b_2(\overline{X}_2 - X_2) - \overline{X}_3b_3 - a_3)(a_3 + b_3X_2) \neq 0$: Now $\Gamma_{080} = 0$ implies $X_3 = 0$. Then Γ_{062} cannot vanish w.c..

ii. $a_3 = [X_2^2(b_3 - b_2) - \overline{X}_2(\overline{X}_3b_3 - \overline{X}_2b_2)]/(\overline{X}_2 - \overline{X}_3)$: Then Γ_{242} can only vanish w.c. for:

- ⋆ $\overline{X}_3 = -X_2$: Now $\Gamma_{206} = 0$ implies $b_2 = B_5$ and $\Gamma_{080} = 0$ yields the contradiction.
- ⋆ $\overline{X}_2 = in$ with $n \in \mathbb{R}$, $\overline{X}_3 + X_2 \neq 0$: Now $\Gamma_{206} = 0$ implies $b_3 = B_5$ and from $\Gamma_{080} = 0$ we get $\overline{X}_3 = -in$. Finally, $\Gamma_{062} = 0$ yields the contradiction.

2. $a_3 = \overline{X}_2b_2 - \overline{X}_3b_3 + a_2$, $\overline{X}_3\overline{X}_2(a_3 - a_2) + \overline{X}_3(b_2 - L_2) - \overline{X}_2(b_3 - L_3) \neq 0$: In this case $\Gamma_{440} = 0$ yields the contradiction.

3.2.3. $F_B \neq 0$, $b_2 \neq b_3$

Now $L_1 = g_4$ has to hold. Then Γ_{080} factors into $G_B[8]H_B[18]^2$.

1. $G_B[8] = 0$: We can express L_1 from $G_B[8] = 0$. Now Γ_{170} can only vanish w.c. for:

a. $a_3 = \overline{X}_2b_2 - \overline{X}_3b_3 + a_2$: We can solve the only non-c. factor of Γ_{620} for h_5. Now we can express L_3 from the only non-c. factor of Γ_{602}. Then we can compute $\overline{x}_5$ from the only non-c. factor of Γ_{260}. Moreover, we express L_2 from the only non-c. factor of Γ_{062}. We compute a_2 from the sum of the only non-c. factors of Γ_{440} and Γ_{404}.

i. $(b_2 + B_5)(b_3 - B_5) \neq 0$: Under this assumption we can solve the only non-c. factor of Γ_{026} for $\overline{X}_3$. Then $\Gamma_{404} = 0$ yields the contradiction.

ii. $B_5 = -b_2$ or $B_5 = b_3$: In both cases $\Gamma_{026} = 0$ cannot vanish w.c..

b. $V_B[18] = 0$, $\overline{X}_2b_2 - \overline{X}_3b_3 - a_3 + a_2 \neq 0$: We can compute h_5 from $V_B[18] = 0$. Then we can express L_3 from the only non-c. factor of Γ_{062}. Now we can also solve the only non-c. factor of Γ_{620} for $\overline{x}_5$. Moreover, we can compute L_2 from the only non-c. factor of Γ_{602}. We compute a_2 from the sum of the only non-c. factors of Γ_{440} and Γ_{404}.

i. $X_2B_5(b_3 - b_2) - \overline{X}_3b_3(B_5 - b_2) + \overline{X}_2b_2(B_5 + b_3) \neq 0$: Under this assumption we can express a_3 from the only non-c. factor of Γ_{026}. Then $\Gamma_{350} = 0$ yields the contradiction.

ii. $X_2B_5(b_3 - b_2) - \overline{X}_3b_3(B_5 - b_2) + \overline{X}_2b_2(B_5 + b_3) = 0$: As for $B_5 = -b_2$ this equation cannot vanish w.c., we can assume $B_5 + b_2 \neq 0$. Now we can compute $\overline{X}_3$ from this equation. Then $\Gamma_{026} = 0$ yields the contradiction.

2. $H_B[18] = 0$, $G_B[8] \neq 0$:

a. $\overline{X}_2a_2 - \overline{X}_3a_3 \neq 0$: Under this assumption we can compute h_5 from $H_B[18] = 0$. Then we can express $\overline{x}_5$ from the only non-c. factor of Γ_{620}. Now we can compute L_2 from the only non-c. factor of Γ_{602}. Moreover, we can solve the sum of the only non-c. factors of Γ_{440} and Γ_{404} for $\overline{X}_3$.

i. $X_2(B_5^2 - b_2b_3) - \overline{X}_2B_5(B_5 + b_3) + a_3(B_5 + b_2) \neq 0$: Under this assumption we can express a_2 from the only non-c. factor of Γ_{404}.

- ⋆ $a_3 \neq 0$: Under this assumption we can compute B_5 from the only non-c. factor of Γ_{206}. Then we can compute L_3 from the only non-c. factor of Γ_{422}. We can solve the only non-c. factor of Γ_{224} for b_2. Now $\Gamma_{242} = 0$ implies $a_3 = -X_2b_3$. Then it can easily be seen, that Γ_{062} and Γ_{026} cannot vanish w.c..
- ⋆ $a_3 = 0$: Now $\Gamma_{206} = 0$ yields the contradiction.

ii. $X_2(B_5^2 - b_2b_3) - \overline{X}_2B_5(B_5 + b_3) + a_3(B_5 + b_2) = 0$: As for $B_5 = -b_2$ this equation cannot vanish w.c., we can assume $B_5 + b_2 \neq 0$. Now we can solve this equation for a_3. Then $\Gamma_{404} = 0$ implies $b_2 = 0$. Finally, $\Gamma_{206} = 0$ yields the contradiction.

b. $a_2 = \overline{X}_3 a_3/\overline{X}_2$: Now H_B can only vanish w.c. for $\overline{x}_5 = -\overline{X}_3 a_3/(X_2 B_5)$. Then we can solve the only non-c. factor of Γ_{620} for h_5. Now we can compute L_2 from the only non-c. factor of Γ_{602}. Moreover, we can express a_3 from the sum of the only non-c. factors of Γ_{440} and Γ_{404}. Then we compute the difference D of the only non-c. factors of Γ_{260} and Γ_{062}. As for $b_2 = -B_5$ the expression D cannot vanish w.c., we can assume $b_2 + B_5 \neq 0$. Now we can solve the only non-c. factor of D for $\overline{X}_3$. Then $\Gamma_{260} = 0$ yields the contradiction.

3.3. For the collinearity of type C

We can solve the only non-c. factor of Γ_{620} for B_5. Then we can express L_2 from the only non-c. factor of Γ_{602}. Moreover, we can compute L_3 from the sum of the only non-c. factors of Γ_{440} and Γ_{404}.

1. $(\overline{X}_2 - \overline{X}_3)(A_5 - \overline{x}_5 b_3) - (\overline{X}_2 + \overline{X}_3)(a_2 - a_3) \neq 0$: Under this assumption we can express g_4 from the only non-c. factor of Γ_{404}. Then $\Gamma_{422} = 0$ cannot vanish w.c..

2. $A_5 = (\overline{X}_2 + \overline{X}_3)(a_2 - a_3)/(\overline{X}_2 - \overline{X}_3) + \overline{x}_5 b_3$: Then Γ_{404} can only vanish w.c. for:

   a. $a_2 = b_3(\overline{X}_2 - \overline{X}_3) + a_3$: Then we can compute h_5 from the only non-c. factor of Γ_{422}.

      i. $x_5 \neq 0$: Under this assumption we can solve the only non-c. factor of Γ_{206} for a_3. Then $\Gamma_{260} = 0$ implies $X_3 = 0$ and $\Gamma_{224} = 0$ yields the contradiction.

      ii. $x_5 = 0$: Now $\Gamma_{206} = 0$ implies $X_3 = 0$ and $\Gamma_{224} = 0$ yields the contradiction.

   b. $a_2 = b_3(\overline{X}_3 - \overline{X}_2) + a_3$, $b_3(\overline{X}_2 - \overline{X}_3) - a_2 + a_3 \neq 0$: Then we can compute h_5 from the only non-c. factor of Γ_{422}.

      i. $x_5 \neq 0$: Under this assumption we can solve the only non-c. factor of Γ_{206} for a_3. Then $\Gamma_{260} = 0$ implies $X_3 = 0$ and $\Gamma_{242} = 0$ yields the contradiction.

      ii. $x_5 = 0$: Now $\Gamma_{206} = 0$ implies $X_3 = 0$ and $\Gamma_{242} = 0$ yields the contradiction.

   c. $P_C[12] = 0$, $(b_3(\overline{X}_2 - \overline{X}_3) - a_2 + a_3)(b_3(\overline{X}_3 - \overline{X}_2) - a_2 + a_3) \neq 0$: We can solve $P_C[12] = 0$ for h_5.

      i. $X_3 \neq x_5$: Under this assumption we can express a_3 from the only non-c. factor of Γ_{260}. Then $\Gamma_{206} = 0$ implies $X_3 = 0$. Then we can compute L_1 from the only non-c. factor of Γ_{062}. Now $\Gamma_{242} = 0$ implies $X_2 = x_5$ and $\Gamma_{224} = 0$ yields the contradiction.

      ii. $X_3 = x_5$: Now Γ_{260} can only vanish w.c. for:

         ⋆ $x_5 = 0$: We compute L_1 from the only non-c. factor of Γ_{062}. Then $\Gamma_{242} = 0$ implies $a_3 = \overline{X}_2 b_3$ and $\Gamma_{224} = 0$ yields the contradiction.

         ⋆ $a_2 = \overline{x}_5 b_3$: Now Γ_{206} cannot vanish w.c..

3.4. For the collinearity of type D

Γ_{800} can only vanish w.c. for $L_1 = g_4$ or if $F_D[8] = 0$ is fulfilled identically.

3.4.1. $F_D[8] = 0$

We can express L_1 from $F_D[8] = 0$.

1. $L_1 \neq g_4$: Now Γ_{710} has only one non-c. factor, which can be solved for a_3. Then Γ_{620} cannot vanish w.c..

2. $L_1 = g_4$: We can express h_5 from the only non-c. factor of Γ_{602}.

   a. $a_3(b_2 - b_3) + b_3(\overline{X}_2 b_2 - \overline{X}_3 b_3) \neq 0$: Now Γ_{530} has only one non-c. factor, which can be solved for L_3. Then we can compute A_5 from the only non-c. factor of Γ_{422}. Now we can express L_2 from the only non-c. factor of Γ_{314}. Then we can solve the only non-c. factor of Γ_{350} for $\overline{x}_5$. Finally, $\Gamma_{206} = 0$ yields the contradiction.

   b. $a_3 = b_3(\overline{X}_2 b_2 - \overline{X}_3 b_3)/(b_3 - b_2)$: Then we can compute L_3 from the only non-c. factor of Γ_{440}. Now we can express A_5 from the only non-c. factor of Γ_{260}. Then we can solve the only non-c. factor of Γ_{422} for L_2.

      i. $\overline{x}_5(b_2 - b_3) + \overline{X}_2 b_2 - \overline{X}_3 b_3 \neq 0$: Under this assumption Γ_{206} has only one non-c. factor, which can be solved for $\overline{x}_5$. Then $\Gamma_{026} = 0$ yields the contradiction.

      ii. $\overline{x}_5 = (\overline{X}_3 b_3 - \overline{X}_2 b_2)/(b_2 - b_3)$: Now $\Gamma_{224} = 0$ yields the contradiction.

3.4.2. $F_D[8] \neq 0$

Now $L_1 = g_4$ has to hold. Then Γ_{080} factors into $G_D[8]H_D[18]^2$.

1. $G_D[8] = 0$: We can express L_1 from $G_D[8] = 0$. Now Γ_{170} can only vanish w.c. for:
   a. $a_3 = b_3(\overline{X}_3 b_3 - \overline{X}_2 b_2)/(b_2 - b_3)$: We can solve the only non-c. factor of Γ_{620} for h_5. Now we can express A_5 from the only non-c. factor of Γ_{440}.
      i. $\overline{x}_5(b_2 - b_3) + \overline{X}_2 b_2 - \overline{X}_3 b_3 \neq 0$: Under this assumption Γ_{260} has only one non-c. factor, which can be solved for $\overline{x}_5$. Then we can compute L_3 from the only non-c. factor of Γ_{602}. Finally, $\Gamma_{404} = 0$ yields the contradiction.
      ii. $\overline{x}_5 = (\overline{X}_3 b_3 - \overline{X}_2 b_2)/(b_2 - b_3)$: Then we can compute L_3 from the only non-c. factor of Γ_{206}. Now we can express L_2 from the only non-c. factor of Γ_{062}. Finally, $\Gamma_{422} = 0$ yields the contradiction.
   b. $V_D[18] = 0$, $a_3(b_2 - b_3) + b_3(\overline{X}_2 b_2 - \overline{X}_3 b_3) \neq 0$: We can express h_5 from $V_D[18] = 0$.
      i. $x_5 \neq 0$: Under this assumption we can compute A_5 from the only non-c. factor of Γ_{620}. Moreover, we can solve the only non-c. factor of Γ_{602} for L_3. Then we can express L_2 from the only non-c. factor of Γ_{062}.
         ⋆ $a_3 - \overline{x}_5 b_3 \neq 0$: Under this assumption Γ_{026} has only one non-c. factor, which can be solved for $\overline{x}_5$. Then $\Gamma_{404} = 0$ yields the contradiction.
         ⋆ $a_3 = \overline{x}_5 b_3$: Again $\Gamma_{404} = 0$ yields the contradiction.
      ii. $x_5 = 0$: As for $\overline{X}_3 = \overline{X}_2 b_2/b_3$ the expression Γ_{620} cannot vanish w.c., we can assume $\overline{X}_2 b_2 - \overline{X}_3 b_3 \neq 0$. Now we can express B_5 from the only non-c. factor of Γ_{620}. Then we can compute L_3 from the only non-c. factor of Γ_{062}. Now we can express L_2 from the only non-c. factor of Γ_{602}. Moreover, we can solve the only non-c. factor of Γ_{440} for A_5. Then we can compute a_3 from the only non-c. factor of Γ_{404}. Finally, $\Gamma_{206} = 0$ yields the contradiction.

2. $H_D[18] = 0$, $G_D[8] \neq 0$: We distinguish the following three cases.
   a. $a_3(\overline{X}_2 b_2 - \overline{X}_3 b_3) \neq 0$: Under this assumption we can compute h_5 from $H_D[18] = 0$. Then we can compute B_5 from the only non-c. factor of Γ_{620}. Now we can express L_2 from the only non-c. factor of Γ_{602}. As for $a_3 = \overline{x}_5 b_3$ the expression Γ_{404} cannot vanish w.c. we can assume $a_3 - \overline{x}_5 b_3 \neq 0$. Under this assumption Γ_{440} has only one non-c. factor, which can be solved for a_3. Then we can express A_5 from the only non-c. factor of Γ_{404}. Finally, $\Gamma_{206} = 0$ yields the contradiction.
   b. $a_3 = 0$: Now H_D can only vanish w.c. for:
      i. $x_5 = 0$: We can compute h_5 from the only non-c. factor of Γ_{620}. Then we can express L_2 from the only non-c. factor of Γ_{602} and A_5 from the only non-c. factor of Γ_{260}. Finally, $\Gamma_{206} = 0$ yields the contradiction.
      ii. $A_5 = 0$: We can compute h_5 from the only non-c. factor of Γ_{620}. Then we can express L_2 from the only non-c. factor of Γ_{602}. Now we can solve the only non-c. factor of Γ_{260} for $\overline{x}_5$. Then $\Gamma_{206} = 0$ implies $\overline{X}_3 = \overline{X}_2 b_2/b_3$ and $\Gamma_{062} = 0$ yields the contradiction.
   c. $\overline{X}_3 = \overline{X}_2 b_2/b_3$, $a_3 \neq 0$: Now $H_D = 0$ implies $a_3 = -A_5 b_3 \overline{x}_5/(\overline{X}_2 b_2)$. We can compute h_5 from the only non-c. factor of Γ_{620}. Then we can express L_2 from the only non-c. factor of Γ_{602}. Now Γ_{260} can only vanish for $\overline{X}_2 = -B_5 \overline{x}_5/b_2$ or $A_5 = -\overline{x}_5 b_2$. In both cases $\Gamma_{404} = 0$ yields the contradiction.

4. Proving the special case $\Omega_{2000}\Pi_{3000} = 0$ of Theorem 3

We do not discuss the special case for all four types A–D separately (as done for the general case) in order to shorten the proof. We distinguish between the different types only if this is necessary during the study of cases. Therefore this discussion is a generalization of the one given in the corresponding technical report of [15].

For the discussion of the special cases we can assume $X_2(X_2 - X_3) \neq 0$, because this also holds true for each of the four types A–D (cf. subsection 2.1–2.4).

If we set e_i equal to zero for any $i \in \{0, \dots, 3\}$, then Ω and Π have to be fulfilled identically. It can immediately be seen, that the conditions implied by $\Omega = 0$ already yield a contradiction. Therefore we can assume $e_0 e_1 e_2 e_3 \neq 0$ for this section of the proof.

4.1. $\Omega_{2000} = 0, \Omega_{1000}\Pi_{3000} \neq 0$

From $\Omega_{2000} = 0$ we can express L_1. Moreover, we can compute e_0 from $\Omega = 0$ and plug the resulting expression into Π, which yields in the numerator a homogeneous polynomial $\Gamma[10058]$ of degree 7 in e_1, e_2, e_3. Now Γ_{700} can only vanish w.c. for:

1. $b_2 = b_3$: Then Γ_{430} can only vanish for:
   a. $a_2 = a_3 + b_3(\overline{X}_3 - \overline{X}_2)$: Now the only non-c. factor of Γ_{322} can be solved for g_4. Finally, $\Gamma_{070} = 0$ yields the contradiction.
   b. $g_4 = (L_2 - L_3 + \overline{X}_2 a_2 - \overline{X}_3 a_3)/(\overline{X}_2 - \overline{X}_3)$, $a_2 - a_3 - b_3(\overline{X}_3 - \overline{X}_2) \neq 0$: Now $\Gamma_{340} = 0$ yields the contradiction.
2. $g_4 = (L_2 - L_3 + b_2 - b_3 + \overline{X}_2 a_2 - \overline{X}_3 a_3)/(\overline{X}_2 - \overline{X}_3)$, $b_2 \neq b_3$: Then Γ_{610} cannot vanish w.c..

4.2. $\Omega_{2000} = \Pi_{3000} = 0, \Omega_{1000}\Pi_{2000} \neq 0$

Again we express L_1 from $\Omega_{2000} = 0$. It can immediately be seen from $\Omega = 0$, that all coefficients of $\Pi_{3000} = 0$ with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can compute g_4 and h_5 from $\Pi_{3100} = 0$ and $\Pi_{3010} = 0$, respectively. We solve $\Omega = 0$ for e_0 and plug it into Π which yields in the numerator a homogeneous polynomial $\Gamma[1666]$ of degree 5 in e_1, e_2, e_3. Now Γ_{500} can only vanish w.c. for:

1. $b_2 = b_3$: Then the only non-c. factor of Γ_{302} can be solved for L_3. Now Γ_{320} can only vanish w.c. for:
   a. $a_2 = a_3 + b_3(\overline{X}_3 - \overline{X}_2)$: Then the only non-c. factor of Γ_{050} can be solved for L_2. We can express A_5 from the only non-c. factor of Γ_{140}. Now $\Gamma_{230} = 0$ implies $B_5 = -b_3$.
      i. $x_5 \neq 0$: Under this assumption we can compute a_3 from the only non-c. factor of Γ_{212}. Then $\Gamma_{104} = 0$ implies $X_3 = 0$ and from $\Gamma_{014} = 0$ we get $X_2 = -x_5$. Finally, $\Gamma_{122} = 0$ yields the contradiction.
      ii. $x_5 = 0$: Now $\Gamma_{212} = 0$ implies $X_3 = 0$ and from $\Gamma_{014} = 0$ we get $a_3 = -\overline{X}_2 b_3$. Again, $\Gamma_{122} = 0$ yields the contradiction.
   b. $a_2 = a_3 + B_5(\overline{X}_2 - \overline{X}_3)$, $a_2 - a_3 - b_3(\overline{X}_3 - \overline{X}_2) \neq 0$: We distinguish two cases:
      i. $a_3 + \overline{X}_2 B_5 \neq 0$: Under this assumption the only non-c. factor of Γ_{050} can be solved for L_2.
         - ⋆ $a_3 - \overline{X}_2 b_3 \neq 0$: Under this assumption the only non-c. factor of Γ_{140} can be solved for $\overline{X}_3$.
           (α) $x_5 \neq 0$: Under this assumption we can express a_3 from the only non-c. factor of Γ_{212}. Then $\Gamma_{104} = 0$ implies $A_5 = \overline{x}_5 b_3 + \overline{X}_2 B_5$ and from $\Gamma_{014} = 0$ we get $X_2 = -x_5$. Finally, $\Gamma_{122} = 0$ yields the contradiction.
           (β) $x_5 = 0$: Now $\Gamma_{212} = 0$ implies $A_5 = \overline{X}_2 B_5$ and from $\Gamma_{014} = 0$ we get $a_3 = -\overline{X}_2 b_3$. Again, $\Gamma_{122} = 0$ yields the contradiction.
         - ⋆ $a_3 = \overline{X}_2 b_3$: Then Γ_{140} can only vanish w.c. for:
           (α) $X_2 = x_5$: Now $\Gamma_{212} = 0$ implies $A_5 = \overline{x}_5 b_3 + B_5(\overline{X}_3 + \overline{x}_5)$ and from $\Gamma_{104} = 0$ we get $X_3 = 0$. Finally, $\Gamma_{014} = 0$ yields the contradiction.
           (β) $A_5 = \overline{X}_2 B_5$, $X_2 \neq x_5$: Now $\Gamma_{212} = 0$ implies $\overline{X}_3 = -\overline{x}_5 b_3/B_5$ and from $\Gamma_{104} = 0$ we get $x_5 = 0$. Again, $\Gamma_{014} = 0$ yields the contradiction.
      ii. $a_3 = -\overline{X}_2 B_5$: Then $\Gamma_{140} = 0$ implies $\overline{X}_3 = \overline{X}_5 A_5/(\overline{X}_2 B_5)$ and from $\Gamma_{032} = 0$ we get $L_2 = \overline{x}_5 A_5 - B_5$. Then Γ_{104} can only vanish w.c. for:
         - ⋆ $x_5 = 0$: Now $\Gamma_{212} = 0$ implies $A_5 = \overline{X}_2 b_3$. Finally, $\Gamma_{230} = 0$ yields the contradiction.
         - ⋆ $A_5 = 0$, $x_5 \neq 0$: Now $\Gamma_{212} = 0$ implies $X_2 = -x_5$. Again, $\Gamma_{230} = 0$ yields the contradiction.
2. $L_3 = \overline{X}_3(L_2 + b_2)/\overline{X}_2 + \overline{X}_3(a_2 - a_3) - b_3$, $b_2 \neq b_3$: Now the only non-c. factor of Γ_{410} can be solved for L_2. Moreover, we can express A_5 from the only non-c. factor of Γ_{320}.
   a. $B_5 \neq 0$: Under this assumption we can compute $\overline{x}_5$ from the only non-c. factor of Γ_{302}. Then the difference of the only non-c. factors of Γ_{104} and Γ_{230} can only vanish w.c. for $\overline{X}_2 a_2 b_3 - \overline{X}_3 a_3 b_2 = 0$.
      i. $b_3 \neq 0$: Under this assumption we can express a_2 from this equation.

- ⋆ $(B_5a_3 - \overline{X}_2b_3^2)(\overline{X}_2B_5b_3 - a_3b_2) \neq 0$: Under this assumption we can compute $\overline{X}_3$ from the only non-c. factor of Γ_{230}. Then $\Gamma_{014} = 0$ yields the contradiction.
- ⋆ $\overline{X}_2 = B_5a_3/b_3^2$: Now Γ_{104} can only vanish w.c. for:
  (α) $B_5 = b_3$: Again $\Gamma_{014} = 0$ yields the contradiction.
  (β) $B_5 = -b_3$: Now $\Gamma_{212} = 0$ yields the contradiction.
- ⋆ $B_5 = a_3b_2/(\overline{X}_2b_3)$: Then $\Gamma_{104} = 0$ implies $a_3 = -\overline{X}_2b_3$ and $\Gamma_{212} = 0$ yields the contradiction.

ii. $b_3 = 0$: Now the last equation implies $X_3 = 0$ and from $\Gamma_{104} = 0$ we get $a_2 = a_3 - \overline{X}_2B_5$. Finally, $\Gamma_{014} = 0$ yields the contradiction.

b. $B_5 = 0$: Now $\Gamma_{302} = 0$ implies $b_3 = 0$ and from $\Gamma_{104} = 0$ we get $x_5 = -X_3$. Then $\Gamma_{230} = 0$ yields $X_3 = 0$ and Γ_{014} cannot vanish w.c..

4.3. $\Omega_{2000} = \Pi_{3000} = \Pi_{2000} = 0$, $\Omega_{1000}\Pi_{1000} \neq 0$

Again we express L_1 from $\Omega_{2000} = 0$. It can immediately be seen from $\Omega = 0$, that all coefficients of $\Pi_{i000} = 0$ (for $i = 2, 3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we compute g_4 and h_5 from $\Pi_{3100} = 0$ and $\Pi_{3010} = 0$, respectively. Moreover, we can solve $\Pi_{2101} = 0$ and $\Pi_{2011} = 0$ for L_3 and L_2, respectively. We solve $\Omega = 0$ for e_0 and plug it into Π which yields in the numerator a homogeneous polynomial $\Gamma[191]$ of degree 5 in e_1, e_2, e_3. Now Γ_{410} can only vanish w.c. for:

1. $b_2 = b_3$: Then $\Gamma_{302} = 0$ implies $B_5 = b_3$ and from $\Gamma_{320} = 0$ we get $a_2 = a_3 + b_3(\overline{X}_3 - \overline{X}_2)$. Then $\Gamma_{212} = 0$ yields $a_3 = b_3(2\overline{x}_5 - \overline{X}_2)$ and from $\Gamma_{140} = 0$ we get $A_5 = b_3(\overline{X}_2 - \overline{X}_3 - 3\overline{x}_5)$. Now Γ_{122} can only vanish w.c. for:

   a. $X_3 = -x_5$: Then the difference of the only non-c. factors of Γ_{050} and Γ_{014} cannot vanish w.c..

   b. $X_2 = -x_5$: Now $\Gamma_{050} = 0$ implies $X_3 = -4x_5$ and $\Gamma_{014} = 0$ yields the contradiction.

2. $b_2 = -B_5$, $b_2 \neq b_3$: Then $\Gamma_{320} = 0$ implies $a_2 = B_5(\overline{X}_2 - \overline{x}_5) - A_5$ and from $\Gamma_{230} = 0$ we get $A_5 = \overline{X}_2B_5$. Now $\Gamma_{302} = 0$ implies $a_3 = b_3(\overline{X}_3 - \overline{X}_2 + \overline{x}_5)$ and from $\Gamma_{104} = 0$ we get $x_5 = -X_3$. Finally, $\Gamma_{212} = 0$ yields the contradiction.

4.4. $\Pi_{3000} = 0$, $\Omega_{2000}\Pi_{2000} \neq 0$

It can immediately be seen from $\Omega = 0$ that all coefficients of $\Pi_{3000} = 0$ with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express g_4 and h_5 from $\Pi_{3100} = 0$ and $\Pi_{3010} = 0$, respectively. Then we compute the resultant of $\Omega[40]$ and $\Pi[44]$ with respect to e_0 which yields a homogeneous polynomial $\Gamma[15153]$ of degree 8 in e_1, e_2, e_3. Γ_{080} can only vanish w.c. in the following two cases:

1. $L_2 = L_1(\overline{X}_2 - \overline{X}_3) + L_3 + \overline{X}_2a_2 - \overline{X}_3a_3 - b_2 + b_3$: Then we can solve the only non-c. factor of Γ_{602} for B_5. Now Γ_{170} can only vanish w.c. for:

   a. $a_2 = -\overline{x}_5A_5/\overline{X}_2$: Then $\Gamma_{062} = 0$ implies $L_1 = 0$ and Γ_{620} can only vanish w.c. for $(b_2-b_3+\overline{X}_3a_3+\overline{x}_5A_5)F[7] = 0$:

      i. $b_2 = b_3 - \overline{X}_3a_3 - \overline{x}_5A_5$: Now $\Gamma_{530} = 0$ implies $L_3 = b_3 - \overline{X}_3a_3$ and Γ_{026} can only vanish w.c. for:
      - ⋆ $X_2 = -x_5$: Then $\Gamma_{422} = 0$ yields the contradiction.
      - ⋆ $b_3 = \overline{X}_3a_3 + \overline{x}_5A_5 - A_5\overline{X}_2$, $X_2 \neq -x_5$: Now $\Gamma_{242} = 0$ implies $\overline{x}_5 = 1/\overline{X}_2$ and from $\Gamma_{422} = 0$ we get $\overline{X}_3 = 1/\overline{X}_2$. Then $\Gamma_{206} = 0$ yields the contradiction.

      ii. $F[7] = 0$, $b_2 - b_3 + \overline{X}_3a_3 + \overline{x}_5A_5 \neq 0$: As for $b_2 = b_3$ the expression F cannot vanish w.c., we can assume $b_2 \neq b_3$. Under this assumption we can express L_3 from $F[7] = 0$. Then Γ_{026} can only vanish w.c. for:
      - ⋆ $X_2 = -x_5$: Moreover, $\Gamma_{440} = 0$ implies $A_5 = \overline{x}_5b_2$ and from $\Gamma_{422} = 0$ we get $a_3 = -\overline{X}_3b_3$. Then $\Gamma_{242} = 0$ implies $X_3 = x_5$ and $\Gamma_{206} = 0$ yields the contradiction.
      - ⋆ $A_5 = \overline{X}_2b_2$, $X_2 \neq -x_5$: Now $\Gamma_{422} = 0$ implies $a_3 = b_3(\overline{x}_5 + \overline{X}_2 - \overline{X}_3)$. Finally, $\Gamma_{314} = 0$ yields the contradiction.

   b. $a_2 = a_3 - \overline{X}_2b_2 + \overline{X}_3b_3$, $\overline{X}_2a_2 + \overline{x}_5A_5 \neq 0$: Then Γ_{260} cannot vanish w.c..

2. $a_2 = -\overline{x}_5 A_5/\overline{X}_2$, $L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 + \overline{X}_2 a_2 - \overline{X}_3 a_3 - b_2 + b_3 \neq 0$: Then we can solve the only non-c. factor of Γ_{260} for b_2. Now Γ_{440} can only vanish w.c. for:

   a. $X_2 = x_5$: As for $B_5 = 0$, the condition $\Gamma_{602} = 0$ yields $L_2 = \overline{x}_5(L_1 + A_5)$ and $\Gamma_{206} = 0$ the contradiction we can assume $B_5 \neq 0$. Under this assumption we can solve $\Gamma_{602} = 0$ for L_3. Then Γ_{404} can only vanish w.c. for:

      i. $a_3 = \overline{x}_5 B_5 + \overline{X}_3 b_3 - A_5$: We distinguish further two cases:

      ⋆ $\overline{X}_3 B_5 + \overline{x}_5 b_3 \neq 0$: Under this assumption we can express A_5 from the only non-c. factor of Γ_{206}. Then we can compute L_1 from the only non-c. factor of Γ_{062}. Now $\Gamma_{026} = 0$ implies $X_3 = -x_5$ and finally $\Gamma_{422} = 0$ yields the contradiction.

      ⋆ $b_3 = -\overline{X}_3 B_5/\overline{x}_5$: Now Γ_{206} can only vanish w.c. for:
      (α) $X_3 = 0$: Then $\Gamma_{422} = 0$ implies $L_1 = 2\overline{x}_5 B_5 - 2A_5$ and finally $\Gamma_{062} = 0$ yields the contradiction.
      (β) $X_3 = -x_5$, $X_3 \neq 0$: We can express L_2 from the only non-c. factor of Γ_{422}. Then $\Gamma_{062} = 0$ implies $B_5 = A_5(2\overline{x}_5^2 + 1)/\overline{x}_5$ and $\Gamma_{026} = 0$ yields the contradiction.

      ii. $L_2 = \overline{x}_5(L_1 + A_5) - B_5$, $\overline{x}_5 B_5 + \overline{X}_3 b_3 - A_5 - a_3 \neq 0$: Then we can compute a_3 from the only non-c. factor of Γ_{422}. Now $\Gamma_{314} = 0$ implies $L_1 = -2A_5 - 2\overline{x}_5 B_5$.

      ⋆ $\overline{X}_3 B_5 + \overline{x}_5 b_3 \neq 0$: Under this assumption we can express A_5 from the only non-c. factor of Γ_{242}. Now $\Gamma_{206} = 0$ implies $X_3 = 0$ and finally $\Gamma_{062} = 0$ yields the contradiction.

      ⋆ $b_3 = -\overline{X}_3 B_5/\overline{x}_5$: Now $\Gamma_{242} = 0$ implies $X_3 = 0$. Finally, $\Gamma_{224} = 0$ yields the contradiction.

   b. $A_5 = \overline{X}_2 B_5$, $X_2 \neq x_5$: From $\Gamma_{602} = 0$ we can express L_3. Then Γ_{062} can only vanish w.c. for:

      i. $L_1 = 0$: Then $\Gamma_{422} = 0$ implies $a_3 = b_3(\overline{x}_5 + \overline{X}_2 - \overline{X}_3)$ and from $\Gamma_{242} = 0$ we get $X_3 = x_5$. Now $\Gamma_{314} = 0$ implies $X_2 = -x_5$ and from $\Gamma_{206} = 0$ we get $L_2 = B_5(\overline{x}_5^2 - 1)$. Finally, $\Gamma_{404} = 0$ yields the contradiction.

      ii. $\overline{x}_5 = -\overline{X}_3 a_3/(\overline{X}_2 B_5)$, $L_1 \neq 0$: Now the only non-c. factor of Γ_{044} can be solved for B_5. Finally, $\Gamma_{026} = 0$ yields the contradiction.

4.5. $\Pi_{3000} = \Pi_{2000} = 0$, $\Omega_{2000}\Pi_{1000} \neq 0$

It can immediately be seen from $\Omega = 0$ that all coefficients of $\Pi_{i000} = 0$ (for $i = 2, 3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express g_4 and h_5 from $\Pi_{3100} = 0$ and $\Pi_{3010} = 0$, respectively. Moreover, we can compute L_2 and L_1 from $\Pi_{2101} = 0$ and $\Pi_{2011} = 0$, respectively. Then we solve $\Pi = 0$ for e_0 and plug it into Ω which yields in the numerator a homogeneous polynomial $\Gamma[2408]$ of degree 8 in e_1, e_2, e_3. $\Gamma_{602} = 0$ implies $B_5 = b_2$. Then Γ_{620} can only vanish for:

1. $b_2 = 0$: Now Γ_{404} can only vanish w.c. for:

   a. $A_5 = -a_2$: Now $\Gamma_{206} = 0$ implies $X_2 = -x_5$ and from $\Gamma_{260} = 0$ we get $L_3 = b_3 + \overline{X}_3(2a_2 - a_3)$. Then $\Gamma_{080} = 0$ implies $b_3 = \overline{X}_3 a_3 + \overline{x}_5 a_2$ and from $\Gamma_{062} = 0$ we get $a_2 = a_3(1 - \overline{X}_3\overline{x}_5)/\overline{x}_5^2$. Finally, $\Gamma_{152} = 0$ yields the contradiction.

   b. $L_3 = b_3 + \overline{X}_3(a_2 - a_3 - A_5)$, $A_5 \neq -a_2$: Now $\Gamma_{350} = 0$ implies $a_2 = a_3 + \overline{X}_3 b_3$ and $\Gamma_{260} = 0$ yields the contradiction.

2. $G[8] = 0$, $b_2 \neq 0$: We can solve $G[8] = 0$ for L_3. Then $\Gamma_{530} = 0$ implies $a_2 = \overline{X}_3 b_3 - \overline{X}_2 b_2 + a_3$. Finally Γ_{440} cannot vanish w.c..

4.6. $\Omega_{2000} = \Omega_{1000} = 0$

We can express L_1 and a_2 from $\Omega_{2000} = 0$ and $\Omega_{1001} = 0$, respectively. As Ω_{0002} cannot vanish w.c. we proceed as follows:

1. $\Pi_{0003} \neq 0$: Now we compute the resultant of Ω and Π with respect to e_3 which yields a homogeneous polynomial $\Gamma[87839]$ of degree 8 in e_0, e_1, e_2. In the following we denote the coefficients of e_1^i, e_2^j, e_0^k of Γ by Γ_{ijk}. Now Γ_{080} equals $(b_2 - b_3)[\overline{X}_2(\overline{X}_2 b_2 - \overline{X}_3 b_3) + a_3(\overline{X}_2 - \overline{X}_3)]H[10]$.

   a. $b_2 = b_3$: Then we can express L_2 from the only non-c. factor of Γ_{206}. Now we can compute L_3 from the only non-c. factor of Γ_{710}. Then $\Gamma_{062} = 0$ implies $B_5 = -b_3$ and $\Gamma_{314} = 0$ yields the contradiction.

b. $a_3 = -\overline{X}_2(\overline{X}_2 b_2 - \overline{X}_3 b_3)/(\overline{X}_2 - \overline{X}_3)$, $b_2 \neq b_3$: Then we can express L_2 from the only non-c. factor of Γ_{206}. Now Γ_{170} can only vanish w.c. for:

i. $g_4 = 0$: We distinguish two cases:

- ⋆ $x_5 \neq 0$: Under this assumption we can express A_5 from the only non-c. factor of Γ_{800}. Then $\Gamma_{404} = 0$ implies $h_5 = B_5 - b_3 - L_3$. Now we can express L_3 from the only non-c. factor of Γ_{062}. Then Γ_{026} can only vanish w.c. for:
  (α) $X_3 = 0$: Now $\Gamma_{602} = 0$ implies $b_2 = -\overline{x}_5 B_5/\overline{X}_2$. Finally, $\Gamma_{224} = 0$ yields the contradiction.
  (β) $b_2 = \overline{X}_3 b_3/\overline{X}_2$, $X_3 \neq 0$: Now $\Gamma_{602} = 0$ implies $b_3 = -\overline{x}_5 B_5/\overline{X}_3$. Finally, $\Gamma_{620} = 0$ yields the contradiction.
- ⋆ $x_5 = 0$: Now Γ_{800} can only vanish w.c. for:
  (α) $X_3 = 0$: Now $\Gamma_{602} = 0$ implies $b_2 = A_5/\overline{X}_2$ and from $\Gamma_{062} = 0$ we get $L_3 = h_5 - B_5 - b_3$. Finally, $\Gamma_{134} = 0$ yields the contradiction.
  (β) $b_2 = \overline{X}_3 b_3/\overline{X}_2$, $X_3 \neq 0$: Now $\Gamma_{602} = 0$ implies $b_3 = A_5/\overline{X}_3$ and $\Gamma_{224} = 0$ yields the contradiction.

ii. $b_2 = \overline{X}_3 b_3/\overline{X}_2$, $g_4 \neq 0$: Now Γ_{800} can only vanish w.c. for $A_5 = 0$ or $x_5 = 0$. In both cases $\Gamma_{260} = 0$ yields the contradiction.

c. $H[10] = 0$, $(b_2 - b_3)[\overline{X}_2(\overline{X}_2 b_2 - \overline{X}_3 b_3) + a_3(\overline{X}_2 - \overline{X}_3)] \neq 0$: We can solve $H[10] = 0$ for g_4. Then we can express L_2 from the only non-c. factor of Γ_{206}. Moreover, we can compute h_5 from the only non-c. factor of Γ_{404}. Then $\Gamma_{602} = 0$ implies $a_3 = b_3\overline{X}_3 + \overline{x}_5 B_5 - A_5$. Now we can solve the only non-c. factor of Γ_{026} for L_3. Then $\Gamma_{044} = 0$ implies $A_5 = \overline{x}_5 B_5$. Now the difference of the only non-c. factors of Γ_{260} and Γ_{062} can only vanish w.c. for:

i. $b_i = 0$ for $i = 2$ or $i = 3$: In both cases $\Gamma_{260} = 0$ yields the contradiction.

ii. $X_2 = -X_3$, $b_2 b_3 \neq 0$: Now Γ_{260} can only vanish w.c. for:

- ⋆ $X_3 = x_5$: Now $\Gamma_{620} = 0$ implies $B_5 = b_3$ and $\Gamma_{242} = 0$ yields the contradiction.
- ⋆ $X_3 = -x_5$: Now $\Gamma_{620} = 0$ cannot vanish w.c..

2. $\Pi_{0003} = 0$, $\Pi_{0002} \neq 0$: It can immediately be seen from $\Omega = 0$ that all coefficients of $\Pi_{0003} = 0$ with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express h_5 and L_2 from $\Pi_{0103} = 0$ and $\Pi_{0013} = 0$, respectively. Now we compute the resultant of Ω and Π with respect to e_3 which yields $(\overline{X}_2 - \overline{X}_3)^2\Gamma[7821]$, where Γ is a homogeneous polynomial of degree 8 in e_0, e_1, e_2. In the following we denote the coefficients of e_1^i, e_2^j, e_0^k of Γ by Γ_{ijk}. Now Γ_{800} can only vanish w.c. for:

a. $b_2 = b_3$: Then we can express a_3 from the only non-c. factor of Γ_{710}. Now $\Gamma_{602} = 0$ implies $g_4 = 0$ and from $\Gamma_{260} = 0$ we get $B_5 = b_3$. Finally, $\Gamma_{062} = 0$ yields the contradiction.

b. $a_3 = \overline{x}_5 A_5/\overline{X}_2 - \overline{X}_2 b_2 + \overline{X}_3 b_3$, $b_2 \neq b_3$: Now we can express b_2 from the only non-c. factor of Γ_{620}. Then $\Gamma_{206} = 0$ implies $g_4 = 0$ and Γ_{602} can only vanish w.c. for:

i. $x_5 = 0$: Now the only non-c. factor of Γ_{026} can be solved for L_3. Then Γ_{062} can only vanish w.c. for:

- ⋆ $X_3 = 0$: Now $\Gamma_{404} = 0$ implies $A_5 = \overline{X}_2 B_5$ and $\Gamma_{224} = 0$ yields the contradiction.
- ⋆ $A_5 = \overline{X}_3 b_3$, $X_3 \neq 0$: Again, $\Gamma_{224} = 0$ yields the contradiction.

ii. $A_5 = \overline{X}_2 B_5$, $x_5 \neq 0$: We can express L_3 from the only non-c. factor of Γ_{026}. Now $\Gamma_{224} = 0$ implies $X_2 = x_5$ and from $\Gamma_{062} = 0$ we get $X_3 = -x_5$. Finally, $\Gamma_{242} = 0$ yields the contradiction.

3. $\Pi_{0003} = \Pi_{0002} = 0$, $\Pi_{0001} \neq 0$: It can immediately be seen from $\Omega = 0$ that all coefficients of $\Pi_{000i} = 0$ (for $i = 2, 3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express h_5 and L_2 from $\Pi_{0103} = 0$ and $\Pi_{0013} = 0$, respectively. Moreover, we can compute g_4 and L_3 from $\Pi_{1102} = 0$ and $\Pi_{1012} = 0$, respectively. Now we compute the resultant of Ω and Π with respect to e_3 which yields $(\overline{X}_2 - \overline{X}_3)^2\Gamma[1766]$, where Γ is a homogeneous polynomial of degree 8 in e_0, e_1, e_2. In the following we denote the coefficients of e_1^i, e_2^j, e_0^k of Γ by Γ_{ijk}. Now Γ_{800} can only vanish w.c. for:

a. $b_2 = b_3$: Then we can express a_3 from the only non-c. factor of Γ_{710}. Now $\Gamma_{260} = 0$ implies $B_5 = b_3$. Finally, $\Gamma_{062} = 0$ yields the contradiction.

b. $a_3 = \overline{x}_5A_5/\overline{X}_2 - \overline{X}_2b_2 + \overline{X}_3b_3$, $b_2 \neq b_3$: Now we can express b_2 from the only non-c. factor of Γ_{620}. Then $\Gamma_{440} = 0$ implies $A_5 = \overline{X}_2B_5$ and $\Gamma_{062} = 0$ yields the contradiction.

4. $\Pi_{0003} = \Pi_{0002} = \Pi_{0001} = 0$, $\Pi_{0300} \neq 0$: It can immediately be seen from $\Omega = 0$ that all coefficients of $\Pi_{000i} = 0$ (for $i = 1, 2, 3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express h_5 and L_2 from $\Pi_{0103} = 0$ and $\Pi_{0013} = 0$, respectively. Moreover, we can compute g_4 and L_3 from $\Pi_{1102} = 0$ and $\Pi_{1012} = 0$, respectively. $\Pi_{0121} = 0$ implies $B_5 = b_2$. From $\Pi_{2011} = 0$ we get $A_5 = a_3 + \overline{x}_5b_2 - \overline{X}_3b_3$. Now Π_{0211} can only vanish w.c. for:

   a. $X_2 = x_5$: We distinguish two cases:

      i. $B_5 \neq b_3$: Under this assumption $\Omega_{0200} \neq 0$ holds and due to our assumption $\Pi_{0300} \neq 0$ we can compute the resultant of Ω and Π with respect to e_1, which yields $(\overline{X}_3 - \overline{x}_5)^2e_0^2\Gamma[1013]$, where Γ is a homogeneous polynomial of degree 6 in e_0, e_2, e_3. Now the coefficient of e_3^6 of Γ cannot vanish w.c..

      ii. $B_5 = b_3$: In this case Ω_{0100} cannot vanish w.c. and therefore we can again compute the resultant of Ω and Π with respect to e_1, which yields $(\overline{X}_3 - \overline{x}_5)^4e_0\Gamma[50]$, where Γ is a homogeneous polynomial of degree 6 in e_0, e_2, e_3. Again the coefficient of e_3^6 of Γ cannot vanish w.c..

   b. $B_5 = 0$, $X_2 \neq x_5$: Then $\Pi_{0301} = 0$ yields the contradiction.

5. $\Pi_{0003} = \Pi_{0002} = \Pi_{0001} = \Pi_{0300} = 0$: It can immediately be seen from $\Omega = 0$ that all coefficients of $\Pi_{000i} = 0$ (for $i = 1, 2, 3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express h_5 and L_2 from $\Pi_{0103} = 0$ and $\Pi_{0013} = 0$, respectively. Moreover, we can compute g_4 and L_3 from $\Pi_{1102} = 0$ and $\Pi_{1012} = 0$, respectively. $\Pi_{0121} = 0$ implies $B_5 = b_2$. From $\Pi_{2011} = 0$ we get $A_5 = a_3 + \overline{x}_5b_2 - \overline{X}_3b_3$. Now Π_{0211} can only vanish w.c. for:

   a. $X_2 = x_5$: Now $\Pi_{0300} = 0$ implies $a_3 = b_3\overline{X}_3$. We distinguish two cases:

      i. $B_5 \neq b_3$: Under this assumption $\Omega_{0200} \neq 0$ holds and Π_{0200} can also not vanish w.c.. Therefore we can compute the resultant of Ω and Π with respect to e_1, which yields $(\overline{X}_3 - \overline{x}_5)^2B_5^2e_0^2e_2^2\Gamma[87]$, where Γ is a homogeneous polynomial of degree 4 in e_0, e_2, e_3. Now the coefficient of e_3^4 of Γ cannot vanish w.c..

      ii. $B_5 = b_3$: In this case Ω_{0100} and Π_{0200} cannot vanish w.c. and therefore we can again compute the resultant of Ω and Π with respect to e_1, which yields $(\overline{X}_3 - \overline{x}_5)^3B_5^3e_0e_2\Gamma[16]$, where Γ is a homogeneous polynomial of degree 4 in e_0, e_2, e_3. Again the coefficient of e_3^4 of Γ cannot vanish w.c..

   b. $B_5 = 0$, $X_2 \neq x_5$: Then $\Pi_{0301} = 0$ yields the contradiction.

4.7. $\Pi_{3000} = \Pi_{2000} = \Pi_{1000} = 0$

In contrast to $\Pi_{1000} = 0$, it can immediately be seen from $\Omega = 0$ that all coefficients of $\Pi_{i000} = 0$ (for $i = 2, 3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can solve $\Pi_{3100} = 0$ for g_4, $\Pi_{3010} = 0$ for h_5, $\Pi_{2101} = 0$ for L_2 and $\Pi_{2011} = 0$ for L_1.

4.7.1. $\Pi_{1000} = 0$ *does not vanish identically for all* e_1, e_2, e_3

If the coefficient Z of e_3^2 of Π_{1000} vanishes, then $\Pi_{1000} = 0$ only depends on e_1, e_2 and this already yields together with $\Omega = 0$ the contradiction. Therefore we can assume $Z \neq 0$. Now we have to distinguish the following cases:

1. $\Omega_{0002}\Pi_{0003} \neq 0$: We can compute the resultant of Π_{1000} and Ω resp. Π with respect to e_3, which yields R^{Ω} and R^{Π}, respectively. Now R^{Ω} and R^{Π} have to vanish independently of e_0, e_1, e_2, where R^{Π} spits up into $e_2P[8]Q[38]^2$:

   a. $P[8] = 0$: Note that $P = 0$ is a quadratic homogeneous polynomial in the unknowns e_1, e_2. Now the coefficient of e_1^2 implies $b_2 = -B_5$ and from the coefficient of e_2^2 we get $a_2 = -\overline{x}_5A_5/\overline{X}_2$. Then the coefficient of e_1e_2 implies $A_5 = \overline{X}_2B_5$. Finally, $\Pi_{1000} = 0$ yields a contradiction.

   b. $Q[38] = 0$: Note that $Q = 0$ is a quartic homogeneous polynomial in e_1, e_2. We denote the coefficients of $e_1^ie_2^j$ of Q by Q_{ij}. Now Q_{04} can only vanish w.c. for:

      i. $b_2 = -B_5$: Then Q_{40} can only vanish w.c. for:

- ⋆ $B_5 = 0$: As Q is fulfilled identically, we consider R^Ω which splits up into $S[91]T[4]^2$. It can easily be seen that the coefficients of the homogeneous linear polynomial $T[4] = 0$ in e_1, e_2 cannot vanish w.c.. Therefore we set $S[91]$ equal to zero, which is a quartic homogeneous polynomial in e_0, e_1, e_2. The coefficient of e_2^4 of S already yields the contradiction.
- ⋆ $a_2 = B_5(\overline{x}_5 - \overline{X}_2) - A_5$, $B_5 \neq 0$: Now Q_{31} cannot vanish w.c..

ii. $a_2 = \overline{X}_2 b_2 - \overline{x}_5 B_5 + A_5$, $b_2 \neq -B_5$: Now Q_{40} can only vanish w.c. for:

- ⋆ $b_2 = B_5$: Then $Q_{13} = 0$ yields the contradiction.
- ⋆ $A_5 = \overline{x}_5 B_5$, $b_2 \neq B_5$: Now Q_{31} can only vanish w.c. for:
  (α) $X_2 = x_5$: As Q is fulfilled identically, we consider R^Ω which is a homogeneous polynomial of degree 6 in e_0, e_1, e_2. Now the coefficient of e_2^6 implies an expression for L_3. Moreover, the coefficient of $e_1^4 e_2^2$ implies $b_3 = \overline{X}_3 a_3/\overline{x}_5^2$ and from the coefficient of $e_0^2 e_2^4$ we get $X_3 = -x_5$. Finally, the coefficient of $e_1^2 e_2^4$ yields the contradiction.
  (β) $X_2 = -x_5$, $X_2 \neq x_5$: Now $Q_{22} = 0$ yields the contradiction.

2. $\Pi_{0003} = 0$, $\Omega_{0002} \neq 0$: We can compute B_5 from $\Pi_{0103} = 0$.

   a. $X_2 \neq x_5$: Under this assumption we can express b_2 from from $\Pi_{0013} = 0$. Now it can easily be seen that Π_{0002} cannot vanish w.c.. Therefore we can compute the resultant of Π_{1000} and Π with respect to e_3, which yields R^Π. R^Π splits up and can only vanish w.c. for $P[6] = 0$ or $Q[14]$. It can easily be seen that the coefficients of the quadratic homogeneous polynomial $P = 0$ in the unknowns e_1, e_2 cannot vanish w.c.. Therefore we set $Q = 0$ which is also a quadratic polynomial in the unknowns e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Now $Q_{02} = 0$ implies $a_2 = -\overline{x}_5 A_5/\overline{X}_2$. Then Q_{11} can only vanish w.c. for:

      i. $x_5 = 0$: Now $Q_{20} = 0$ yields the contradiction.
      ii. $\overline{X}_2 = -1/\overline{x}_5$, $x_5 \neq 0$: Then $\Pi_{1000} = 0$ yields the contradiction.

   b. $X_2 = x_5$: Now Π_{0013} can only vanish w.c. for:

      i. $a_2 = A_5$: Then Π_{1000} is a factor of Π. Therefore we can only compute the resultant of Π_{1000} and Ω with respect to e_3, which yields a homogeneous polynomial $R^\Omega[674]$ of degree 6 in e_0, e_1, e_2. We denote the coefficients of $e_0^i e_1^j e_2^k$ of Q by R^Ω_{ijk}. Now R^Ω_{060} can only vanish w.c. for:

      - ⋆ $A_5 = \overline{x}_5 b_2$: Then R^Ω_{402} implies $L_3 = -\overline{X}_3 a_3 - b_3$ and from R^Ω_{006} we get $b_3 = \overline{X}_3 a_3/\overline{x}_5^2$. Finally, $R^\Omega_{024} = 0$ yields the contradiction.
      - ⋆ $L_3 = b_3 - 2b_2 - \overline{X}_3 a_3$, $A_5 - \overline{x}_5 b_2 \neq 0$: Now R^Ω_{402} can only vanish w.c. for $b_2 = b_3$ or $b_2 = -\overline{x}_5 A_5$. In both cases $R^\Omega_{006} = 0$ yields the contradiction.

      ii. $x_5 = \mp i$: Now it can easily be seen that Π_{0002} cannot vanish w.c.. Therefore we can compute the resultant of Π_{1000} and Π with respect to e_3, which yields R^Π. Then R^Π can only vanish w.c. for $e_1(\mp a_2 \pm A_5 - 2ib_2) + ie_2(a_2 + A_5) = 0$. It can easily be seen that the coefficients of this linear homogeneous polynomial in the unknowns e_1, e_2 cannot vanish w.c..

3. $\Omega_{0002} = 0$, $\Omega_{0001}\Pi_{0003} \neq 0$: We can express L_3 from $\Omega_{0002} = 0$. Moreover, we can compute the resultant of Π_{1000} and Ω resp. Π with respect to e_3, which yields R^Ω and R^Π, respectively. Now R^Ω and R^Π have to vanish independently of e_0, e_1, e_2, where R^Π spits up into $e_2 P[8]Q[38]^2$:

   a. $P[8] = 0$: Note that $P = 0$ is a quadratic homogeneous polynomial in the unknowns e_1, e_2. Now the coefficient of e_1^2 implies $b_2 = -B_5$ and from the coefficient of e_2^2 we get $a_2 = -\overline{x}_5 A_5/\overline{X}_2$. Then the coefficient of $e_1 e_2$ implies $A_5 = \overline{X}_2 B_5$. Finally, $\Pi_{1000} = 0$ yields a contradiction.

   b. $Q[38] = 0$: Note that $Q = 0$ is a quartic homogeneous polynomial in e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Now Q_{04} can only vanish w.c. for:

      i. $b_2 = -B_5$: Then Q_{40} can only vanish w.c. for:

      - ⋆ $B_5 = 0$: As Q is fulfilled identically, we consider R^Ω which splits up into $S[47]T[4]$. It can easily be seen that the coefficients of the homogeneous linear polynomial $T[4] = 0$ in e_1, e_2 cannot vanish w.c.. Therefore we set $S[47]$ equal to zero, which is a quartic homogeneous polynomial in e_0, e_1, e_2. The coefficient of e_2^4 of S already yields the contradiction.

- ⋆ $a_2 = B_5(\overline{x}_5 - \overline{X}_2) - A_5$, $B_5 \neq 0$: Now Q_{31} cannot vanish w.c..

ii. $a_2 = \overline{X}_2 b_2 - \overline{x}_5 B_5 + A_5$, $b_2 \neq -B_5$: Now Q_{40} can only vanish w.c. for:

- ⋆ $b_2 = B_5$: Then $Q_{13} = 0$ yields the contradiction.
- ⋆ $A_5 = \overline{x}_5 B_5$, $b_2 \neq B_5$: Now Q_{31} can only vanish w.c. for:
  (α) $X_2 = x_5$: As Q is fulfilled identically, we consider R^Ω which is a homogeneous polynomial of degree 5 in e_0, e_1, e_2. Now the coefficient of e_2^5 yields the contradiction.
  (β) $X_2 = -x_5$, $X_2 \neq x_5$: Now $Q_{22} = 0$ yields the contradiction.

4. $\Omega_{0002} = \Omega_{0001} = 0$, $\Pi_{0003} \neq 0$: We can express L_3 from $\Omega_{0002} = 0$ and a_2 from $\Omega_{1001} = 0$. Then we compute the resultant of Π_{1000} and Π with respect to e_3, which yields R^Π. Now R^Π splits up and can only vanish w.c. for $P[11] = 0$ or $Q[49] = 0$.

a. $P[11] = 0$: Now the coefficients of this equation can only vanish w.c. for $b_2 = -B_5$, $A_5 = \overline{X}_2 B_5$ and $a_3 = B_5(\overline{X}_2 - \overline{x}_5) + \overline{X}_3 b_3$. But then $\Pi_{1000} = 0$ yields the contradiction.

b. $Q[49] = 0$: This is a quartic polynomial in the unknowns e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Then Q_{04} can only vanish w.c. for:

i. $b_2 = -B_5$: Now Q_{40} can only vanish w.c. for:

- ⋆ $B_5 = 0$: Then Q is fulfilled identically. As now the coefficient of the highest exponent of e_2 in Π_{1000} and $\Omega = 0$ cannot vanish w.c., we can compute the resultant of $\Pi_{1000} = 0$ and Ω with respect to e_2, which yields R^Ω. Moreover, $R^\Omega = 0$ factors into $S[34]T[59] = 0$. As the coefficient of e_3^4 of S already yields a contradiction, we set T equal to zero. The coefficient of e_0^2 of T implies $a_3 = \overline{X}_3 b_3 - \overline{x}_5 A_5/\overline{X}_2$. Finally, the coefficient of e_1^2 of T yields the contradiction.
- ⋆ $a_3 = \overline{X}_3 b_3 + \overline{x}_5 B_5 - A_5$, $B_5 \neq 0$: Then $Q_{31} = 0$ yields the contradiction.

ii. $a_3 = \overline{X}_3 b_3 - \overline{x}_5 B_5 + A_5$, $b_2 + B_5 \neq 0$: Now Q_{40} can only vanish w.c. for:

- ⋆ $b_2 = B_5$: Then $Q_{13} = 0$ yields the contradiction.
- ⋆ $A_5 = \overline{x}_5 B_5$, $b_2 - B_5 \neq 0$: Now Q_{13} can only vanish w.c. for $X_2 = \pm x_5$. As for $X_2 = -x_5$ the equation $Q_{22} = 0$ yields the contradiction, we set $X_2 = x_5$. Then Q is fulfilled identically.
  (α) $b_2 \neq b_3$: Under this assumption the highest exponent of e_2 in Ω and Π_{1000} cannot vanish w.c.. Therefore we can compute the resultant of Ω and the only non-c. factor of Π_{1000} with respect to e_2, which yields $R^\Omega[87]$. Then the coefficient of e_3^4 of R^Ω yields the contradiction.
  (β) $b_2 = b_3$: Now Ω_{0020} is fulfilled identically but Ω_{0010} and the highest exponent of e_2 in $\Pi_{1000} = 0$ cannot vanish w.c.. Therefore we can compute the resultant of Ω and the only non-c. factor of Π_{1000} with respect to e_2, which yields $R^\Omega[26]$. Then the coefficient of e_1^4 of R^Ω implies $\overline{X}_3 = -\overline{x}_5 - 2/\overline{x}_5$. Finally, the coefficient of e_0^4 of R^Ω yields the contradiction.

5. $\Omega_{0002} = \Pi_{0003} = 0$, $\Omega_{0001} \neq 0$: We can express L_3 from $\Omega_{0002} = 0$ and $\Pi_{0103} = 0$ implies $b_2 = \overline{x}_5 A_5 + B_5 - \overline{X}_2 a_2$.

a. $X_2 \neq x_5$: Under this assumption we can express B_5 from $\Pi_{0013} = 0$. As Π_{0002} cannot vanish w.c. we can compute the resultant of Π_{1000} and Π with respect to e_3, which yields R^Π. Now R^Π splits up and can only vanish w.c. for $P[6] = 0$ or $Q[14] = 0$. As it can easily be seen, that the coefficients of $P[6] = 0$ cannot vanish w.c., we set $Q[14]$ equal to zero, which is a quadratic polynomial in the unknowns e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Then $Q_{02} = 0$ implies $a_2 = -\overline{x}_5 A_5/\overline{X}_2$. Now $Q_{11} = 0$ can only vanish w.c. for:

i. $X_2 = -x_5$: Then Q_{20} can only vanish for $x_5 = \pm 1$. In both cases $\Pi_{1000} = 0$ yields the contradiction.

ii. $\overline{x}_5 = -1/\overline{X}_2$, $X_2 + x_5 \neq 0$: Again, $\Pi_{1000} = 0$ yields the contradiction.

b. $X_2 = x_5$: Now Π_{0013} can only vanish w.c. for:

i. $a_2 = A_5$: Now Π_{1000} is a factor of Π. Therefore we compute the resultant of Π_{1000} and Ω with respect to e_3, which yields $R^\Omega[244]$. Moreover, $R^\Omega = 0$ is a homogeneous equation of degree 5 in e_0, e_1, e_2. We denote the coefficients of $e_0^i e_1^j e_2^k$ of R^Ω by R^Ω_{ijk}. Then R^Ω_{005} can only vanish w.c. for:

- ⋆ $B_5 = b_3$: Then $R^{\Omega}_{400} = 0$ implies $A_5 = \overline{X}_3 a_3/\overline{x}_5$. Now R^{Ω}_{032} can only vanish w.c. for:
  (α) $a_3 = \overline{x}_5 b_3$: Then $R^{\Omega}_{203} = 0$ implies $X_3 = 0$ and $R^{\Omega}_{212} = 0$ yields the contradiction.
  (β) $b_3 = \overline{X}_3 a_3/\overline{x}_5^2$, $B_5 \neq b_3$: Now $R^{\Omega}_{212} = 0$ yields the contradiction.
- ⋆ $B_5 = -\overline{x}_5 A_5$, $B_5 \neq b_3$: Then $R^{\Omega}_{400} = 0$ implies $b_3 = -\overline{X}_3 a_3$ and $R^{\Omega}_{014} = 0$ yields the contradiction.

ii. $x_5 = \pm i$, $a_2 \neq A_5$: In this case we can compute the resultant of Ω and the only non-c. factor of Π_{1000} with respect to e_3, which yields $R^{\Omega}[207]$. Moreover, $R^{\Omega} = 0$ is a homogeneous equation of degree 4 in e_0, e_1, e_2. We denote the coefficients of $e_0^i e_1^j e_2^k$ of R^{Ω} by R^{Ω}_{ijk}. Then $R^{\Omega}_{004} = 0$ imply $a_2 = A_5 \pm b_3 i \mp B_5 i$ and from $R^{\Omega}_{040} = 0$ we get $A_5 = (\pm B_5 \mp b_3 \mp \overline{X}_3 a_3)i$. Finally, $R^{\Omega}_{022} = 0$ yields the contradiction.

6. $\Omega_{0002} = \Omega_{0001} = \Pi_{0003} = 0$: We can express L_3 from $\Omega_{0002} = 0$ and $\Pi_{0001} = 0$ implies $a_2 = a_3 + \overline{X}_2 b_2 - \overline{X}_3 b_3$. Moreover we get $a_3 = A_5 \overline{X}_3 b_3 - \overline{x}_5 B_5$ from $\Pi_{0013} = 0$.

   a. $X_2 \neq x_5$: Under this assumption we can express A_5 from $\Pi_{0103} = 0$. As Π_{0002} cannot vanish w.c. we can compute the resultant of Π_{1000} and Π with respect to e_3, which yields R^{Π}. Now R^{Π} splits up and can only vanish w.c. for $P[6] = 0$ or $Q[14] = 0$. As it can easily be seen, that the coefficients of $P[6] = 0$ cannot vanish w.c., we set $Q[14]$ equal to zero, which is a quadratic polynomial in the unknowns e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Then $Q_{20} = 0$ implies $B_5 = -b_2$. Now $Q_{02} = 0$ can only vanish w.c. for:

      i. $X_2 = -x_5$: Then $Q_{11} = 0$ yields the contradiction.

      ii. $\overline{x}_5 = -1/\overline{X}_2$, $X_2 + x_5 \neq 0$: Now $\Pi_{1000} = 0$ yields the contradiction.

   b. $X_2 = x_5$: Now Π_{0103} can only vanish w.c. for:

      i. $b_2 = B_5$: Now Π_{1000} is a factor of Π. The coefficient of e_2^3 of Π_{1000} resp. Ω_{0020} equals $\overline{X}_2 A_5$ resp. $(b_3 - B_5)$:
      - ⋆ $A_5(b_3 - B_5) \neq 0$: Under this assumption we can compute the resultant of Π_{1000} and Ω with respect to e_2, which yields $R^{\Omega}[792]$. The coefficient of e_0^6 of R^{Ω} cannot vanish w.c..
      - ⋆ $A_5 = 0$, $b_3 \neq B_5$: Now the coefficient of e_2^2 of Π_{1000} cannot vanish w.c., and therefore we can compute the resultant of Π_{1000} and Ω with respect to e_2, which yields $R^{\Omega}[80]$. Then the coefficient of e_0^4 of R^{Ω} yields the contradiction.
      - ⋆ $b_3 = B_5$, $A_5 \neq 0$: Now Ω_{0010} cannot vanish w.c., and therefore we can compute the resultant of Π_{1000} and Ω with respect to e_2, which yields $R^{\Omega}[167]$. The coefficient of e_0^6 of R^{Ω} cannot vanish w.c..
      - ⋆ $A_5 = 0$, $b_3 = B_5$: Now the coefficient of e_2^2 of Π_{1000} and Ω_{0010} cannot vanish w.c.. Therefore we can compute the resultant of Π_{1000} and Ω with respect to e_2, which yields $R^{\Omega}[22]$. Then the coefficient of e_0^4 of R^{Ω} yields the contradiction.

      ii. $X_2 = \pm i$, $b_2 \neq B_5$: We distinguish two cases:
      - ⋆ $2A_5 \pm b_2 i \mp B_5 i \neq 0$: Under this assumption the highest exponent of e_2 in Π and Π_{1000} cannot vanish w.c.. Therefore we can compute the resultant of the only non-c. factors of Π_{1000} and Π with respect to e_2, which yields R^{Π}. It can immediately be seen, that R^{Π} cannot vanish w.c..
      - ⋆ $A_5 = (\pm B_5 i \mp b_2 i)/2$: Now Π_{1000} can only vanish w.c. for $e_1 = \mp e_3^2 i/e_2$. Then it can immediately be seen, that $\Pi = 0$ yields the contradiction.

4.7.2. $\Pi_{1000} = 0$ *vanishes identically for all* e_1, e_2, e_3

Now $\Pi_{1210} = 0$ implies $b_2 = -B_5$ and from $\Pi_{1012} = 0$ we get $a_2 = -A_5 \overline{x}_5/\overline{X}_2$. Then $\Pi_{1120} = 0$ and $\Pi_{1102} = 0$ can only vanish w.c. for $X_2 = x_5$. Now we distinguish the following cases:

1. $\Omega_{0200} \neq 0$: We distinguish two cases:

   a. $b_2 \neq 0$: Due to this assumption $\Pi_{0300} \neq 0$ holds and we can compute the resultant of Ω and Π with respect to e_1 which yields $e_3^2 \Gamma[3200]$, where Γ is a homogeneous polynomial of degree 6 in e_0, e_2, e_3. In the following we denote the coefficients of e_0^i, e_2^j, e_3^k of Γ by Γ_{ijk}. We can solve the only non-c. factor of Γ_{600} for L_3 and $\Gamma_{303} = 0$ implies $a_3 = \overline{X}_3 b_3 + B_5 \overline{x}_5 - A_5$.

      i. $\overline{X}_3 B_5 + \overline{x}_5 b_3 \neq 0$: Under this assumption we can compute A_5 from the only non-c. factor of Γ_{006}.

- ⋆ $\overline{X}_3 - 2\overline{X}_3^2\overline{x}_5 - \overline{x}_5 \neq 0$: Now we can solve the only non-c. factor of $\Gamma_{024} = 0$ for b_3. Then $\Gamma_{042} = 0$ yields the contradiction.
- ⋆ $\overline{x}_5 = \overline{X}_3 - 2\overline{X}_3^2\overline{x}_5$: Now $\Gamma_{024} = 0$ yields the contradiction.

ii. $A_5 = -\overline{X}_3^2 b_2/\overline{x}_5$: Then Γ_{006} can only vanish w.c. for:
- ⋆ $X_3 = 0$: Now $\Gamma_{024} = 0$ implies $b_2 = -b_3$ and $\Gamma_{042} = 0$ yields the contradiction.
- ⋆ $X_3 = -x_5$, $X_3 \neq 0$: Then $\Gamma_{024} = 0$ cannot vanish w.c..

b. $b_2 = 0$: Now Π_{0300} and Π_{0200} vanish but Π_{0100} cannot vanish w.c.. Therefore we can compute the resultant of Ω and Π with respect to e_1 which yields $e_3^2\Gamma[62]$, where Γ is a homogeneous polynomial of degree 6 in e_0, e_2, e_3. In the following we denote the coefficients of e_0^i, e_2^j, e_3^k of Γ by Γ_{ijk}. We can solve the only non-c. factor of Γ_{006} for L_3. Then $\Gamma_{024} = 0$ imply $a_3 = \overline{x}_5 b_3$ and from $\Gamma_{204} = 0$ we get $A_5 = -b_3(\overline{X}_3\overline{x}_5 + 1)/\overline{x}_5$. Now $\Gamma_{105} = 0$ yields the contradiction.

2. $\Omega_{0200} = 0$, $\Omega_{0100} \neq 0$: We can express L_3 from $\Omega_{0200} = 0$.

   a. $b_2 \neq 0$: Due to this assumption $\Pi_{0300} \neq 0$ holds and we can compute the resultant of Ω and Π with respect to e_1 which yields $e_3\Gamma[621]$, where Γ is a homogeneous polynomial of degree 6 in e_0, e_2, e_3. In the following we denote the coefficients of e_0^i, e_2^j, e_3^k of Γ by Γ_{ijk}. Then $\Gamma_{600} = 0$ implies $b_2 = b_3$ and from $\Gamma_{006} = 0$ we get $A_5 = -\overline{X}_3 a_3/\overline{x}_5$. Finally, $\Gamma_{303} = 0$ yields the contradiction.

   b. $b_2 = 0$: Now Π_{0300} and Π_{0200} vanish but Π_{0100} cannot vanish w.c.. Therefore we can compute the resultant of Ω and Π with respect to e_1 which yields $e_3\Gamma[22]$, where Γ is a homogeneous polynomial of degree 4 in e_0, e_2, e_3. Then the coefficient of $e_0^2e_3^2$ of Γ cannot vanish w.c..

3. $\Omega_{0200} = \Omega_{0100} = 0$, $\Pi_{0030} \neq 0$: We can express L_3 from $\Omega_{0200} = 0$ and a_3 from $\Omega_{0110} = 0$. Now it can easily be seen that Ω_{0020} cannot vanish w.c.. Due to this fact an the assumption $\Pi_{0020} \neq 0$ we can compute the resultant of Ω and Π with respect to e_2 which yields $e_3^2\Gamma[838]$, where Γ is a homogeneous polynomial of degree 6 in e_0, e_1, e_3. The coefficient of e_0^6 of Γ implies $b_2 = b_3$ and from the coefficient of $e_0^3e_3^3$ we get the contradiction.

4. $\Omega_{0200} = \Omega_{0100} = \Pi_{0030} = 0$: We can express L_3 from $\Omega_{0200} = 0$ and a_3 from $\Omega_{0110} = 0$. Now $\Pi_{0030} = 0$ implies $A_5 = B_5\overline{x}_5$. Then it can easily be seen that Ω_{0020} as well as Π_{0020} cannot vanish w.c.. Therefore we can compute the resultant of Ω and Π with respect to e_2, which yields $e_1^2e_3^2\Gamma[100]$, where Γ is a homogeneous polynomial of degree 4 in e_0, e_1, e_3. The coefficient of e_0^4 of Γ implies $b_2 = b_3$ and the coefficient of e_3^4 of Γ yields the contradiction.

Due to the structure[4] of Ω it can easily be seen, that Ω and Π can only have a common factor, which does not depend on e_0 (cf. footnote 2) if $\Omega = 0$ has this property too. As this case was already treated in subsection 4.6 we remain with the discussion of those cases excluded by the assumption $e_0e_2 - e_1e_3 \neq 0$ (cf. footnote 1). This discussion is done in the next section.

5. Proving the special case $e_0e_2 - e_1e_3 \neq 0$ of Theorem 3

We split up the proof of this section into the following three cases:

1. As $e_0 = e_1 = e_2 = e_3 = 0$ does not correspond with an Euclidean motion, we start the case study by considering the following four cases:

$$e_0 = e_1 = e_2 = 0, \quad e_0 = e_1 = e_3 = 0, \quad e_0 = e_2 = e_3 = 0, \quad e_1 = e_2 = e_3 = 0.$$

   We only discuss the case $e_0 = e_1 = e_2 = 0$ in more detail because the other three cases can be done analogously. Now $\Psi = 0$ implies $f_3 = 0$. Then $\Omega_1 = 0$ yields an expression for f_2 and $\Omega_2 = 0$ implies an expression for f_1. This cannot yield a two-parametric self-motion as only the homogeneous parameters e_3 and f_0 are free.

[4] $\Omega : \sum_{i=0}^{3} c_i e_i^2 + c_4 e_0 e_3 + c_5 e_1 e_2$ where $c_0, \ldots, c_5$ only depend on the geometry of the SG platform.

2. In this part we discuss the following four special cases:

   a. $e_0 = e_1 = 0$: Due to item 1 we can assume $e_2e_3 \neq 0$. We can compute f_2 from $\Psi = 0$. Then Ω_1 implies $f_3 = -L_1e_2/2$. Then Π_4 can only vanish w.c. for $g_4 = -L_1$. Moreover, we can express f_1 from Π_5. Finally the coefficients of $e_2 f_0$ of Ω_2 and Ω_3 cannot vanish w.c..
   b. $e_2 = e_3 = 0$: This case can be done analogously to the last one.
   c. $e_0 = e_3 = 0$: Due to item 1 we can assume $e_1e_2 \neq 0$. We can compute f_1 from $\Psi = 0$. Then we can express f_0 from $\Pi_4 = 0$. Moreover, we can compute f_3 from $\Pi_5 = 0$. Now Ω_1, Ω_2 and Ω_3 have to vanish independently of the choice of the unknowns e_1, e_2, f_2.

      The coefficient of e_1^4 of Ω_2 implies an expression for h_5. Then we get L_2 from the coefficient of $e_1^1e_2^3$ of Ω_2 and L_3 from the coefficient of e_1^4 of Ω_3. Then the coefficients of e_1^4 and e_2^4 of Ω_1 imply $L_1 = g_4 = 0$. Now we can compute a_2 from the coefficient of $e_1^1e_2^3$ of Ω_1. Moreover, the coefficient of $e_1^3e_2^1$ of Ω_1 implies $B_5 = \overline{x}_5A_5$ and from the coefficient of $e_1^1e_2^3$ of Ω_3 we get $a_3 = A_5(1 + \overline{x}_5^2) - \overline{X}_3b_3$. Then the coefficient of $e_1^2e_2^2$ of Ω_1 can only vanish w.c. for $x_5 = \mp i$. Then the coefficient of e_2^4 of Ω_2 implies $X_2 = \pm i$. Finally, the coefficient of e_2^4 of Ω_3 yields the contradiction.
   d. $e_1 = e_2 = 0$: This case can be done analogously to the last one.

3. Due to the discussion of the special cases in item 1 and item 2, we can assume $e_0e_1e_2e_3 \neq 0$. Therefore we can solve $e_0e_2 - e_1e_3 = 0$ for e_2. Moreover, we can solve $\Psi, \Omega_1, \Pi_4, \Pi_5$ for f_0, f_1, f_2, f_3.

   Now Ω_2 and Ω_3 have to vanish independently of the choice of the unknowns e_0, e_1, e_3. Therefore the coefficient of e_0^6 of Ω_2 implies $L_1 = g_4$. Then the coefficient of $e_0^5e_3$ of Ω_2 yields an expression for L_2. Now we get $g_4 = 2a_2 - 2\overline{X}_2b_2$ from the coefficient of $e_0^4e_3^2$ of Ω_2. Moreover, we get $a_2 = \overline{X}_2b_2$ from the coefficient of $e_1^2e_3^4$ of Ω_2. Finally the coefficient of $e_0e_1^2e_3^3$ of Ω_2 cannot vanish w.c.. This finishes the proof of Theorem 3. □

6. Geometric interpretation of the necessary conditions

As noted in [15], the equations Eq. (2) and Eq. (3) arise from the condition that Ω of subsection 2.5 does not depend on e_0 and e_3 or e_1 and e_2, respectively. By computing $\Omega_{2000} + \Omega_{0002}$, $\Omega_{2000} - \Omega_{0002}$ and Ω_{1001} it can immediately be seen that the conditions of Eq. (2) can also be written as:

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0, \quad \overline{X}_2a_2 - \overline{X}_3a_3 + b_2 - b_3 = 0, \quad \overline{X}_2b_2 - \overline{X}_3b_3 - a_2 + a_3 = 0. \tag{4}$$

By computing $\Omega_{0200} + \Omega_{0020}$, $\Omega_{0200} - \Omega_{0020}$ and Ω_{0110} it can immediately be seen that Eq. (3) can be rewritten as:

$$L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0, \quad \overline{X}_2a_2 - \overline{X}_3a_3 - b_2 + b_3 = 0, \quad \overline{X}_2b_2 - \overline{X}_3b_3 + a_2 - a_3 = 0. \tag{5}$$

In the following we give the geometric interpretation of Eq. (4), which is sketched in Figure 1(a):

I. $L_1(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 = 0$ expresses that the three lines $\mathsf{t}_i \in \Sigma_0$ $(i = 1, 2, 3)$ with homogeneous line coordinates $[L_i : \overline{X}_i : \overline{Y}_i]$ have a common point T ($\Rightarrow$ the three Darboux planes belong to a pencil of planes).

II. $\overline{X}_2b_2 - \overline{X}_3b_3 - a_2 + a_3 = 0$ expresses that the three lines $\mathsf{s}_i := [\mathsf{u}_i, \overline{\mathsf{U}}_i]$ $(i = 1, 2, 3)$ with $\overline{\mathsf{U}}_i = (0 : \overline{X}_i : \overline{Y}_i)$ have a common point S.

III. $\overline{X}_2a_2 - \overline{X}_3a_3 + b_2 - b_3 = 0$ expresses that the three lines $\mathsf{s}_i^{\perp} := [\mathsf{u}_i, \overline{\mathsf{U}}_i^{\perp}]$ $(i = 1, 2, 3)$ with $\overline{\mathsf{U}}_i^{\perp} = (0 : -\overline{Y}_i : \overline{X}_i)$ have a common point $\mathsf{S}^{\perp}$.

Note that the items II and III only hold if the coordinate systems of the platform and base are chosen according to Lemma 1 and if these two coordinate systems coincide.

The geometric interpretation of Eq. (5) is equivalent with the one given above, if one rotates the platform about the x-axis with angle π. Therefore the two triples of necessary conditions are connected by this rotation, which is represented in the Euler parameter space by the transformation (cf. [10]): $(e_0, e_1, e_2, e_3) \mapsto (-e_1, e_0, -e_3, e_2)$.

Remark 1. It is interesting to note, that the given necessary conditions only arise from the three Darboux constraints. A purely geometric proof of the necessity of these conditions for a type II DM self-motion of a general planar SG platform seems to be a complicated task. ⋄

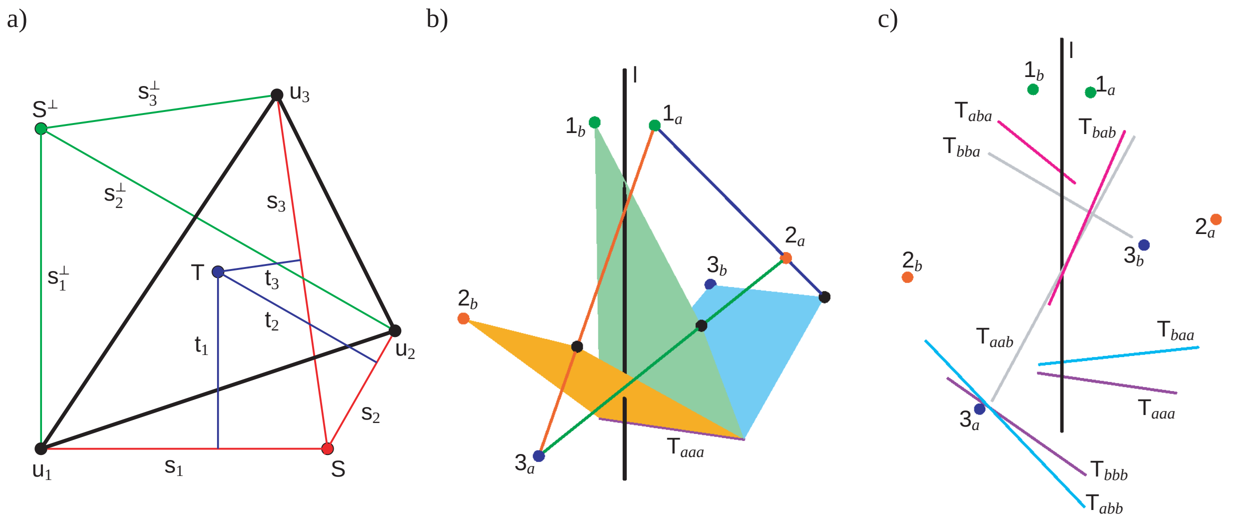

Figure 1: a) Sketch of the geometric interpretation of the necessary conditions.
b,c) Line-symmetric Bricard octahedron: $1_a = (1, 0, 0)$, $2_a = (5, 3, -6)$, $3_a = (-2, -7, -9)$ and the line of symmetry is the z-axis.

6.1. Line-symmetric Bricard octahedra

We denote the vertices of the line-symmetric Bricard octahedron [3] by 1_a, 1_b, 2_a, 2_b, 3_a, 3_b, where v_a and v_b are symmetric with respect to the line l for $v \in \{1, 2, 3\}$. Moreover, ε_{ijk} denotes the face spanned by $1_i, 2_j, 3_k$ with $i, j, k \in \{a, b\}$. Under consideration of this notation we can formulate the following theorem, which is illustrated in Fig. 1b,c:

Theorem 4. *Every line-symmetric Bricard octahedron has the property that the following three planes, orthogonal to ε_{ijk}, have a common line T_{ijk}:*

⋆ *plane orthogonal to $[1_i, 2_j]$ though $3_{k'}$ where $k \neq k' \in \{a, b\}$,*

⋆ *plane orthogonal to $[2_j, 3_k]$ though $1_{i'}$ where $i \neq i' \in \{a, b\}$,*

⋆ *plane orthogonal to $[3_k, 1_i]$ though $2_{j'}$ where $j \neq j' \in \{a, b\}$.*

Proof: It was already proven by the author in Corollary 1 of [16] that the continuous flexion of a line-symmetric Bricard octahedron is a type II DM self-motion. Then the theorem follows immediately by item I. □

7. Conclusion

In this article we have proven the necessity of three conditions for obtaining a type II DM self-motion of a general planar SG platform (cf. Theorem 3). Moreover, we also gave a geometric interpretation of these conditions cf. section 6), which identified a property of line-symmetric Bricard octahedra, which was not known until now, to the best knowledge of the author (cf. Theorem 4).

Finally, it should be noted that Theorem 3 is the key for the determination of all planar SG platforms with a type II DM self-motion, which was already done in [17].

Acknowledgment

The work of the author is supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

References

[1] Borel, E.: *Mémoire sur les déplacements à trajectoires sphériques*, Mém. présenteés par divers savants, Paris(2), 33, 1–128 (1908).

[2] Borras, J., Thomas, F., Torras, C.: *Singularity-invariant leg rearrangements in doubly-planar Stewart-Gough platforms*, In Proc. of Robotics Science and Systems, Zaragoza, Spain (2010).

[3] Bricard, R.: Mémoire sur la théorie de l'octaèdre articulé, Journal de Mathématiques pures et appliquées, Liouville **3** 113–148 (1897).

[4] Bricard, R.: *Mémoire sur les déplacements à trajectoires sphériques*, Journ. École Polyt.(2), 11, 1–96 (1906).

[5] Husty, M.: *E. Borel's and R. Bricard's Papers on Displacements with Spherical Paths and their Relevance to Self-Motions of Parallel Manipulators*, Int. Symp. on History of Machines and Mechanisms (M. Ceccarelli ed.), 163–172, Kluwer (2000).

[6] Husty, M.L., Karger, A.: *Self motions of Stewart-Gough platforms: an overview*, Proc. of the workshop on fundamental issues and future research directions for parallel mechanisms and manipulators (C.M. Gosselin, I. Ebert-Uphoff eds.), 131–141 (2002).

[7] Husty, M., Mielczarek, S., Hiller, M.: *A redundant spatial Stewart-Gough platform with a maximal forward kinematics solution set*, Advances in Robot Kinematics: Theory and Applications (J. Lenarcic, F. Thomas eds.), 147–154, Kluwer (2002).

[8] Karger, A.: *Architecture singular planar parallel manipulators*, Mechanism and Machine Theory **38** (11) 1149–1164 (2003).

[9] Karger, A.: *Architecturally singular non-planar parallel manipulators*, Mechanism and Machine Theory **43** (3) 335–346 (2008).

[10] Karger, A.: *Parallel manipulators and Borel-Bricard problem*, Computer Aided Geometric Design, Special Issue: Advances in Applied Geometry (B. Jüttler, M. Lavicka, O. Röschel eds.) **27** (8) 669–680 (2010).

[11] Ma, O., Angeles, J.: *Architecture Singularities of Parallel Manipulators*, Int. J. of Robotics and Automation **7** (1) 23-29 (1992).

[12] Mielczarek, S., Husty, M.L., Hiller, M.: *Designing a redundant Stewart-Gough platform with a maximal forward kinematics solution set*, In Proc. of the International Symposion of Multibody Simulation and Mechatronics (MUSME), Mexico City, Mexico, September 2002.

[13] Nawratil, G.: *On the degenerated cases of architecturally singular planar parallel manipulators*, J. Geom. Graphics **12** (2) 141–149 (2008).

[14] Nawratil, G.: *A new approach to the classification of architecturally singular parallel manipulators*, Computational Kinematics (A. Kecskemethy, A. Müller eds.), 349–358, Springer (2009).

[15] Nawratil, G.: *Basic result on type II DM self-motions of planar Stewart Gough platforms*, Mechanisms, Transmissions, Applications (E.Chr. Lovasz, B. Corves eds.), 235–244, Springer (2011).

[16] Nawratil, G.: *Types of self-motions of planar Stewart Gough platforms*, under review.

[17] Nawratil, G.: *Planar Stewart Gough platforms with a type II DM self-motion*, Technical Report No. 220, Geometry Preprint Series, TU Vienna (2011).

[18] Röschel, O., Mick, S.: *Characterisation of architecturally shaky platforms*, Advances in Robot Kinematics: Analysis and Control (J. Lenarcic, M.L. Husty eds.), 465–474, Kluwer (1998).

[19] Vogler, H.: *Bemerkungen zu einem Satz von W. Blaschke und zur Methode von Borel-Bricard*, Grazer Mathematische Berichte **352** 1–16 (2008).

[20] Wohlhart, K.: *From higher degrees of shakiness to mobility*, Mechanism and Machine Theory **45** (3) 467–476 (2010).