Self-motions of planar projective Stewart Gough platforms

G. Nawratil

Abstract In this paper, we study self-motions of non-architecturally singular parallel manipulators of Stewart Gough type, where the planar platform and the planar base are related by a projectivity. By using mainly geometric arguments, we show that these manipulators have either so-called elliptic self-motions or pure translational self-motions. In the latter case, the projectivity has to be an affinity $\mathbf{a}+\mathbf{A}\mathbf{x}$, where the singular values s_1 and s_2 of the 2×2 transformation matrix $\mathbf{A}$ with $0 < s_1 \leq s_2$ fulfill the condition $s_1 \leq 1 \leq s_2$.

Key words: Self-motion, Stewart Gough platform, Borel Bricard problem

1 Introduction

The geometry of a planar Stewart Gough (SG) platform is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i)^T$ with respect to the xy-plane π_{M} of the fixed system Σ_0 and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i)^T$ with respect to the xy-plane π_{m} of the moving system Σ. If the geometry of the manipulator is given as well as the six leg lengths R_i, then the SG platform is in general rigid, but under particular conditions the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion. Note that such motions are also solutions to the famous Borel Bricard problem (cf. [1, 2, 3]).

It is well known that planar SG platforms which are singular in every possible configuration, possess self-motions in each pose. These so-called architecturally singular planar SG platforms were extensively studied in [4, 5, 6, 7]. Therefore, we are only interested in self-motions of planar SG platforms, which are not architecturally singular.

Georg Nawratil
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Austria,
e-mail: nawratil@geometrie.tuwien.ac.at

In this paper, we discuss the case where the base anchor points M_i and the platform anchor points m_i are related by a non-singular projectivity κ. [1] For the remainder of this article we call such manipulators planar projective SG platforms. Note that a projectivity is the most general linear mapping between two projective extended planes, and that κ is uniquely determined by corresponding quadrangles.

It is well known (cf. Chasles [8]), that a planar projective SG platform is architecturally singular if and only if one set of anchor points is located on a conic section. Under consideration of this result the theorem given by Karger in Sec. 3 of [9] can be rewritten as follows:

Theorem 1. *A singular configuration of a planar projective SG platform, which is not architecturally singular, does not depend on the distribution of the anchor points in the platform and the base, but only on the mutual position of the planes π_{M} and π_{m} and on the correspondence between them. The configuration is singular iff either one of the legs can be replaced by a leg of zero length or two legs can be replaced by aligned legs.*

A non-singular projectivity which maps ideal points onto ideal points is a non-singular affinity. The subcase of planar parallel manipulators of SG type with affinely equivalent platform and base (= planar affine SG platforms) was studied by Karger in [9, 10, 11]. It should also be noted that according to Mielczarek et al. [12], one can attach a two-parametric set of additional legs to planar affine SG platforms without restricting the direct kinematics, whereas the correspondence between the anchor points is given by the affinity itself.

As we want to study planar projective SG platforms we have to consider the projective extension of the carrier planes of the platform and base anchor points, i.e.

$$(a_i, b_i) \mapsto (w_i : x_i : y_i) \quad \text{and} \quad (A_i, B_i) \mapsto (W_i : X_i : Y_i). \tag{1}$$

Note that ideal points are characterized by $w_i = 0$ and $W_i = 0$, respectively.

2 Basic results

Lemma 1. *One can attach a two-parametric set of additional legs to planar projective SG platforms without changing the forward kinematics and singularity surface.*

Proof. For the proof we can use the homogenized version of the criterion given in Eq. (12) of [13] which corresponds with the criterion for the solvability of the inhomogeneous system of equations given in Eq. (30) of [12].

Assume a non-architecturally singular planar manipulator $\mathsf{m}_1, \ldots, \mathsf{M}_6$ is given. Then one can add a further leg (with anchor points m_7 and M_7) to the originally legs (without changing the direct kinematics and the singularity surface) if the following rank condition holds (see also Remark 1 of Röschel and Mick [6]):

[1] If κ is singular, one set of anchor points would collapse into a line or a point, which yields trivial cases of architecturally singular manipulators.

$$\begin{aligned} rk(\mathbf{Q}) = 6 \quad \text{with} \quad \mathbf{Q} = (\mathbf{q}_1, \ldots, \mathbf{q}_7)^T \quad \text{and} \\ \mathbf{q}_i = (w_iW_i, w_iX_i, w_iY_i, x_iW_i, x_iX_i, x_iY_i, y_iW_i, y_iX_i, y_iY_i)^T. \end{aligned} \tag{2}$$

Now it can easily be checked by the use of MAPLE that $rk(\mathbf{Q}) = 6$ holds true for

$$(W_i, X_i, Y_i)^T := \mathbf{P}(w_i, x_i, y_i)^T \quad \text{for} \quad i = 1, \ldots, 7 \tag{3}$$

where $\mathbf{P}$ is the matrix of the projectivity ($\mathbf{P}$ is a regular 3×3 matrix). □

Remark 1. Due to Lemma 1 it is clear why a singular configuration of a planar projective SG platform does not depend on the distribution of the anchor points in π_{M} and π_{m} (cf. Thm. 1). ⋄

Theorem 2. *Self-motions of planar projective SG platforms, which are not architecturally singular, can only be of the following type:*

1. *Spherical self-motion with rotation center* $\mathsf{m}\kappa = \mathsf{m}$,
2. *Schönflies self-motion, where the direction of the rotation axis is parallel to the planes* π_{M} *and* π_{m},
3. *Elliptic self-motion.*

Proof. We start by denoting the line of intersection of π_{M} and π_{m} by s in the projective extension of the Euclidean 3-space. As in any pose of a self-motion of a planar projective SG platform, the manipulator has to be in a singular configuration, we can apply Thm. 1. Therefore the manipulator is singular if and only if one of the following cases hold:

a. π_{M} and π_{m} coincide,
b. $\mathsf{S} = \mathsf{S}\kappa$ holds, where S is the intersection point of s and $\mathsf{s}\kappa$,
c. $\mathsf{s} = \mathsf{s}\kappa$.

It is well known that every projectivity of the projective extension of the Euclidean plane onto itself has at least one real fixed point $\mathsf{F} = \mathsf{F}\kappa$. Therefore, if one pose of the self-motion is singular due to item (a), this already implies item (1) if F is a finite point or item (2) if F is an ideal point. Clearly, this also holds for item (b) with respect to the fixed point $\mathsf{S} = \mathsf{S}\kappa$. For the study of item (c) we consider again only one singular configuration of the self-motion. As $\mathsf{s} = \mathsf{s}\kappa$ holds the projectivity from s onto itself can be (i) hyperbolic, (ii) parabolic or (iii) elliptic.

Item (i) immediately implies that the self-motion can only be a pure rotation about the finite axis s which is a special case of item (1) and (2), respectively. If s is the ideal line ($\Rightarrow \pi_{\mathsf{M}} \parallel \pi_{\mathsf{m}}$) then the self-motion is a pure translation, which is a special case of item (2).

For item (ii) we have one fixed point and we end up with item (2) and (1), respectively, depending on the circumstance if the fixed point is an ideal point or not.

Item (iii) corresponds to the case of Thm. 1, where two legs can be replaced by collinear legs, as we cannot attach a leg with zero length (over $\mathbb{R}$) without changing the direct kinematics and singularity surface. Therefore the following definition finishes the proof of Thm. 2.

Definition 1. A self-motion of a planar projective SG platform is called *elliptic*, if each pose of this motion is singular due to item (c,iii). □

Due to Thm. 2 we only have to investigate spherical self-motions with rotation center $\mathsf{m}\kappa = \mathsf{m}$ (cf. Sec. 3), Schönflies self-motions with the rotation axis parallel to π_{M} and π_{m} (cf. Sec. 4) and elliptic self-motions (cf. Sec. 5).

3 Spherical self-motions

If a planar projective SG platform has a spherical self-motion about $\mathsf{m}\kappa = \mathsf{m}$, then the spherical image of this manipulator with respect to the unit sphere S^2 centered in $\mathsf{m}\kappa = \mathsf{m}$ also has to have a self-motion. Therefore the problem reduces to the determination of non-degenerated[2] spherical 3-dof RPR manipulators with self-motions, where the three base and platform anchor points are located on great circles. The following result is proven in Appendix A:

Lemma 2. *A non-degenerated spherical 3-dof RPR manipulator, where the base anchor points* M_1°, M_2°, M_3° *and the platform anchor points* m_1°, m_2°, m_3° *are located on great circles, can only have a self-motion if two platform points* $\mathsf{m}_1^\circ = \mathsf{m}_3^\circ$ *coincide (after relabeling of anchor points and interchange of platform and base) and if the spherical lengths* R_i° *of the legs equal* $R_1^\circ = \overline{\mathsf{M}_1^\circ\mathsf{M}_2^\circ}$, $R_2^\circ = \overline{\mathsf{m}_1^\circ\mathsf{m}_2^\circ}$, $R_3^\circ = \overline{\mathsf{M}_3^\circ\mathsf{M}_2^\circ}$.

The self-motion of the manipulator given in Lemma 2 is a pure rotation about the axis $\mathsf{a} := [\mathsf{m}\kappa = \mathsf{m}, \mathsf{m}_1^\circ = \mathsf{m}_3^\circ = \mathsf{M}_2^\circ]$ (cf. Fig. 1a). Trivially, we can only add an additional leg (with anchor points m_4° and M_4°) to this manipulator without restricting the self-motion if $\mathsf{m}_4^\circ = \mathsf{m}_1^\circ$ or $\mathsf{M}_4^\circ = \mathsf{M}_2^\circ$ holds. This has the following consequence for the corresponding planar projective SG platform: κ has to map all platform anchor points $\notin \mathsf{a}$ on points of a. Therefore κ cannot be a bijection and we get the contradiction. This proves the following theorem:

Theorem 3. *Planar projective SG platforms, which are not architecturally singular, do not have spherical self-motions with rotation center* $\mathsf{m}\kappa = \mathsf{m}$.

4 Schönflies self-motions

The Schönflies motion group is a four-dimensional subgroup of the Euclidean motion group and consists of all translations combined with all rotations about a fixed direction d, which in our case is parallel to π_{M} and π_{m}. Moreover, it is well known (e.g. [14]) that platform points being on lines parallel to d have congruent trajectories in a Schönflies motion. Therefore we can translate every leg of the manipulator in direction d during a Schönflies self-motion without changing this motion. This property is important for the following argumentation.

[2] Neither all platform anchor points nor all base anchor points collapse into one point.

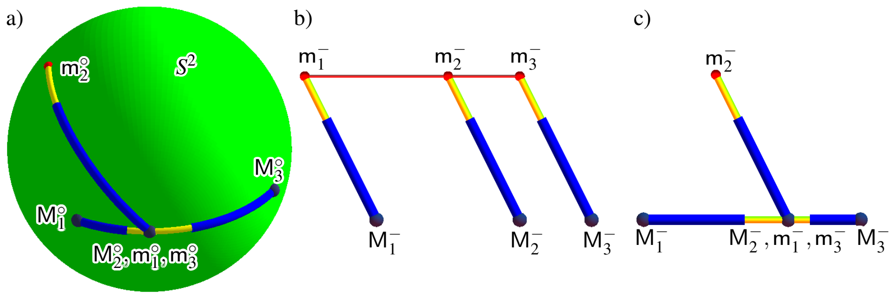

Fig. 1 a) Non-degenerated spherical 3-dof RPR manipulator with a self-motion. b,c) Non-degenerated planar 3-dof RPR manipulators with self-motions: Circular translation (b) and a pure rotation about the point $\mathsf{M}_2^- = \mathsf{m}_1^- = \mathsf{m}_3^-$ (c), which is the planar analogue of (a).

We choose the y-axis of the moving and the fixed frame parallel to the direction d. Moreover, we choose a line $\mathsf{g} \in \pi_{\mathsf{m}}$ which is orthogonal to d (cf. Fig. 2a). Under the projectivity κ the platform anchor points $\mathsf{m}_g \in \mathsf{g}$ are mapped to the corresponding base anchor points $\mathsf{M}_g := \mathsf{m}_g\kappa$ on the line $\mathsf{g}\kappa \in \pi_{\mathsf{M}}$, which cannot be parallel to d (cf. Fig. 2b). Note that the lines $[\mathsf{m}_g, \mathsf{M}_g]$ belong to a regulus $\mathscr{R}$.

Now we choose a platform point $\mathsf{m} \notin \mathsf{g}$ and denote the footpoint on g with respect to m by m_f. Then τ denotes the signed distance of m_f and m with respect to the direction d. Due to the above considerations we can also add the leg $[\mathsf{m}, \mathsf{m}\tau]$ (beside the leg $[\mathsf{m}, \mathsf{m}\kappa]$) without restricting the self-motion, where $\mathsf{m}\tau$ denotes the point which we get by translating $\mathsf{m}_f\kappa$ about τ in direction d. If this construction is done for all points of a line $\mathsf{h} \parallel \mathsf{g}$ through m we get the line $\mathsf{h}\tau$. We distinguish two cases:

- $\mathsf{h}\kappa \neq \mathsf{h}\tau$: Now every point $\mathsf{m} \in \mathsf{h}$ (with exception of $\mathsf{m}_e := \{\mathsf{h}\kappa \cap \mathsf{h}\tau\}\kappa^{-1}$) can only rotate about the line $[\mathsf{m}\tau, \mathsf{m}\kappa] \parallel \mathsf{d}$ (cf. Fig. 2a,b). Therefore the platform cannot move in direction d during the self-motion and the problem reduces to the following planar one: Determine all non-degenerated 3-dof RPR manipulators with collinear platform anchor points m_1^-, m_2^-, m_3^- and collinear base anchor points M_1^-, M_2^-, M_3^- possessing a self-motion.
  It is well known, that there only exists the so-called *circular translation* (cf. Fig. 1b) beside the planar analogue (cf. Fig. 1c) of the spherical self-motion given in Lemma 2, which yields for the same arguments as in the spherical case no solution to our problem. The *circular translation* implies that the projectivity κ with matrix $\mathbf{P} = (p_{ij})$ has to be an affinity of the following form:

$$\mathbf{M}_i = \begin{pmatrix} p_{21} \\ p_{31} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ p_{32} & p_{33} \end{pmatrix} \mathbf{m}_i \quad \text{with} \quad p_{33} \in \mathbb{R} \setminus \{0,1\} \quad \text{and} \quad p_{21}, p_{31}, p_{32} \in \mathbb{R}. \tag{4}$$

  This can be seen as follows: As the pencil of lines through the ideal point of d has to be mapped onto an identical pencil of lines through the ideal point of d, the ideal line has to be mapped onto itself ($\Rightarrow p_{12} = p_{13} = 0$). As the entries of $\mathbf{P}$ are still homogeneous, we can set $p_{11} = 1$ without loss of generality (w.l.o.g.), as $p_{11} = 0$ implies that κ is singular. Moreover, the fact that the above mentioned

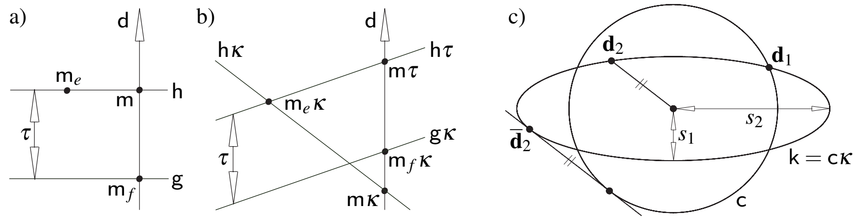

Fig. 2 Sketch of the mappings κ and τ between the platform (a) and the base (b). In (c) the proof of Thm. 5 is illustrated: common point and tangent of an ellipse k and the unit circle c.

pencils are identical yields $p_{22}=1$. Finally, we get $p_{23}=0$ from the condition that the ideal point in direction d is fixed under κ. Moreover, $p_{33}\neq\{0,1\}$ has to hold, because otherwise the affinity is singular resp. hκ = hτ holds, a contradiction.

- hκ = hτ: Now this has to hold for all distances τ because otherwise we get the above case. As a consequence the projectivity κ with matrix $\mathbf{P}=(p_{ij})$ has to be an affinity of the following form:

$$\mathbf{M}_i=\begin{pmatrix}p_{21}\\p_{31}\end{pmatrix}+\begin{pmatrix}p_{22}&0\\p_{32}&1\end{pmatrix}\mathbf{m}_i \quad \text{with} \quad p_{22}\in\mathbb{R}\setminus\{0\} \quad \text{and} \quad p_{21},p_{31},p_{32}\in\mathbb{R}. \tag{5}$$

This can be seen as follows: The condition that the ideal point of the y-axis of the moving frame is mapped onto the ideal point of the y-axis of the fixed frame yields $p_{13}=p_{23}=0$. Now we invest the property that the anchor points of a leg, during its translation in direction d, always have to correspond one another within the projectivity. This can be expressed as follows:

$$\mathbf{P}\begin{pmatrix}1\\u\\v\end{pmatrix}+\begin{pmatrix}0\\0\\\tau\end{pmatrix}=\mathbf{P}\begin{pmatrix}1\\u\\v+\tau\end{pmatrix}. \tag{6}$$

As the first two rows are fulfilled identically, only the third row yields a condition, which reads as follows: $\tau(p_{11}-p_{33}+p_{12}u)=0$. This equation can only be fulfilled for all $u\in\mathbb{R}$ if $p_{12}=0$ ($\Rightarrow$ κ is an affinity) and $p_{11}=p_{33}$ hold. As the entries of $\mathbf{P}$ are still homogeneous, we can set $p_{11}=1$ w.l.o.g., as $p_{11}=0$ implies that κ is singular. Moreover, $p_{22}\neq 0$ has to hold, because otherwise the affinity is also singular.

These considerations prove the first two sentences of the following theorem:

Theorem 4. *A planar projective SG platform, which is not architecturally singular, can only have a Schönflies self-motion with the direction* d *of the rotation axis parallel to π_{M} and π_{m}, if it belongs to the subset of planar affine SG platforms. Moreover, if we choose the y-axis of the moving and the fixed frame in direction of* d*, the affinity κ has to be of the form given in Eqs. (4) or (5). In addition, all self-motions of these manipulators are pure translations and the self-motion is two-dimensional only if the platform and the base are congruent and all legs have equal length.*

The last sentence of this theorem, which was already known to Karger [9], can easily be proved by direct computations, which are given in Appendix B. Moreover, we can give a geometric characterization of all non-architecturally singular planar affine SG platforms with self-motions:

Theorem 5. *Assume a planar affine SG platform, which is not architecturally singular, is determined by* $\mathbf{M}_i = \mathbf{a} + \mathbf{A}\mathbf{m}_i$. *Then this manipulator has a self-motion if and only if the singular values* s_1 *and* s_2 *of* $\mathbf{A}$ *with* $0 < s_1 \leq s_2$ *fulfill* $s_1 \leq 1 \leq s_2$.

Proof. First of all, we prove that a planar affine SG platform cannot have an elliptic self-motion. If $\mathsf{s} = \mathsf{s}\kappa$ is not the ideal line, then the projectivity from s onto itself has at least one fixed point, namely the ideal point of $\mathsf{s} = \mathsf{s}\kappa$. Therefore $\mathsf{s} = \mathsf{s}\kappa$ has to be the ideal line during the whole self-motion. This implies that the elliptic self-motion is a Schönflies motion, where the direction of the rotation axis is orthogonal to $\pi_\mathsf{M} \parallel \pi_\mathsf{m}$. As all points of the platform have to run on spherical paths, this Schönflies motion can only be the Borel Bricard motion (cf. [1, 2]) due to [14]. Therefore the corresponding points of the platform and base have to be related by an inversion. As an inversion is no projectivity, we get a contradiction.

Under consideration of this result and Thms. 2 and 3, planar affine SG platforms can only have self-motions given in Thm. 4. We consider the image of the unit vectors $\mathbf{c} = (\cos\varphi, \sin\varphi) \in \pi_\mathsf{m}$ for $\varphi \in [0, 2\pi]$ under κ. Clearly, the tie points of the vectors $\mathbf{Ac}$ are located on an ellipse k (including the special case of a circle).

- ⋆ The necessary and sufficient condition for an affinity of the form Eq. (5) is that a vector $\mathbf{d}_1$ of $\mathbf{Ac}$ has length 1. This corresponds geometrically to the common points of k and the unit circle c (cf. Fig. 2c).
- ⋆ The necessary and sufficient condition for an affinity of the form Eq. (4) is that a vector $\mathbf{d}_2$ of $\mathbf{Ac}$ exists, which has distance 1 from the ellipse tangent in its conjugate point $\overline{\mathbf{d}}_2$ on k. This corresponds geometrically to the determination of common tangents of k and c (cf. Fig. 2c).

If we choose a new coordinate system in the base and platform such that the y-axis is parallel to $\mathbf{d}_i$ and $\mathbf{A}^{-1}\mathbf{d}_i$, respectively, we end up with an affinity of the form given in Eq. (5) for $i = 1$ resp. Eq. (4) for $i = 2$. Clearly, we only get real common points and tangents of k and c if the singular values s_1 and s_2 of $\mathbf{A}$ fulfill $s_1 \leq 1 \leq s_2$. □

Remark 2. Note, that Thm. 5 also implies the result of [10] that planar equiform SG platforms cannot have a self-motion if they are not architecturally singular, as $s_1 = s_2 \neq 1$ holds. Finally, it should also be mentioned that all planar affine SG platforms given in Eq. (4) and Eq. (5) are Schönflies-singular manipulators due to item (3) and item (2), respectively, of Thm. 3 given by the author in [15]. ⋄

Example 1. We verify Thm. 5 at hand of the planar affine SG platform with a self-motion given by Karger on page 162 of [9]. The first three pairs of anchor points are determined by $a_1 = b_1 = b_2 = A_1 = B_1 = B_2 = 0$, $a_2 = 1$, $a_3 = 5$, $b_3 = -4$ and $A_2 = A_3 = B_3 = 2$. For this example $\mathbf{a}$, $\mathbf{A}$, s_1 and s_2 are given by:

$$\mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 2 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad s_1 = \frac{\sqrt{41} - 5}{4} \approx 0.35, \quad s_2 = \frac{\sqrt{41} + 5}{4} \approx 2.85.$$

5 Conclusion and future research

We proved that non-architecturally singular planar projective SG platforms have either elliptic self-motions (Def. 1) or pure translational self-motions (Thms. 2–4). The latter are the only self-motions of planar affine SG platforms (Thm. 5).

The study of elliptic self-motions is dedicated to future research. It remains open whether these self-motions even exist, as no example is known to the author so far.

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Appendix A

In the following we give the proof of Lemma 2:

Proof. W.l.o.g. we can assume that the origin O of Σ_0 is the center of rotation. Moreover, the constraint $\Gamma_i(\mathsf{m}_i, \mathsf{M}_i, R_i)$ that $\mathsf{m}_i \neq \mathsf{O}$ is located on a sphere with radius R_i and midpoint $\mathsf{M}_i \neq \mathsf{O}$ can be replaced by the constraint $\Gamma_k(\mathsf{m}_k, \mathsf{M}_k, R_k)$ if $\sphericalangle(\mathsf{M}_i, \mathsf{O}, \mathsf{m}_i) = \sphericalangle(\mathsf{M}_k, \mathsf{O}, \mathsf{m}_k)$, $\{\mathsf{O}, \mathsf{M}_i, \mathsf{M}_k\}$ collinear and $\{\mathsf{O}, \mathsf{m}_i, \mathsf{m}_k\}$ collinear.

If $\left\{\mathsf{m}_i, \mathsf{m}_j, \mathsf{O}\right\}$ are collinear and $\left\{\mathsf{M}_i, \mathsf{M}_j, \mathsf{O}\right\}$ are not collinear, then one can replace Γ_i by Γ_k such that $\mathsf{m}_k = \mathsf{m}_j$ holds. Therefore the self-motion can only be a pure rotation about the line $\mathsf{a} = [\mathsf{O}, \mathsf{m}_k = \mathsf{m}_j]$, as the tetrahedron $\left\{\mathsf{O}, \mathsf{m}_k = \mathsf{m}_j, \mathsf{M}_k, \mathsf{M}_j\right\}$ forms a rigid structure. For pure rotations we distinguish two cases:

I. $\mathsf{a} \notin \pi_{\mathsf{M}}$: We can add all legs $[\mathsf{M}_p, \mathsf{m}_p]$ with $\mathsf{m}_p \in \mathsf{a}$
II. $\mathsf{a} \in \pi_{\mathsf{M}}$: We can add all legs $[\mathsf{M}_p, \mathsf{m}_p]$ with $\mathsf{M}_p \in \mathsf{a}$ or $\mathsf{m}_p \in \mathsf{a}$

without disturbing the rotational self-motion. Note that for $R_i = 0$ ($\Rightarrow \mathsf{m}_i = \mathsf{M}_i$) the self-motion is also a pure rotation about the line $\mathsf{a} = [\mathsf{O}, \mathsf{m}_i = \mathsf{M}_i]$ (cf. item II). Item I implies an architecturally singular manipulator as all platform anchor points have to be located on a. This corresponds to the degenerated case of the spherical 3-dof RPR manipulator (cf. footnote 2). Item II yields the self-motion given in Lemma 2.

Clearly, the same argumentation can be done for the case that $\left\{\mathsf{M}_i, \mathsf{M}_j, \mathsf{O}\right\}$ are collinear and $\left\{\mathsf{m}_i, \mathsf{m}_j, \mathsf{O}\right\}$ are not collinear.

Due to these considerations we can assume w.l.o.g. that the triples $\left\{\mathsf{M}_i, \mathsf{M}_j, \mathsf{O}\right\}$ and $\left\{\mathsf{m}_i, \mathsf{m}_j, \mathsf{O}\right\}$ are not collinear for $i, j \in \{1,2,3\}$ and $i \neq j$. Therefore we can set:

$$B_1 = b_1 = 0, \quad A_1 = B_2 = B_3 = a_1 = b_2 = b_3 = 1, \quad (A_2 - A_3)(a_2 - a_3)R_1R_2R_3 \neq 0.$$

According to Brunnthaler et al.[3] the constraints Γ_i can be written as:

$$\begin{aligned} \Gamma_i: \quad & 2a_iA_i(e_2 +^2 + e_3^2 - e_0^2 - e_1^2) + A_i^2 + a_i^2 - R_i^2 + \\ & 4(e_1^2 + e_3^2) + 4e_0e_3(a_i - A_i) - 4e_1e_2(a_i + A_i) = 0, \end{aligned} \tag{7}$$

for $i = 2,3$ and $\Gamma_1 : 4(e_2^2 + e_3^2) - R_1^2 = 0$, where $e_0, \ldots, e_3$ are the Euler parameters. Now Γ_1, Γ_2 and Γ_3 represent three quadrics in the Euler parameter space and a self-motion corresponds to a common intersection curve of them.

Part [A] $a_2A_2a_3A_3 \neq 0$:
Under this assumption we can compute the resultant $\Gamma_{23}[193]$[4] of Γ_2 and Γ_3 with respect to e_0 w.l.o.g.. The coefficient of e_2^2 of Γ_{23} splits up into F_1F_2 with $F_1 = Ge_1 + He_3$, $F_2 = Ge_1 - He_3$ and

$$G = a_2a_3(A_2 - A_3) + A_2A_3(a_2 - a_3), \quad H = a_2a_3(A_3 - A_2) + A_2A_3(a_2 - a_3). \tag{8}$$

[3] Brunnthaler, K., Schröcker, H.-P., Husty, M.: Synthesis of spherical four-bar mechanisms using spherical kinematic mapping, Advances in Robot Kinematics: Mechanisms and Motion (J. Lenarcic, B. Roth eds.), 377–384, Springer (2006)

[4] We write the number of terms of the not explicitly given factors into brackets.

1. $F_1F_2 \neq 0$: Under this assumption we can compute the resultant $\Gamma_{123}[3497]$ of Γ_{23} and Γ_1 with respect to e_2. In order to obtain a self-motion Γ_{123} has to be fulfilled independently of the remaining Euler parameters. Therefore we denote the coefficients of $e_1^i e_3^j$ of Γ_{123} by Γ_{123}^{ij} and set them equal to zero. Then Γ_{123}^{80} implies $a_2 = a_3A_3/A_2$ and from Γ_{123}^{08} we get $A_2 = -a_3$. Then Γ_{123}^{44} cannot vanish without contradiction (w.c.).
2. $F_1 = 0$, $G \neq 0$: Now we can compute e_1 from $F_1 = 0$. Moreover, we denote the coefficient of e_2 of Γ_{23} by W.
   a. $W \neq 0$: Now we can compute the resultant $\Gamma_{123}[28809]$ of Γ_{23} and Γ_1 with respect to e_2 w.l.o.g.. We denote the coefficients of e_3^i of Γ_{123} by Γ_{123}^i. Now Γ_{123}^0 and Γ_{123}^4 imply $R_i^2 = a_i^2 + A_i^2$ for $i = 2,3$. Then Γ_{123}^6 cannot vanish w.c..
   b. $W = 0$: If Γ_{23} is not fulfilled identically, $W = 0$ or Γ_{23} implies an expression for e_3. Finally, Γ_1 would imply an expression for e_2 and we cannot get a self-motion. Therefore there can only exist a self-motion if Γ_{23} is fulfilled independently of the Euler parameters. We denote the coefficients of $e_2^i e_3^j$ of Γ_{23} by Γ_{23}^{ij}. Now Γ_{23}^{11} can only vanish w.c. for $HJ[6] = 0$.
      i. $H = 0$ ($\Rightarrow e_1 = 0$): As for $a_3 = A_2A_3/(A_2 - A_3)$ the expression H cannot vanish w.c. we can assume $A_2A_3 + a_3(A_3 - A_2) \neq 0$. Therefore we can express a_2 from $H = 0$. Then Γ_{23}^{04} implies $A_2 = -a_3$ and from Γ_{23}^{00} we get $R_2^2 = R_3^2$ ($\Rightarrow \Gamma_2 = \Gamma_3$). We can always express e_0 from $\Gamma_2 = \Gamma_3$ due to $A_2A_3a_2a_3 \neq 0$ but Γ_1 is not fulfilled identically ($\Rightarrow$ no self-motion).
      ii. $J = 0, H \neq 0$: W.l.o.g. we can solve $J = 0$ for R_2. Moreover, we can solve the only non-contradicting factor of Γ_{23}^{02} for R_3 which yields $R_i^2 = a_i^2 + A_i^2$ for $i = 2,3$. Now the resultant Γ_{23}^{04} and Γ_{23}^{13} with respect to a_3 can only vanish w.c. for $a_2 = \pm A_3$ over $\mathbb{R}$. In both cases the back substitution into Γ_{23}^{04} and Γ_{23}^{13} yields the contradiction.
3. $F_1 = 0$, $G = 0$: Now F_1 can only vanish for $e_3 = 0$ or $H = 0$. As it can easily be seen that $G = H = 0$ yields a contradiction, we are left with the case $H \neq 0$ and $e_3 = 0$. In this case Γ_{23} only depends on e_2. Therefore there can only be a self-motion if Γ_{23} is fulfilled independently of e_2.
   a. $A_2A_3 + a_3(A_2 - A_3) \neq 0$: Under this assumption we can express a_2 from $G = 0$. Then the coefficient of e_2^2 of Γ_{23} implies $A_2 = a_3$ and from the coefficient of e_2^0 of Γ_{23} we get $R_2^2 = R_3^2$. Now again $\Gamma_2 = \Gamma_3$ holds and we get the same contradiction as in item (2bi).
   b. $a_3 = A_2A_3/(A_3 - A_2)$: Now $G = 0$ cannot vanish w.c..
4. $F_2 = 0$: This can be done analogously to item (2) and item (3).

Part [B] $a_2A_2a_3A_3 = 0$:
First we assume $a_2 = A_2 = 0$. As a consequence $a_3A_3 \neq 0$ has to hold. Moreover, we get $\Gamma_2 = 4(e_1^2 + e_3^2) - R_2^2$. Therefore we can always express e_1 from Γ_2 and e_2 from Γ_1. Due to $a_3A_3 \neq 0$ we can also solve Γ_3 for e_0 which already shows that this cannot yield a self-motion.

Therefore we can assume w.l.o.g. that $A_2 = 0$ and $a_2A_3 \neq 0$ hold (under consideration of exchanging the platform and the base). We distinguish two cases:

1. $e_3 = 0$: For this condition we can always solve Γ_2 for e_1 and Γ_1 for e_2. Therefore we cannot get a self-motion in this case.
2. $e_3 \neq 0$: We have to distinguish two cases:
   a. $a_3 = 0$: W.l.o.g. we can compute Γ_{123}. Then the coefficient of e_3^6 can only vanish w.c. for $a_2 = -A_3$. Then the coefficient of e_3^4 yields the contradiction.
   b. $a_3 \neq 0$: We distinguish further two cases:
      i. $e_1 \neq \pm e_3$: Under this assumption the coefficient of e_2^2 of Γ_{23} cannot vanish. Therefore we can compute Γ_{123}. Now the coefficient of e_1^8 of Γ_{123} cannot vanish w.c..
      ii. $e_1 = \pm e_3$: In these two special cases the coefficient of e_2 of Γ_{23} can only vanish for $e_3^2(a_2 - a_3) + a_3(R_2^2 - a_2^2) = 0$. As this equation cannot be fulfilled independently of e_3 without yielding a contradiction, we can also compute Γ_{123} w.l.o.g.. Now the coefficient of e_1^8 of Γ_{123} cannot vanish w.c. over $\mathbb{R}$. This finishes the proof of Lemma 2. $\square$

Remark 3. It can be proven analogously, that general non-degenerated spherical 3-dof RPR mechanisms (platform anchor points and base anchor points have not to be located on great circles) can only have a self-motion if $\mathsf{m}_1^\circ = \mathsf{m}_3^\circ$ holds (after relabeling of anchor points and interchange of platform and base) and if the spherical lengths R_i° of the legs equal $R_1^\circ = \overline{\mathsf{M}_1^\circ \mathsf{M}_2^\circ}$, $R_2^\circ = \overline{\mathsf{m}_1^\circ \mathsf{m}_2^\circ}$, $R_3^\circ = \overline{\mathsf{M}_3^\circ \mathsf{M}_2^\circ}$. $\diamond$

Appendix B

By using the Study parameters $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$ to parametrize Euclidean displacements, Husty[5] showed that the condition for m_i to be located on a sphere with center M_i and radius R_i can be expressed by the following homogeneous quadratic equation:

$$
\begin{aligned}
\Lambda_i: \quad & (A_i^2 + B_i^2 + a_i^2 + b_i^2 - R_i^2)K + 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) \\
& + 2(e_3^2 - e_0^2)(A_i a_i + B_i b_i) + 2(e_2^2 - e_1^2)(A_i a_i - B_i b_i) \\
& + 4[(f_0 e_2 - e_0 f_2)(B_i - b_i) + (e_1 f_3 - f_1 e_3)(B_i + b_i) \\
& + (f_2 e_3 - e_2 f_3)(A_i + a_i) + (f_0 e_1 - e_0 f_1)(A_i - a_i) \\
& + e_0 e_3 (A_i b_i - B_i a_i) - e_1 e_2 (A_i b_i + B_i a_i)] = 0.
\end{aligned}
$$

Moreover, we abbreviate the difference $\Lambda_i - \Lambda_j$ by $\Lambda_{i,j}$ and denote the equation $\sum_{i=0}^{3} e_i f_i = 0$ of the Study quadric by Ψ. Based on this preparatory work we can prove the last sentence of Thm. 4:

[5] Husty, M.L.: An algorithm for solving the direct kinematics of general Stewart-Gough platforms, Mechanism and Machine Theory **31**(4) 365–380 (1996)

Proof. As the self-motion induced by the circular translation is trivially a pure translation, we only have to prove that the self-motions of the manipulators given by Eq. (5) are pure translations. This can be done by direct computations as follows:

We choose the y-axis of the moving and the fixed frame parallel to the direction d. Therefore the Schönflies motion group is determined by $e_1 = e_3 = 0$.

Due to the considerations given in Sec. 4 we can restrict ourselves to the motion implied by the regulus $\mathscr{R}$, which is determined by three pairwise distinct legs $\in \mathscr{R}$. W.l.o.g. we can choose a coordinate system in the moving system Σ such that the platform anchor points of the three legs are given by $\mathbf{m}_1 = (0,0)$ and $\mathbf{m}_j = (a_j, 0)$ for $j = 2,3$ and $0 \neq a_2 \neq a_3 \neq 0$. Moreover, we can assume that M_1 is located in the origin of the fixed system ($\Rightarrow p_{21} = p_{31} = 0$). Therefore we get $\mathbf{M}_1 = (0,0)$ and $\mathbf{M}_j = (p_{22}a_j, p_{32}a_j)$ for $j = 2,3$ with $p_{22} \neq 0$.

In the following we give the discussion of cases:

I. $p_{32} \neq 0$: Under this assumption we can solve Ψ and $\Lambda_{2,1}$ for f_0 and f_2. Plugging the obtained expressions into $\Lambda_{3,1}$ yields a homogeneous quadratic equation of the form $c_0e_0^2 + c_2e_2^2 = 0$ where the c_i's only depend on the geometry of the manipulator. Moreover, as $c_0 - c_2 = 4p_{22}a_2a_3(a_2 - a_3) \neq 0$ holds, $\Lambda_{3,1}$ cannot be fulfilled identically. Therefore this equation determines already the orientation of the platform, which is constant during the self-motion, as $c_0e_0^2 + c_2e_2^2 = 0$ do not depend on f_1 and f_3.
   Now we only have to show that we cannot get a two-dimensional self-motion, which corresponds to the fact that the remaining equation Λ_1 cannot be fulfilled identically. This can be seen as follows: The difference of the coefficients of f_1^2 and f_3^2 of Λ_1 splits up into:

$$16a_2^2(p_{22}(e_0 - e_2) - e_0 - e_2)(p_{22}(e_0 + e_2) - e_0 + e_2). \tag{9}$$

   a. $p_{22}(e_0 - e_2) - e_0 - e_2 = 0$: As $e_0 \neq e_2$ has to hold (otherwise we get $e_0 = e_2 = 0$, a contradiction) we can solve this equation for p_{22}. Moreover, we can express p_{32} from the only non-contradicting factor of the coefficient of f_1^2 of Λ_1, which yields:

$$\pm \frac{2e_0e_2}{(e_0 - e_2)\sqrt{-e_0^2 - e_2^2}}. \tag{10}$$

   As this cannot yield an real number for $e_0, e_2 \in \mathbb{R}$ we are done.
   b. $p_{22}(e_0 + e_2) - e_0 + e_2 = 0$: This can be done analogously to item (a).

II. $p_{32} = 0$: We have to distinguish the following two cases:

   a. $p_{22} \neq 1$: We distinguish further two cases:
      i. $e_0 \neq 0$: Under this assumption we can solve Ψ and $\Lambda_{2,1}$ for f_0 and f_1. Plugging the obtained expressions into $\Lambda_{3,1}$ yields again a homogeneous quadratic equation of the form $c_0e_0^2 + c_2e_2^2 = 0$ where the c_i's only depend on the geometry of the manipulator. Moreover, as again

$c_0 - c_2 = 4p_{22}a_2a_3(a_2 - a_3) \neq 0$ holds, $\Lambda_{3,1}$ cannot be fulfilled identically. Therefore this equation determines already the orientation of the platform, which is again constant during the self-motion.
Moreover, the coefficient of f_2^2 of Λ_1 factors into $16a_2^2(p_{22}-1)^2(e_0^2 + e_2^2) \neq 0$ and therefore we cannot obtain a two-dimensional self-motion.

ii. $e_0 = 0$ ($\Rightarrow e_2 = 1$): Now Ψ implies $f_2 = 0$.
  - ⋆ $p_{22} \neq -1$: Under this assumption we can express f_3 from $\Lambda_{2,1}$. Then $\Lambda_{3,1}$ only depends on the geometry of the manipulator and therefore this is a so-called *assembly condition*.
    Moreover, the coefficient of f_1^2 of Λ_1 factors into $16a_2^2(p_{22}+1)^2 \neq 0$ and therefore we cannot obtain a two-dimensional self-motion.
  - ⋆ $p_{22} = -1$ ($\Rightarrow$ platform and base are congruent): Now $\Lambda_{2,1}$ and $\Lambda_{3,1}$ are assembly conditions, which imply $R_1 = R_2 = R_3$. We are only left with $\Lambda_1 : 4(f_0^2 + f_1^2 + f_3^2) - R_1^2 = 0$, which yields the two-dimensional self-motion.

b. $p_{22} = 1$ ($\Rightarrow$ platform and base are congruent): We distinguish three cases:

i. $e_0e_2 \neq 0$: Under this assumption we can solve Ψ and $\Lambda_{2,1}$ for f_0 and f_3. Plugging the obtained expressions into $\Lambda_{3,1}$ yields again a homogeneous quadratic equation of the form $c_0e_0^2 + c_2e_2^2 = 0$ where the c_i's only depend on the geometry of the manipulator. Moreover, as again $c_0 - c_2 = 4a_2a_3(a_2 - a_3) \neq 0$ holds, $\Lambda_{3,1}$ cannot be fulfilled identically. Therefore this equation determines already the orientation of the platform, which is again constant during the self-motion.
Moreover, the coefficient of f_2^2 of Λ_1 factors into $64a_2^2e_2^2(e_0^2 + e_2^2) \neq 0$ and therefore we cannot obtain a two-dimensional self-motion.

ii. $e_2 = 0$ ($\Rightarrow e_0 = 1$): Now Ψ implies $f_0 = 0$. Then $\Lambda_{2,1}$ and $\Lambda_{3,1}$ are assembly conditions, which imply $R_1 = R_2 = R_3$. We are left with $\Lambda_1 : 4(f_1^2 + f_2^2 + f_3^2) - R_1^2 = 0$, which yields the two-dimensional self-motion.

iii. $e_0 = 0$ ($\Rightarrow e_2 = 1$): Now Ψ implies $f_2 = 0$. Moreover, we can solve $\Lambda_{2,1}$ for f_3 w.l.o.g.. Then $\Lambda_{3,1}$ is an assembly condition.
Moreover, the coefficient of f_0^2 of Λ_1 equals $64a_2^2 \neq 0$ and therefore we cannot obtain a two-dimensional self-motion. This finishes the proof of the last sentence of Thm. 4. □

GEORG NAWRATIL

Institute of Discrete Mathematics and Geometry,
Research Unit on Differential Geometry and Geometric Structures,
Vienna University of Technology,
Wiedner Hauptstrasse 8-10/104,
A-1040 Vienna, Austria

Phone (+43)-(0)1-58801/104362
Fax (+43)-(0)1-58801/10493
Email nawratil@geometrie.tuwien.ac.at