A global approach for the redefinition of higher-order flexibility and rigidity

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Introduction

Fundamentals

Graph G of a framework

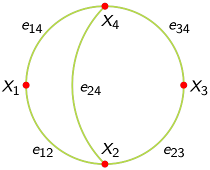
consists of a knot-set $\mathcal{K} = \{X_1, \ldots, X_w\}$, where knots X_i and X_j are connected by edges e_{ij} ($\Rightarrow$ combinatorial structure).

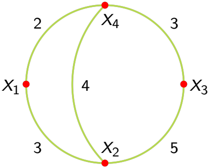
Inner geometry

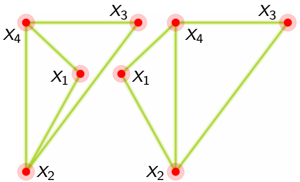
is determined by assigning to each edge e_{ij} a length $L_{ij} > 0$ ($\Leftrightarrow$ fixing intrinsic metric).

Realization $G(\mathbf{X})$

with $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_w) \in \mathbb{R}^{wd}$ corresponds to the embedding of the framework with fixed inner geometry into the Euclidean d-space. Let's assume $d = 2$.







Algebraic approach to rigidity theory

The relation that two knots X_i and X_j are edge-connected can also be expressed algebraically as $\|\mathbf{x}_i - \mathbf{x}_j\|^2 - L_{ij}^2 = 0$.

In addition we can add 3 linear conditions to eliminate isometries. We end up with l algebraic conditions in $m = 2w$ unknowns $z_1, \ldots, z_m$ constituting an algebraic variety $V(c_1, \ldots, c_l)$.

Def.: *A realization is* **flexible**

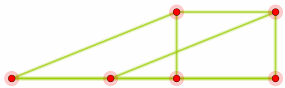
if it belongs to a real positive-dimensional component of $V(c_1, \ldots, c_l)$. For $l \geq m$ the motion is called **paradox**.

Def.: *A realization is* **rigid**

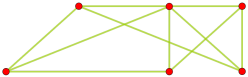
if it corresponds to a real isolated solution of $V(c_1, \ldots, c_l)$.

Algebraic approach to rigidity theory

Example: planar parallel mechanism



∃ paradox mobile realization



Rigid realization

Algebraic approach to rigidity theory

We can compute in a realization the tangent-hyperplane to each of the hypersurfaces $c_i = 0$ in $\mathbb{R}^m$ for $i = 1, \ldots, l$. The normal vectors ∇c_i of these tangent-hyperplanes constitute the columns of the $m \times l$ **rigidity matrix** $\mathbf{R}_{G(\mathbf{X})}$ of the realization $G(\mathbf{X})$; i.e.

$$\mathbf{R}_{G(\mathbf{X})} = (\nabla c_1, \nabla c_2, \ldots, \nabla c_l)$$

For $rk(\mathbf{R}_{G(\mathbf{K})}) = m$ the realization $G(\mathbf{K})$ is **infinitesimal rigid**.

For $rk(\mathbf{R}_{G(\mathbf{K})}) < m$ the realization $G(\mathbf{K})$ is **infinitesimal flexible**; i.e. the hyperplanes have a positive-dimensional affine subspace in common. Therefore the intersection multiplicity of the l hypersurfaces is at least two in an infinitesimal flexible realization.

The zero-set of the ideal generated by all $m \times m$ minors of $\mathbf{R}_{G(\mathbf{X})}$ is called **shakiness variety**.

Review

Review on higher-order flexibility and rigidity

1980 Connelly gave a definition of second-order flexibility and rigidity for frameworks.

Connelly, R.: The rigidity of certain cabled frameworks and the second-order rigidity of arbitrarily triangulated convex surfaces. Advances in Mathematics **37**:272–299 (1980)

1989 An exhaustive treatment of higher-order flexion and rigidity of surfaces was done by Sabitov, in which also a section is devoted to discrete structures.

Sabitov, I.Kh.: Local Theory of Bendings of Surfaces. Geometry III, 179–250, Springer (1992)

1989 Tarnai gave a definition of higher-order infinitesimal mechanisms relying on the power-series expansion of the elongation of the bar in terms of the displacement.

Tarnai, T.: Higher-order infinitesimal mechanisms. Acta technica Academiae Scientiarum Hungaricae **102**:363–378 (1989)

Review on higher-order flexibility and rigidity

According to Stachel* all these approaches to higher-order flexible frameworks can be unified to the so-called *classical* definition:

Classical Definition: *A framework has a* **flex of order n**

if for each vertex $\mathbf{x}_i$ $(i = 1, \ldots, w)$ there is a polynomial function

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1} t + \ldots + \mathbf{x}_{i,n} t^n \quad \text{with} \quad n > 0$$

such that

1. the replacement of $\mathbf{x}_i$ by $\mathbf{x}_i'$ in the equation of the edge lengths gives stationary values of multiplicity $\geq n+1$ at $t = 0$; i.e.

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 - L_{ij}^2 = 0 \quad \Longrightarrow \quad \|\mathbf{x}_i' - \mathbf{x}_j'\|^2 - L_{ij}^2 = o(t^n)$$

2. the velocity vectors $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ do not originate from a rigid body motion (incl. standstill) of the complete framework.

* Stachel, H.: A proposal for a proper definition of higher-order rigidity. Tensegrity Workshop, La Vacquerie, France (2007)

Review on higher-order flexibility and rigidity

Koiter's idea of replacing the bar elongation in Tarnai's approach by their strain energies, also results in an equivalent definition; cf.

Salerno, G.: How to recognize the order of infinitesimal mechanisms: a numerical approach. International Journal for Numerical Methods in Engineering **35**:1351–1395 (1992)

Another definition of higher-flexibility was given by Kuznetsov

Kuznetsov, E.N.: Underconstrained structural systems. Springer (1991)

which relies on the Taylor expansion of the constraint equations of the framework. Exactly the same approach was used by Chen

Chen, C.: The order of local mobility of mechanisms. Mechanism and Machine Theory **46**:1251–1264 (2011)

to define the local mobility of a mechanism. A closer look at

Rameau, J-F., Serre, P.: Computing mobility condition using Groebner basis. Mechanism and Machine Theory **91**:21–38 (2015)

shows that these (identical) definitions of Kuznetsov and Chen are again equivalent to the classical one.

Review on higher-order flexibility and rigidity

Based on the classical definition of n^{th}-order flex one can define n^{th}-order rigidity as follows according to Connelly & Servatius*:

Classical Definition: *A framework is* **rigid of order n**

if every n^{th}-order flex has $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ trivial as a first-order flex; i.e. it originates from a rigid body motion (incl. standstill) of the complete framework.

The **double-Watt mechanism** of Connelly & Servatius* raises some problems concerning these classical definitions, as they attest the mechanism in a certain configuration a 3rd-order rigidity which conflicts with its continuous flexibility; i.e. a proper definition should imply rigidity from n^{th}-order rigidity.

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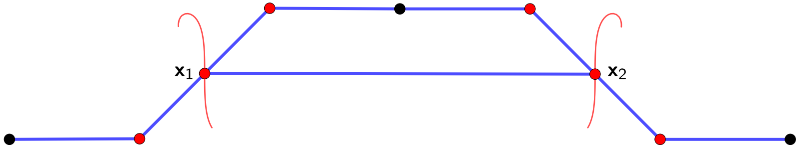
* Connelly, R., Servatius, H.: Higher-order rigidity – What is the proper definition? Discrete & Comp. Geometry **11**:193–200 (1994)

Double-Watt mechanism of Connelly & Servatius



The dimensions of each Watt mechanism:

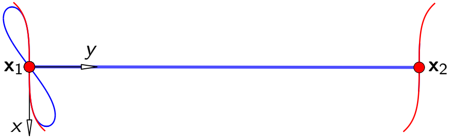
The arms have length 1 and the couplers length $\sqrt{2}$. The midpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ of both couplers are connected by a bar of length 3.

The problematic configuration corresponds to a cusp in the configuration space; i.e. the mechanism has an instantaneous standstill.

Further cusp mechanisms were given by Lopez-Custodio et al. in

Lopez-Custodio, P.C., Müller, A., Rico, J.M., Dai, J.S.: A synthesis method for 1-dof mechanisms with a cusp in the configuration space. Mechanism and Machine Theory **132**:154–175 (2019)

Analysis of cusp configuration according to Stachel*



The coupler-curve of the point $\mathbf{x}_1$ is given by the algebraic equation

$$x^6+3x^4y^2+3x^2y^4+y^6+3x^4+6x^3y-2x^2y^2+6xy^3-5y^4-6xy+8y^2=0.$$

The branch where the x-axis is the tangent to the inflection point can be parametrized locally by means of Puiseux series as:

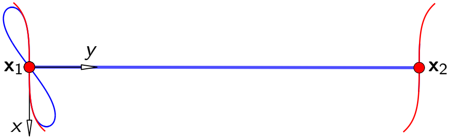
$$\mathbf{x}_1=\begin{pmatrix}\tau_1\\ \frac{1}{2}\tau_1^3+\tau_1^5+\frac{9}{4}\tau_1^7+\frac{13}{2}\tau_1^9+\ldots\end{pmatrix}$$

The path of $\mathbf{x}_2$ can be parametrized in the same way yielding:

$$\mathbf{x}_2=\begin{pmatrix}\tau_2\\ 3-\frac{1}{2}\tau_2^3-\tau_2^5-\frac{9}{4}\tau_2^7-\frac{13}{2}\tau_2^9-\ldots\end{pmatrix}$$

* Stachel, H.: A (3,8)-flexible bar-and-joint framework? AIM Workshop rigidity & polyhedral combinatorics, Palo Alto, USA (2007)

Analysis of cusp configuration according to Stachel



Two-point guidance problem, where the time dependence of τ_i is set up by

$$\tau_i = v_{i,1}t + v_{i,2}t^2 + v_{i,3}t^3 + \dots.$$

For an n-th order flex at $t = 0$ the $v_{i,j}$ have to be adjusted in order to fulfill

$$F := \|\mathbf{x}_2 - \mathbf{x}_1\|^2 - 3^2 = o(t^n)$$

We consider the coefficients f_i of t^i in F:

$$f_1 = 0$$

$$f_2 = (v_{1,1} - v_{2,1})^2 \quad \Longrightarrow \quad v_{2,1} = v_{1,1}$$

$$f_3 = -6v_{1,1}^3 \quad \xRightarrow{v_{1,1} \neq 0} \quad \text{2nd order flexible and 3rd-order rigid}$$

Attempts to resolve the dilemma: Gaspar & Tarnai

Gaspar and Tarnai suggested to use fractional exponents in

Gaspar, Z., Tarnai, T.: Finite mechanisms have no higher-order rigidity. Acta technica Academiae Scientiarum Hungaricae **106**:119–125 (1994)

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}t + \ldots + \mathbf{x}_{i,n}t^n \quad \Longrightarrow$$

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}t + \mathbf{x}_{i,\frac{3}{2}}t^{\frac{3}{2}} + \mathbf{x}_{i,2}t^2 + \mathbf{x}_{i,\frac{5}{2}}t^{\frac{5}{2}} \ldots + \mathbf{x}_{i,n}t^n$$

where $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ is non-trivial. This solves the particular problem of the double-Watt mechanism, but not the parametrization problem according to

Tarnai, T., Lengyel, A.: A remarkable structure of Leonardo and a higher-order infinitesimal mechanism. Journal of Mechanics of Materials and Structures **6**:591–604 (2011)

where it is also written that "*a very promising approach was presented recently by [Stachel 2007]*".

Attempts to resolve the dilemma: Stachel

Stachel's approach follows the more general notation of (k, n)-flexibility suggested by Sabitov and was presented in

Stachel, H.: A proposal for a proper definition of higher-order rigidity. Tensegrity Workshop, La Vacquerie, France (2007)

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1} t + \ldots + \mathbf{x}_{i,n} t^n \quad \Longrightarrow$$
$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,k} t^k + \ldots + \mathbf{x}_{i,n} t^n \quad \text{with} \quad n \geq k > 0$$

where $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{w,k}$ is non-trivial. In addition the (k, n)-flex has to be irreducible; this means that the flex does not result from a polynomial parameter substitution

$$t = \bar{t}^p (a_0 + a_1 \bar{t} + a_2 \bar{t}^2 + \ldots) \quad \text{with} \quad a_0 \neq 0 \quad \text{and} \quad p > 1$$

of a lower-order flex.

Continuing Stachel's analysis of the cusp mechanism

$$f_1 = 0$$

$$f_2 = (v_{1,1} - v_{2,1})^2 \quad \Longrightarrow \quad v_{2,1} = v_{1,1}$$

$$f_3 = -6v_{1,1}^3 \quad \xRightarrow{v_{1,1} \neq 0} \quad (1,2)\text{-flexible}$$

But we can also set $v_{1,1} = 0$ and continue

$$f_4 = (v_{1,2} - v_{2,2})^2 \quad \Longrightarrow \quad v_{2,2} = v_{1,2}$$

$$f_5 = 0$$

$$f_6 = -6v_{1,2}^3 + v_{1,3}^2 - 2v_{1,3}v_{2,3} + v_{2,3}^2 \quad \Longrightarrow \quad v_{2,3} = v_{1,3} \pm \sqrt{6v_{1,2}^3}$$

Remark: Sign $\pm$ corresponds to the two ways out of the cusp.

$$f_7 = \ldots \quad \Longrightarrow \quad (2,\infty)\text{-flexible}$$

But is the obtained $(2,\infty)$-flex irreducible?

Continuing Stachel's analysis of the cusp mechanism

We only have to check that the $(2,\infty)$-flex was not obtained by the $(1,2)$-flex by a polynomial parameter substitution of the form

$$t = \overline{t}^p(a_0 + a_1\overline{t} + a_2\overline{t}^2 + \dots) \quad \text{with} \quad a_0 \neq 0 \quad \text{and} \quad p > 1$$

For $p = 2$ we get:

$$\begin{aligned}
&\overline{f}_1 = \overline{f}_2 = \overline{f}_3 = 0 \\
&\overline{f}_4 = a_0^2(v_{1,1} - v_{2,1})^2 \quad \Longrightarrow \quad v_{2,1} = v_{1,1} \\
&\overline{f}_5 = 0 \\
&\overline{f}_6 = -6a_0^3 v_{1,1}^3 \quad \xRightarrow{v_{1,1} \neq 0} \quad (2,5)\text{-flexible}
\end{aligned}$$

$\Longrightarrow$ Substitution converts the $(1,2)$-flex into a reducible $(2,5)$-flex.

$\Longrightarrow$ **$(2,\infty)$-flex is irreducible.**

Remark: Applying this parameter substitution to Gaspar & Tarnai's ansatz of fractional exponents yields the $(2,\infty)$-flexibility of Stachel.

A new dilemma arises

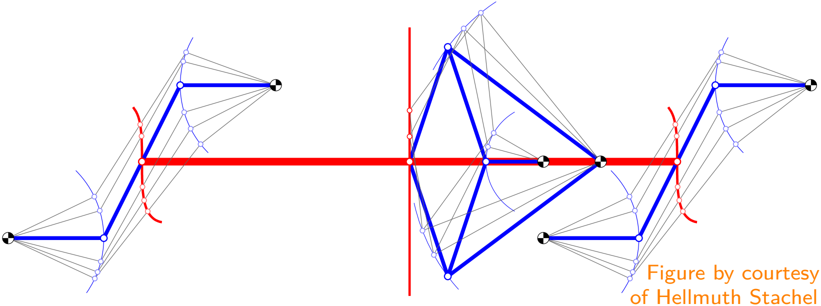
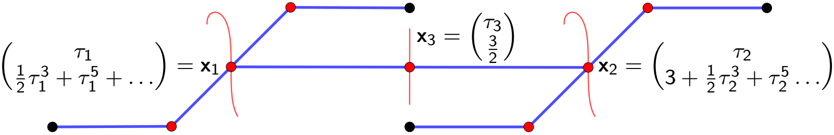


Figure by courtesy of Hellmuth Stachel

Stachel's proposal was only presented at the Tensegrity Workshop in 2007. It remained unpublished as another dilemma arose; namely no unique flexion order can be identified for another double-Watt mechanism extended by a Kempe-mechanism as demonstrated in

Stachel, H.: A (3,8)-flexible bar-and-joint framework? AIM Workshop rigidity & polyhedral combinatorics, Palo Alto, USA (2007)

Stachel's extended double-Watt mechanism



We do not use a Kempe-mechanism for the generation of the straight line motion of the midpoint $\mathbf{x}_3$ of $\mathbf{x}_1$ and $\mathbf{x}_2$ but a point guidance*. For an n-th order flex at $t = 0$ the following conditions have to hold:

$$F := \|\mathbf{x}_2 - \mathbf{x}_1\|^2 - 3^2 = o(t^n)$$
$$\begin{pmatrix} G \\ H \end{pmatrix} := \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 = \mathbf{o}(t^n)$$

by adjusting the $v_{i,j}$ of $\tau_i = v_{i,1}t + v_{i,2}t^2 + v_{i,3}t^3 + \ldots$.

*This can also be interpreted in the terms of a bar-joint framework, where the corresponding pin-joint is an ideal point.

Stachel's extended double-Watt mechanism

We consider the coefficients f_i, g_i and h_i of t^i in F, G and H.

$$g_i = v_{1,i} + v_{2,i} - 2v_{3,i} \quad \Longrightarrow \quad v_{3,i} = \tfrac{v_{1,i}+v_{2,i}}{2} \quad \text{for} \quad i = 1, 2, \ldots$$

$$f_1 = h_1 = h_2 = 0$$

$$f_2 = (v_{1,1} - v_{2,1})^2 \quad \Longrightarrow \quad v_{2,1} = v_{1,1}$$

$$f_3 = 0, \quad h_3 = v_{1,1}^3 \quad \xRightarrow{v_{1,1} \neq 0} \quad (1,2)\text{-flexible}$$

But we can also set $v_{1,1} = 0$ and continue

$$f_4 = (v_{1,2} - v_{2,2})^2 \quad \Longrightarrow \quad v_{2,2} = v_{1,2}$$

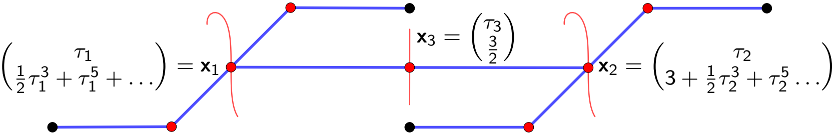
$$h_4 = f_5 = h_5 = 0$$

$$f_6 = (v_{1,3} - v_{2,3})^2 \quad \Longrightarrow \quad v_{2,3} = v_{1,3}$$

$$h_6 = v_{1,2}^3 \quad \xRightarrow{v_{1,2} \neq 0} \quad (2,5)\text{-flexible}$$

But we can also set $v_{1,2} = 0$ and continue ...

Stachel's extended double-Watt mechanism



This procedure yields the flexion order sequence $(k, 3k-1)$, $k = \mathbb{N}$. It can be shown that all the obtained orders are irreducible.

According to Stachel the following question remained open:
Which is the correct order?

Therefore the problem is not yet settled!

Redefinition

Redefinition of a higher-order flexibility and rigidity

Cusp mechanisms demonstrate the basic shortcoming of any local mobility analysis using higher-order constraints according to

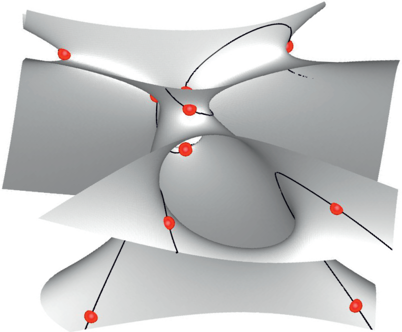
Müller, A.: Higher-order analysis of kinematic singularities of lower pair linkages and serial manipulators. Journal of Mechanisms and Robotics **10**:011008 (2018)

Therefore we present a global approach, which is also inspired by an idea of Sabitov like Stachel's approach; namely by his finite algorithm for testing the bendability of a polyhedron given in

Sabitov, I.Kh.: Local Theory of Bendings of Surfaces. Geometry III, 179–250, Springer (1992)

Let us consider the configuration-set $\mathcal{S}$ of all frameworks having the same connectivity which differ in the intrinsic metric. Note that $\mathcal{S}$ is only a subset of $\mathbb{R}^m$ (as edges are not allowed to have zero length).

Redefinition of a higher-order flexibility and rigidity



$\mathcal{S}_1 \ldots$ surface
$\mathcal{S}_2 \ldots$ curve
$\mathcal{S}_3 \ldots$ isolated points

Let $\mathcal{S}_1 \subset \mathcal{S}$ be the set of points of the already discussed shakiness variety.

Then the sets $\mathcal{S}_j$ with $j > 1$ are defined recursively as follows:
If in a point of $\mathcal{S}_{j-1}$ a non-trivial first order flex exists, which is tangential to $\mathcal{S}_{j-1}$ then this point belongs to the set $\mathcal{S}_j$.
$\Longrightarrow$ hierarchical structure of flexibility of higher-order.

A configuration is called n^{th}-order flexible if it belongs to $\mathcal{S}_n \setminus \mathcal{S}_{n+1}$.

Redefinition of a higher-order flexibility and rigidity

Remarks:

- This approach goes along with a recent result of
  Alexandrov, V.: A note on the first-order flexes of smooth surfaces which are tangent to the set of all nonrigid surfaces. Journal of Geometry **112**:41 (2021)
  for smooth surfaces, who was able to show that a first-order flex tangential to $\mathcal{S}_1$ can be extended to a second-order flex.
- Sabitov assumed that all the appearing sets $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2, \ldots$ are manifolds and submanifolds, respectively.

  A analogous assumption has to be done by Alexandrov in the smooth setting, namely the restriction to regular points of $\mathcal{S}_1$.

  In general $\mathcal{S}_i$ contains singular points, which correspond to the interesting configurations in the study of higher-order flexes.

Redefinition of a higher-order flexibility and rigidity

This approach gives a proper definition of n^{th}-order flexibility for configurations that correspond to points of $\mathbb{R}^m$ which are regular with respect to each of the corresponding varieties $V_1, V_2, \ldots, V_n$.

Lemma

Every regular point of V_1 has to have a single non-trivial flex.

Proof: V_1 is the zero set of the ideal generated by $p_1, \ldots, p_\gamma$, which are all $m \times m$ minors of $\mathbf{R}_{G(\mathbf{X})}$. Let us denote $p_j = \det(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m)$, thus its gradient

$$\nabla p_j = \left(\frac{\partial p_j}{\partial z_1}, \frac{\partial p_j}{\partial z_2}, \ldots, \frac{\partial p_j}{\partial z_m} \right),$$

can be computed due to the following product rule for determinants:

$$\frac{\partial p_j}{\partial z_i} = \det\left(\frac{\partial \mathbf{r}_1}{\partial z_i}, \mathbf{r}_2, \ldots, \mathbf{r}_m\right) + \ldots + \det\left(\mathbf{r}_1, \mathbf{r}_2, \ldots, \frac{\partial \mathbf{r}_m}{\partial z_i}\right).$$

$\Longrightarrow$ point $\mathbf{X}$ of V_1 with $rk(\mathbf{R}_{G(\mathbf{X})}) < m - 1$ implies $\nabla p_j = \mathbf{o}\ \forall j$

$\Longrightarrow$ $\mathbf{X}$ singular point of V_1. $\square$

Redefinition of a higher-order flexibility and rigidity

This Lemma explains Husty's observation for 3-RPR robots given in

Husty, M.L.: On Singularities of Planar 3-RPR Parallel Manipulators. Proceedings of 14th IFToMM World Congress, pp. 2325–2330, IFToMM (2015)

namely "*the surprising property that it* (singularity surface) *has a singularity itself at the point which corresponds to the pose with two dof local mobility*."

Note that it is well known that there exists for each geometric structure an **upper bound** n^* such that the n^*-order flexibility results in a continuous flexion; e.g. see

Sabitov, I.Kh.: Local Theory of Bendings of Surfaces. Geometry III, 179–250, Springer (1992)

Redefinition of a higher-order flexibility and rigidity

Theorem

The n^{th}-order flexibility with $n < n^*$ of a configuration which corresponds to a regular point of each variety $V_1, \ldots, V_n$ is equivalent with the fact that it is a realization of multiplicity $n+1$.

Proof: In a general point $\mathbf{X}$ of V_1 the intersection multiplicity of $V_1 \cap c_1 \cap \ldots \cap c_l$ is 1.
For increasing it a necessary and sufficient condition is that the tangent spaces have a positive-dimensional subspace in common. This is exactly the condition that in $\mathbf{X}$ a 1-dim flex (according to the Lemma) exists, which is tangential to $V_1 \Longrightarrow \mathbf{X} \in V_2$
This line of argumentation can be iterated until we reach the set V_{n^*}, which consists of points having multiplicity ∞.
Thus points of $V_n \setminus V_{n+1}$ with $n < n^*$ have to correspond with realizations of multiplicity $n+1$. $\square$

Redefinition of a higher-order flexibility and rigidity

A redefinition can be based on this property as it can also be extended to singular points of the varieties $V_1, V_2, \ldots$ which are not covered by Sabitov's algorithm.

Redefinition of higher-order flexibility

If a configuration does not belong to a continuous flexion of the framework then we define its order of flexibility by the number of coinciding framework realizations minus 1.

Based on this definition we can also give a redefinition of higher-order rigidity as follows:

Redefinition of higher-order rigidity

Is a configuration n^{th}-order flexible according to the definition above then it is $(n+1)$-rigid.

3-step algorithm for computing the flexion order

1. According to the Lasker–Noether theorem every algebraic set is the union of a finite number of uniquely defined algebraic sets known as irreducible components. They can be computed with an irredundant primary decomposition algorithm.
2. Then one has to test if the given realization is contained in a irreducible composition of dimension 1 or higher. If this is the case the configuration **X** is assigned with the flexion order ∞. If this is not the case then we identify all zero-dimensional primary ideals $I_1, \ldots, I_s$ containing **X**.
3. We compute the intersection multiplicity q_i of **X** with respect to each primary ideal I_i for $i = 1, \ldots, s$.
   $\Longrightarrow$ flexion order equals $q_1 + \ldots + q_s - 1$

Computation of the intersection multiplicity q_i

Let us assume that the zero-dimensional primary ideal I_i is generated by polynomials $g_1, \ldots, g_k$. We distinguish the following two cases:

a) If $k = m$; i.e. I_i is a complete intersection, then we can use the U-resultant method, which works as follows: One adds the so-called U-polynomial

$$g_0 = u_0 + u_1 z_1 + \ldots + u_m z_m$$

to the set $g_1, \ldots, g_m$ and eliminates $z_1, \ldots, z_m$ by means of Macaulay resultant. This results in a homogeneous polynomial

$$\prod_j (\zeta_{j,0} u_0 + \zeta_{j,1} u_1 + \ldots + \zeta_{j,m} u_m)^{q_j} .$$

Then the jth common point of $g_1, \ldots, g_m$ has multiplicity q_j and his coordinates are given by $z_i = \zeta_{j,i}/\zeta_{j,0}$ for $i = 1, \ldots, m$.

Computation of the intersection multiplicity q_i

Let us assume that the zero-dimensional component I_i is generated by polynomials $g_1, \ldots, g_k$. We distinguish the following two cases:

a) If $k = m$; i.e. I_i is a complete intersection, then we can use the U-resultant method; cf.

Macaulay, F.S.: The Algebraic Theory of Modular Systems. Cambridge University Press (1916)

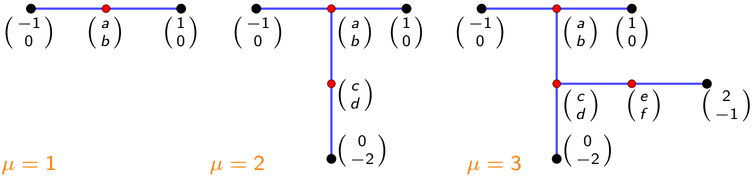
b) If $k > m$ one can use a generalization of the U-resultant method given by Lazard

Lazard, D.: Solving Systems of Algebraic Equations. ACM SIGSAM Bulletin **35**:11–37 (2001)

to end up with an expression of the form

$$\prod_j (\zeta_{j,0} u_0 + \zeta_{j,1} u_1 + \ldots + \zeta_{j,m} u_m)^{q_j} .$$

Tarnai's* Leonardo structure of $(2^\mu - 1)$-order flex



$\mu = 1$
$\mathbf{X} = (1:0:0)$
$(h:a:b)$

$\mu = 2$
$\mathbf{X} = (1:0:0:0:-1)$
$(h:a:b:c:d)$

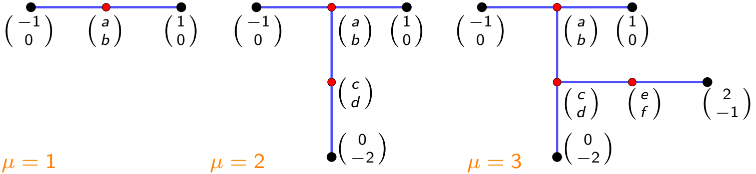
$\mu = 3$
$\mathbf{X} = (1:0:0:0:-1:1:-1)$
$(h:a:b:c:d:e:f)$

The primary decomposition of the ideal of constraint equations only contains one 0-dim component containing the configuration $\mathbf{X}$:

μ	I(Maple 2022)	U-resultant (Macaulay2)
1	$\langle a, b^2 \rangle$	u_0^2
2	$\langle a, b^2, b-2d-2h, bh+c^2 \rangle$	$(u_0-u_4)^4$
3	$\langle a, b, c^2, c-2e+2h, 2e-c+2d,$	$(u_0-u_4+u_5-u_6)^4$
	$e^2+2ef+f^2-2ce-cf \rangle$	wrong!

* Tarnai, T.: Higher-order infinitesimal mechanisms. Acta technica Academiae Scientiarum Hungaricae **102**:363–378 (1989)

Tarnai's Leonardo structure of $(2^\mu - 1)$-order flex



$\mu = 1$
$\mathbf{X} = (1:0:0)$
$(h:a:b)$

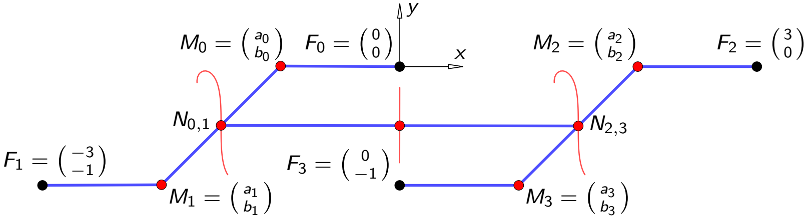
$\mu = 2$
$\mathbf{X} = (1:0:0:0:-1)$
$(h:a:b:c:d)$

$\mu = 3$
$\mathbf{X} = (1:0:0:0:-1:1:-1)$
$(h:a:b:c:d:e:f)$

The primary decomposition of the ideal of constraint equations only contains one 0-dim component containing the configuration $\mathbf{X}$:

μ	I(Maple 2022)	U-resultant (Macaulay2)
1	$\langle a, b^2 \rangle$	u_0^2
2	$\langle a, b^2, b - 2d - 2h, bh + c^2 \rangle$	$(u_0 - u_4)^4$
3	$\langle a, b^2, c^2, c - 2e + 2h, 2e - c + 2d,$	$(u_0 - u_4 + u_5 - u_6)^8$
	$e^2 + 2ef + f^2 - 2ce - cf \rangle$	corrected

Stachel's extended double-Watt mechanism



Using the above coordinatization the 8 constraining equations are:

$$\begin{aligned}
&\|F_i - M_i\|^2 = 1 && i = 0, \dots, 3 \\
&\|M_j - M_{j+1}\|^2 = 2 && j = 0, 2 \\
&\|N_{0,1} - N_{2,3}\|^2 = 9 && \\
&a_0 + a_1 + a_2 + a_3 = 0 &&
\end{aligned}$$

The last equation corresponds with the straight line motion of the midpoint of $N_{0,1} := \frac{M_0+M_1}{2}$ and $N_{2,3} := \frac{M_2+M_3}{2}$.

Stachel's extended double-Watt mechanism

The primary decomposition (operated by MAPLE 2022) yields only one zero-dimensional component I containing the configuration

$$\mathbf{X}: \quad (a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3) = (-1, 0, -2, -1, 2, 0, 1, -1)$$

$$\begin{aligned} I = \langle &(1+a_0)^2, (a_2-2)^2, (3-a_2)^2 + b_2^2 - 1, (a_0+3)a_1 + 5 + (b_0+1)b_1, \\ &(a_3-3)a_2 + 5 + (b_2+1)b_3, a_1 + a_0 + a_3 + a_2, a_0^2 + b_0^2 - 1, \\ &(6-2a_3)a_2 + a_3^2 - 2b_2b_3 + b_3^2 - 10, a_1^2 - 2a_0a_1 - 2b_0b_1 + b_1^2 - 1, \\ &(a_0 - a_2 - a_3 - 3)a_1 + (3 - a_0 + a_3)a_2 + (b_0 - b_2 - b_3 - 1)b_1 + \\ &(b_2 - b_0 - 1)b_3 - a_0a_3 - b_0b_2 - 26\rangle \end{aligned}$$

U-resultant method is not possible as I has more than 8 generators.

$\Longrightarrow$ generalized U-resultant method of Lazard

But we are not aware of any implementation.

Stachel's extended double-Watt mechanism

Therefore we proceed as follows:
The ideal I only has the solution $\mathbf{X}$ and we determined its multiplicity by the Maple command `NumberOfSolutions` yielding 6.

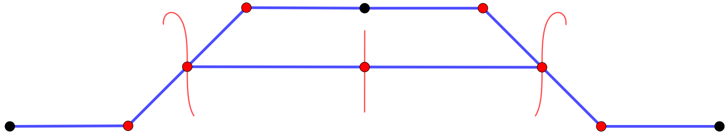
As one cannot trust for sure the `PrimaryDecomposition` command in MAPLE 2022 as demonstrated in the example of Tarnai's Leonardo structure, we also did the following recheck according to

Weil, A.: Foundation of algebraic geometry, American Mathematical Society (1946)

By a slight perturbation of the system of equations one also obtains 6 solutions in the neighborhood of $\mathbf{X}$.

According to the given redefinition this implies flexion order 5.
$\Longrightarrow k = 2$ in Stachel's flexion order sequence $(k, 3k - 1)$

Extension of the original double-Watt mechanism



Stachel's approach yields the sequence of flexion orders:

$$(k, 3k-1) \qquad \text{for odd } k$$

$$(k, 3k+\tfrac{k}{2}-1) \qquad \text{for even } k$$

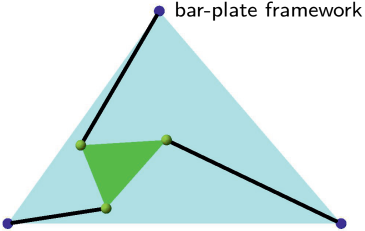
Perturbation approach* shows 7 coinciding realizations.

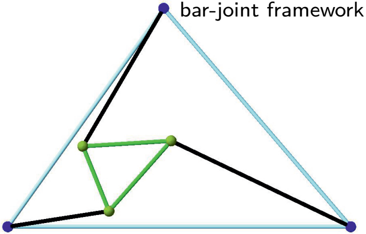
According to the given redefinition this implies flexion order 6.
$\Longrightarrow k = 2$ in Stachel's flexion order sequence

*`PrimaryDecomposition` command in MAPLE 2022 gives again a wrong result (6-fold solution). The `IntersectionMultiplicity` command of Maple fails for all possible $8! = 40320$ permutations.

Application

Computing configurations with a higher-order flexion





We do this exemplarily for a planar 3-RPR manipulators, motivated by the following statement of Husty in

Husty, M.: Multiple Solutions of Direct Kinematics of 3-RPR Parallel Manipulators. Proceedings of 16th IFToMM World Congress, pp. 599–608, Springer (2023)

that 3rd-order flexibility "*can be reached by any design because the three necessary conditions could be imposed on the input parameters only. Unfortunately neither the conditions nor the number of corresponding poses are known*".

Review: 3-RPRs with a higher-order flexion

Wohlhart followed a kinematic version of Kuznetsov's approach in

Wohlhart, K.: Degrees of shakiness. Mechanism Machine Theory **34**:1103–1126 (1999)

for the study of higher-order flexible 3-RPRs (interpreted as bar-plate frameworks). Stachel studied the geometry of higher-order flexible 3-RPRs (interpreted as bar-joint frameworks) in

Stachel, H.: Infinitesimal flexibility of higher order for a planar parallel manipulator. Topics in Algebra, Analysis and Geometry, 343–353, BPR Kiadó (1999)

where the following result for a $(1, n)$-flexible configuration is shown:
If one disconnects the leg $M_i m_i$ from the platform, then the trajectory of the point m_i under the resulting four bar motion has n^{th}-order contact with the circle centered in M_i having radius r_i.
This implies that $(n + 1)$ realizations coincide. Based on this characterization the computation of 3-RPR configurations (interpreted as bar-plate frameworks) with 5th order flex was given by Husty

Husty, M.: Multiple Solutions of Direct Kinematics of 3-RPR Parallel Manipulators. Proceedings of 16th IFToMM World Congress, pp. 599–608, Springer (2023)

3-RPR manipulator as bar-plate framework

We compute V_1 using the approach given by Husty and Gosselin in Husty, M., Gosselin, C.: On the Singularity Surface of Planar 3-RPR Parallel Mechanisms. Mechanics Based Design of Structures and Machines **36**:411–425 (2008)

Using Blaschke-Grünwald parameters $(q_0 : q_1 : q_2 : q_3)$ the condition that a point m_i with coordinates (a_i, b_i) w.r.t. the moving frame is located on a circle with radius r_i around the fixed point M_i with coordinates (A_i, B_i) w.r.t. the fixed frame, can be written as:

$$
\begin{aligned}
c_i :=\,& 2A_i a_i q_1^2 - 2A_i a_i q_0^2 + 4A_i b_i q_0 q_1 - 4B_i a_i q_0 q_1 - 2B_i b_i q_0^2 + 2B_i b_i q_1^2 + \\
& (a_i^2 + b_i^2)(q_0^2 + q_1^2) - 4A_i q_0 q_3 - 4A_i q_1 q_2 + 4B_i q_0 q_2 - 4B_i q_1 q_3 + \\
& 4a_i q_0 q_3 - 4a_i q_1 q_2 - 4b_i q_0 q_2 - 4b_i q_1 q_3 + A_i^2 + B_i^2 + 4q_2^2 + 4q_3^2 - r_i^2
\end{aligned}
$$

Then the framework realizations are obtained as the solutions of

$$
c_1 = c_2 = c_3 = c_4 = 0 \quad \text{with} \quad c_4 := q_0^2 + q_1^2 - 1.
$$

3-RPR manipulator as bar-plate framework

V_1 is obtained as the zero-set of $s := \det\left(\mathbf{R}_{G(\mathbf{x})}\right)$ with

$$\mathbf{R}_{G(\mathbf{x})} = (\nabla c_1, \nabla c_2, \nabla c_3, \nabla c_4)$$

Beside the parametrization singularities (double line $q_0 = q_1 = 0$) V_1 has only singularities for some special designs according to

Kapilavai, A., Nawratil, G.: Singularity Distance Computations for 3-RPR Manipulators using Extrinsic Metrics. Mechanism and Machine Theory **195**:105595 (2024)

In the generic case each point of V_1 sliced along the line $q_0 = q_1 = 0$ is regular $\xRightarrow{\text{Lem.}}$ tangent planes to $c_1, \ldots, c_4$ have a line in common. The orthogonality of this line to ∇s is equivalent to the condition

$$rk(\nabla c_1, \nabla c_2, \nabla c_3, \nabla c_4, \nabla s) = 3 \quad \Longrightarrow$$

$$s_1 := \det(\nabla c_2, \nabla c_3, \nabla c_4, \nabla s) \qquad s_2 := \det(\nabla c_1, \nabla c_3, \nabla c_4, \nabla s)$$
$$s_3 := \det(\nabla c_1, \nabla c_2, \nabla c_4, \nabla s) \qquad s_4 := \det(\nabla c_1, \nabla c_2, \nabla c_3, \nabla s)$$

3-RPR manipulator as bar-plate framework

Thus V_2 is the zero set of the ideal

$$I_2 = \langle s, s_1, s_2, s_3, s_4 \rangle.$$

Iteration of the above procedure yields the conditions:

$$s_{1,i} := \det(\nabla c_2, \nabla c_3, \nabla c_4, \nabla s_i) \qquad s_{2,i} := \det(\nabla c_1, \nabla c_3, \nabla c_4, \nabla s_i)$$
$$s_{3,i} := \det(\nabla c_1, \nabla c_2, \nabla c_4, \nabla s_i) \qquad s_{4,i} := \det(\nabla c_1, \nabla c_2, \nabla c_3, \nabla s_i)$$

for $i = 1, \ldots, 4$. Then V_3 is the zero set of the ideal

$$I_3 = \langle s, s_1, s_2, s_3, s_4, s_{1,1}, \ldots, s_{4,4} \rangle.$$

In addition the singular points of V_2 have to be considered separately. As V_2 is a curve in P^3 a singularity corresponds to the case

$$rk(\nabla s, \nabla s_1, \nabla s_2, \nabla s_3, \nabla s_4) = 1.$$

Example: 3-RPR manipulator as bar-plate framework

The geometry of the platform and base is given by:

$$A_1 = 0, \quad B_1 = 0, \quad A_2 = 3, \quad B_2 = 0, \quad A_3 = 1, \quad B_3 = 3,$$
$$a_1 = 0, \quad b_1 = 0, \quad a_2 = 1, \quad b_2 = 0, \quad a_3 = 2, \quad b_3 = 1.$$

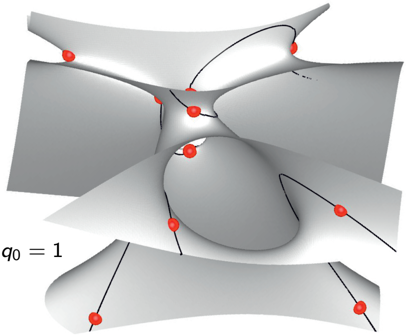
For this values we obtain V_1 by $s = 0$ with

$$s = 5q_0^3 q_2 - 13q_0^2 q_1 q_2 - 4q_0^2 q_1 q_3 + 5q_0^2 q_2^2 + 7q_0 q_1^2 q_2 + 11q_0 q_1^2 q_3 - 6q_0 q_1 q_2^2 - 6q_0 q_1 q_3^2 + 10q_1^3 q_3 - 5q_1^2 q_3^2.$$

In the next step we consider the ideal I_2. It can be verified that V_2 is a curve in P^3 of degree 18, which splitu up into a curve g of degree 14 and the line $q_0 = q_1 = 0$ of multiplicity 4. It can easily be checked that g does not contain any singular points by applying the criterion

$$rk(\nabla s, \nabla s_1, \nabla s_2, \nabla s_3, \nabla s_4) = 1.$$

Example: 3-RPR manipulator as bar-plate framework



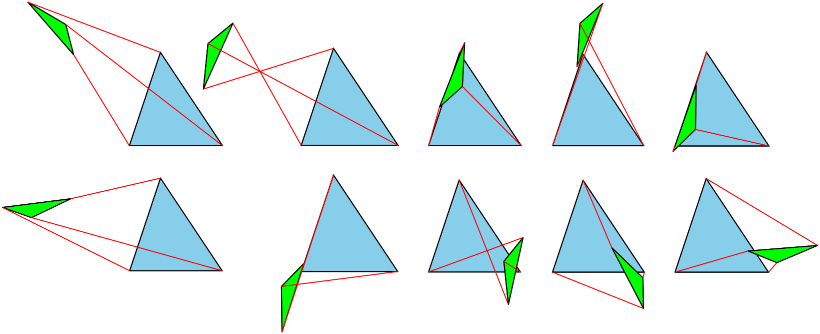
In the last step we consider the ideal I_3. V_3 consists of the line $q_0 = q_1 = 0$ with multiplicity 3 and **32 isolated solutions**.

We can even eliminate q_0 and q_3 from the generators of I_3 to end up with the corresponding polynomial of degree 32:

$$5169694889612648582969770 44q_1^{32} - 92803092139877774194843805 70q_1^{31}q_2 + 4352627023211727183455650 2073q_1^{30}q_2^2 - 45280692730479399589412412168q_1^{29}q_2^3 - \ldots\ldots - 874805860916262711853056q_2^{32} = 0$$

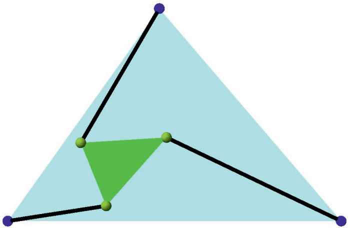
By setting $q_1 = 1$ we can easily check that it has 10 real solutions.

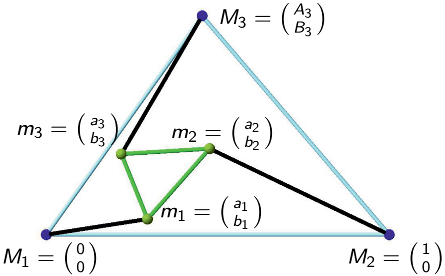
Example: 3-RPR manipulator as bar-plate framework



Remark: It remains unclear if examples with 32 real solutions exist.

3-RPR manipulator as bar-joint framework





A bar-plate framework has 6 realizations but a bar-joint framework has 24, as the platform triangle as well as the base triangle can flip.

Without loss of generality we can assume that the bar between M_1 and M_2 has length one. If the remaining 8 bar lengths are known they imply 8 distance equations $c_1, \ldots, c_8$ for the 8 unknowns. The solutions of $c_1, \ldots, c_8$ correspond to realizations.

3-RPR manipulator as bar-joint framework

V_1 is given by the determinant of the (8×8) rigidity matrix splitting up into $s_1 s_2 s_3$ with:

$$
\begin{aligned}
s_1 =& B_3, \\
s_2 =& a_1 b_2 - a_1 b_3 - a_2 b_1 + a_2 b_3 + a_3 b_1 - a_3 b_2, \\
s_3 =& A_3 a_1 b_2 b_3 - A_3 a_2 b_1 b_3 - B_3 a_1 a_3 b_2 + B_3 a_2 a_3 b_1 - \\
& A_3 b_1 b_2 + A_3 b_1 b_3 + B_3 a_1 b_2 - B_3 a_3 b_1 - a_1 b_2 b_3 + a_3 b_1 b_2
\end{aligned}
$$

Let us denote the varieties $s_i = 0$ by S_i for $i = 1, 2, 3$. Now we can easily identify the following regions of V_1 with different n^* values:

$S_1 \setminus (S_2 \cup S_3)$	$n^* = 2$	$(S_1 \cap S_3) \setminus S_2$	$n^* = 12$
$S_2 \setminus (S_1 \cup S_3)$	$n^* = 2$	$(S_2 \cap S_3) \setminus S_1$	$n^* = 12$
$S_3 \setminus (S_1 \cup S_2)$	$n^* = 6$	$S_1 \cap S_2 \cap S_3$	$n^* = 24$
$(S_1 \cap S_2) \setminus S_3$	$n^* = 4$		

3-RPR bar-joint framework of flexion order 23

Let us assume that the platform and the base triangles degenerate into lines l and L, respectively. A necessary condition for a configuration of flexion order 23, is that l and L coincide.

Remark: Interestingly such a configuration is not only a singular point of V_1, as it is located in the intersection of S_1, S_2 and S_3 but already a singular point of S_3 according to

Kapilavai, A., Nawratil, G.: Singularity Distance Computations for 3-RPR Manipulators using Extrinsic Metrics. Mechanism and Machine Theory **195**:105595 (2024)

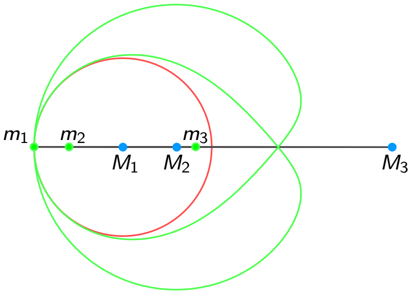
Thus a 23-order flexible bar-joint framework follows from a fifth-order flexible plate-bar framework, where all six anchor points are located on a line. This problem can be solved with Husty's approach

Husty, M.: Multiple Solutions of Direct Kinematics of 3-RPR Parallel Manipulators. Proceedings of 16th IFToMM World Congress, pp. 599–608, Springer (2023)

Final example

Coordinate of the base points w.r.t. the fixed frame:

$$M_1 = (0,0)^T,$$
$$M_2 = (1,0)^T,$$
$$M_3 = (5,0)^T.$$

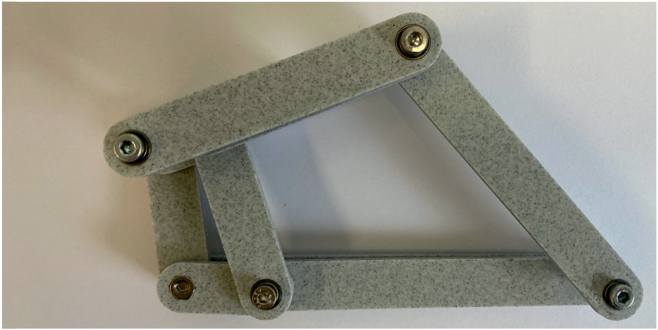


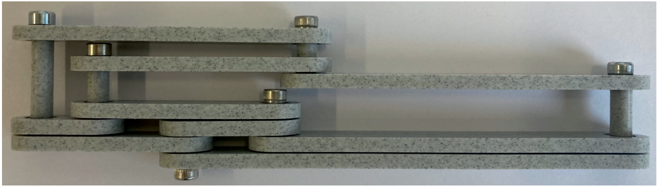
Coordinates of the platform points w.r.t. the fixed frame:

$$m_1 = \left(\frac{\sqrt{120\sqrt{10}-255}}{10} - \frac{3}{2} - \frac{2\sqrt{10}}{5}, 0\right)^T, \quad m_2 = (-1,0)^T, \quad m_3 = m_1 + (3,0)^T.$$

Remark: It should be possible to determine the complete set of these frameworks with flexion order 23.

Final example





Acknowledgment: Thanks to Daniel Huczala for the production of the corresponding model.

Open problems and future work

- It is planned to compute Stewart-Gough configurations (interpreted as bar-body frameworks) with flexion order 6 by the iterative procedure. What is the maximal flexion order?
- The presented approach does not only work for bar-joint frameworks but it can be applied to any framework with algebraic joints. But it remains open to extend it to frameworks with non-algebraic joints?
- The given algorithm for determining the flexion order requires global constructions (primary decomposition, U-resultant method), but the multiplicity is a local property according to

  Kirby, D.: Multiplicity in Algebra and Geometry. Arab Journal of Mathematical Sciences **1**:55–63 (1995)

  Therefore again one can think about using local methods (e.g. Serre's Tor formula) to determine this number. It remains open if these local methods can also detect a continuous flexion and if they work in all cases (like our approach).

Thanks