Singularity Distance Computation for Parallel Manipulators of Stewart Gough Type

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Introduction

The number of applications of parallel robots has increased enormously during the last decades due to their advantages of high speed, stiffness, accuracy, load/weight ratio, etc.

One of the drawbacks of these parallel robots are their singular configurations, where the manipulator has at least one uncontrollable instantaneous degree of freedom. Furthermore, the actuator forces can become very large, which may result in a breakdown of the mechanism.

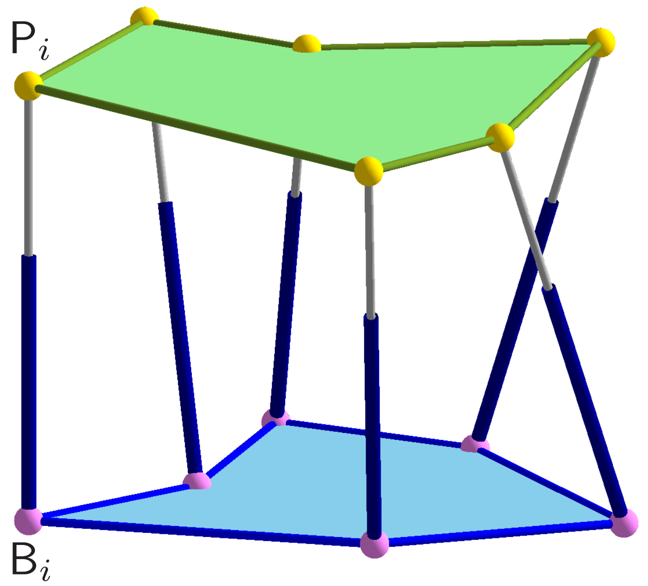
Introduction

We focus on the following three robot architectures, which are subsumed under the term "*parallel manipulators of Stewart Gough (SG) type*":

(A) Hexapod

The moving platform is connected via six spherical-prismatic-spherical (SPS) legs with the base.

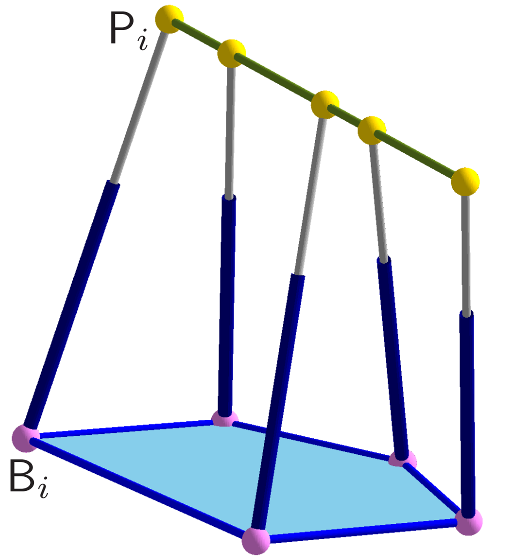
A hexapod is in a singular (*shaky* or *infinitesimal movable*) configuration if and only if the six lines $\mathsf{l}_1, \ldots, \mathsf{l}_6$ spanned by the centers of corresponding spherical joints belong to a linear line complex.



Introduction

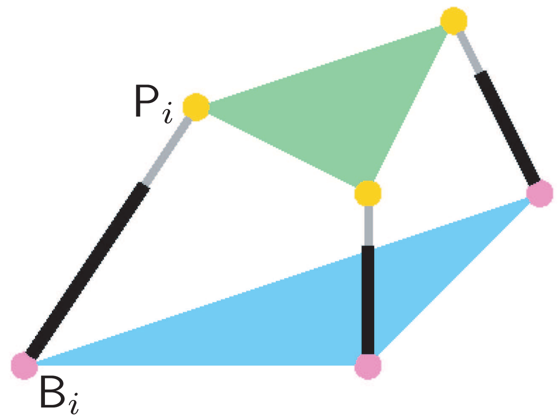
(B) Linear Pentapod

In this case the platform degenerates to a line, which is connected via five SPS-legs to the fixed base.
The linear pentapod is shaky if and only if the five lines $\mathsf{l}_1, \ldots, \mathsf{l}_5$ belong to a congruence of lines.



(C) 3-RPR Manipulator

The platform is connected via three rotational-prismatic-rotational (RPR) legs with the base.
This planar manipulator is shaky if and only if the three lines $\mathsf{l}_1, \mathsf{l}_2, \mathsf{l}_3$ belong to a pencil of lines.



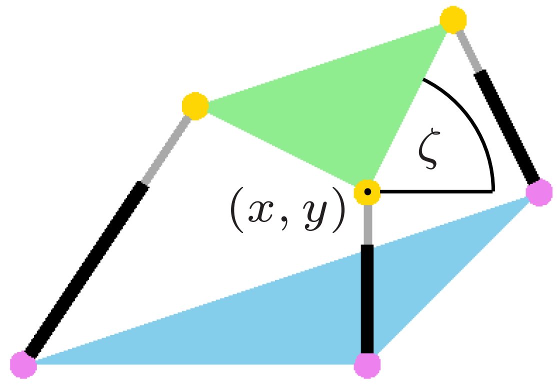
Motivation & Overview

Due to the loss of control, singularities and their vicinity have to be avoided. As a consequence the kinematic/robotic community is highly interested in evaluating the singularity closeness, but a geometric meaningful distance measure between a given manipulator configuration and the next singular configuration is still missing. We introduce such measures for *parallel manipulators of SG type*:

1. **Review on the Determination of the Closest Singularity**
2. **Distance Function**
3. **Singularity Distance**
4. **Results for 3-RPR Manipulators**

1. Review: 3-RPR Manipulator

Li, Gosselin & Richard [8] determined singularity-free zones around non-singular configurations by parametrizing the 3-dimensional configuration space by x, y, ζ, where x, y are the two position variables and ζ the orientation angle.



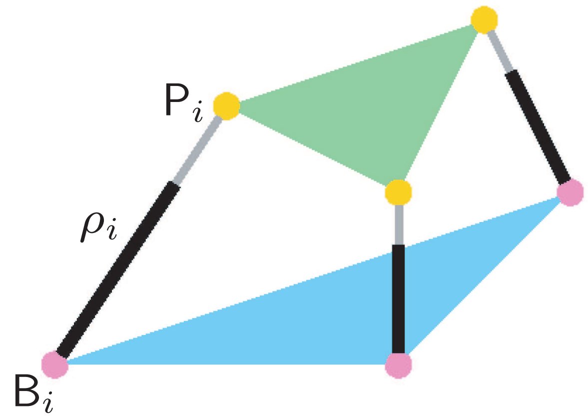
Then the point (x, y, ζ) of the singularity variety which minimizes the function

$$d := (x - x_0)^2 + (y - y_0)^2$$

where (x_0, y_0) corresponds with the position of the given non-singular configuration. The orientation of the given pose is not taken into account thus $\sqrt{d}$ is the radius of the circular directrix centered in (x_0, y_0) of the "*singularity-free cylinder*".

1. Review: 3-RPR Manipulator

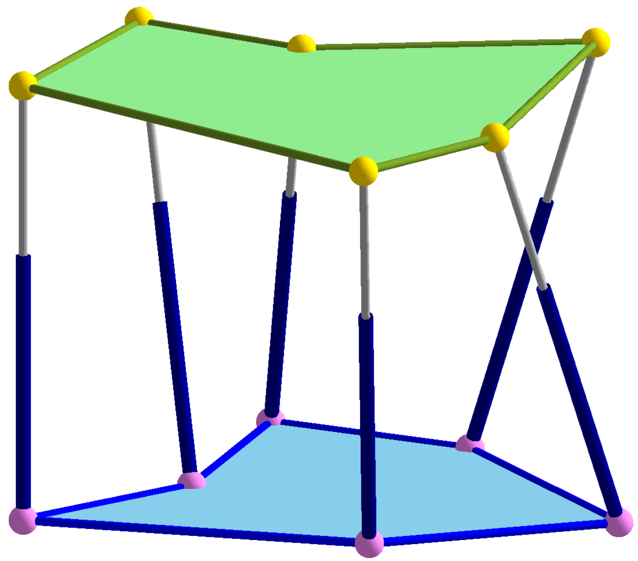
Zein, Wenger & Chablat [20] presented a procedure for the determination of a maximal singularity-free cube in the joint space centered in (ρ_1, ρ_2, ρ_3), where ρ_i is the length of the i-th leg in the given non-singular configuration.



But the edge length e of this cube is not well suited as a closeness index due to the fact that the mapping from the configuration space to the joint space is 6 to 1 (cf. **Husty [4]**). As in general not all six configurations, which correspond to a point on the singularity variety in the joint space, are singular ones, it can be the case that even in a non-singular configuration e equals zero.

1. Review: Hexapod

Li, Gosselin & Richard [9] computed "*maximally singularity-free hyperspheres*" around non-singular configurations by parametrizing the 6-dimensional configuration space by $x, y, z, \theta, \varphi, \psi$, where x, y, z are the three position variables and θ, φ, ψ the Euler angles representing the orientation.
Then they looked for the point $(x, y, z, \theta, \varphi, \psi)$ of the singularity variety which minimizes the function



$$D := W\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right] + \\ (1-W)\left[(\tan\tfrac{\theta}{2} - \tan\tfrac{\theta_0}{2})^2 + (\tan\tfrac{\varphi}{2} - \tan\tfrac{\varphi_0}{2})^2 + (\tan\tfrac{\psi}{2} - \tan\tfrac{\psi_0}{2})^2\right]$$

where $(x_0, y_0, z_0, \theta_0, \varphi_0, \psi_0)$ corresponds with the given non-singular configuration.

1. Review: Hexapod

$W \in [0, 1]$ is a weighting coefficient, which can be used by the designer to "*favour either the position workspace or the orientation workspace*".

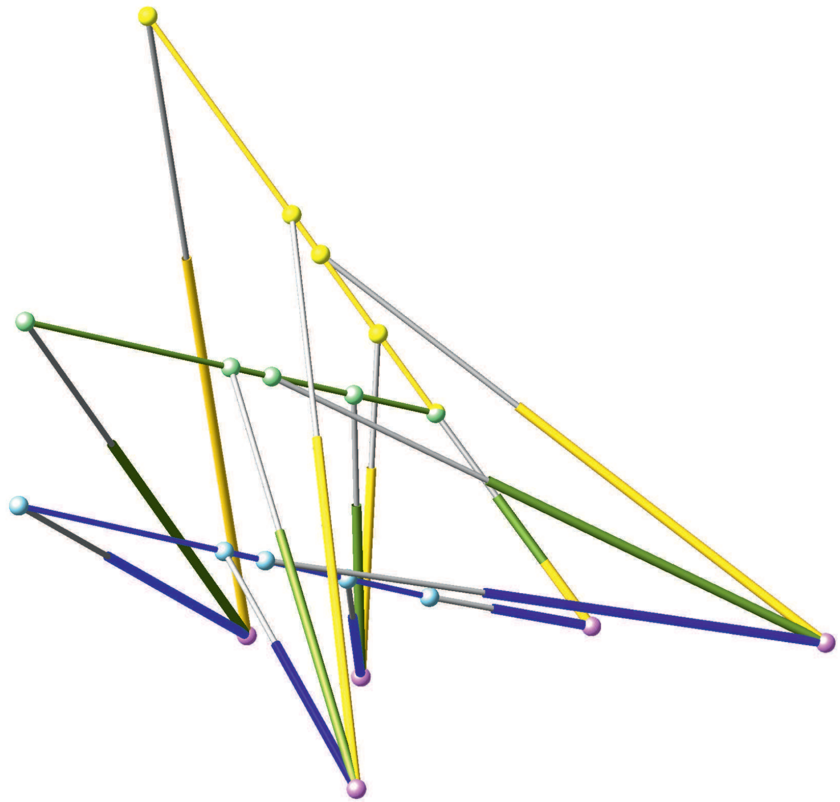
Li et al [9] were aware of the drawbacks of their objective function:

"*...the above formulation poses the problem of defining a distance in the 6-D workspace in order to find the 'closest' point on the singularity manifold. Clearly, an Euclidean distance cannot be defined in this space since it is composed of mixed dimensions (position coordinates and orientation coordinates). Therefore, the above index D cannot be called a distance in the mathematical sense of the term and the singularity-free region obtained cannot properly speaking be termed a hyper-sphere.*"

1. Review

Computing the distance to the next singularity for fixed orientation (blue, e.g. [5,9]) and position (yellow, e.g. [9,13]), respectively, are further concepts known in kinematics but from these two separated informations no conclusion about the closeness of a non-singular configuration (green) to the next singular one within the n-dimensional configuration space can be drawn.

Thus the question of a suitable distance function arises.



Linear Pentapod

2. Distance Function

It is well known (cf. **Park [16]** and **Murray et al [12]**), that there does not exist a bi-invariant (positive-definite) metric on SE(3).

Remark: A metric is called bi-invariant if it is invariant with respect to changes of the fixed frame (left invariant) and the moving frame (right invariant). ◇

Therefore it is not possible to define a geometric meaningful distance between two poses, which reasons the following statement of **Merlet & Gosselin [11]**:

"Measuring closeness between a pose and a singular configuration is a difficult problem: there exists no mathematical metric defining the distance between a prescribed pose and a given singular pose. Hence, a certain level of arbitrariness must be accepted in the definition of the distance to a singularity ..."

2. Distance Function

According to **Park [16]** there is the following alternative to distance metrics on SE(3), which yields a geometric meaningful distance function: One can consider the distance between two poses of the same rigid body, which yields so-called *object depended metrics* firstly studied by **Kazerounian & Rastegar [6]**.

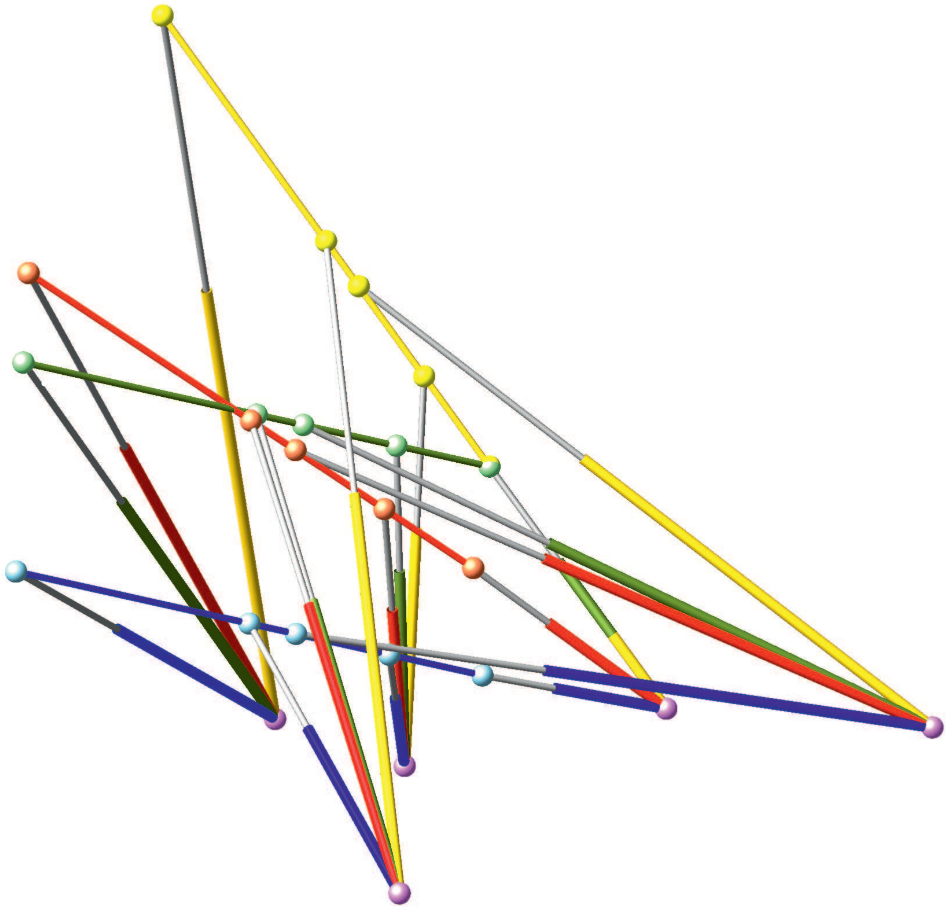
As the moving platform has n exceptionally points (i.e. platform anchor points) it suggests itself to measure the distance between two poses of the moving platform (given pose P_i and transformed pose P_i^{α}) by the distance measure

$$d_n := \sqrt{\frac{1}{n}\sum_{i=1}^{n}\langle \mathsf{P}_i^{\alpha} - \mathsf{P}_i, \mathsf{P}_i^{\alpha} - \mathsf{P}_i\rangle} \quad \text{where } \langle\,,\,\rangle \text{ denotes the standard scalar product.}$$

2. Distance Function

The considerations done so far do not only hold for the configuration space SE(3) of hexapods, but also for the configuration space SE(2) of 3-RPR manipulators as well as the set of oriented line elements of $\mathbb{R}^3$, which is the configuration space of linear pentapods (cf. [15]).

d_n was used by **Rasoulzadeh & Nawratil [18]** to compute the distance of linear pentapods to the next singularity (red).

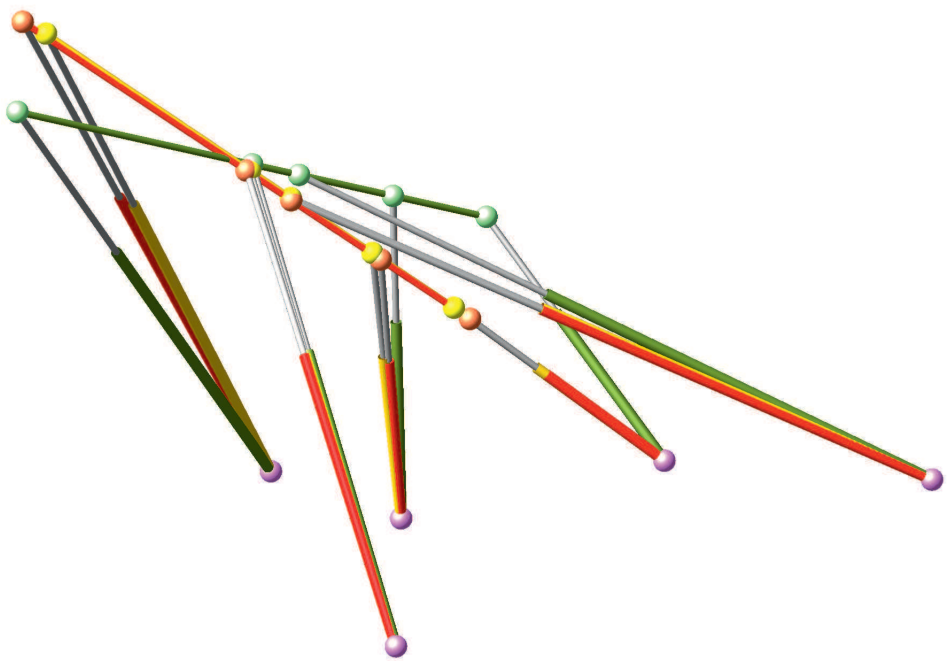


2. Distance Function

The determination of singular configurations yielding *local extrema* of d_n is an algebraic problem of degree 80 (cf. [18]). The computation of the sought *global minimium* can be relaxed by allowing α to belong to the equiform motion group.

The closest singularity (yellow) under equiform motions results from an algebraic problem of degree 28.

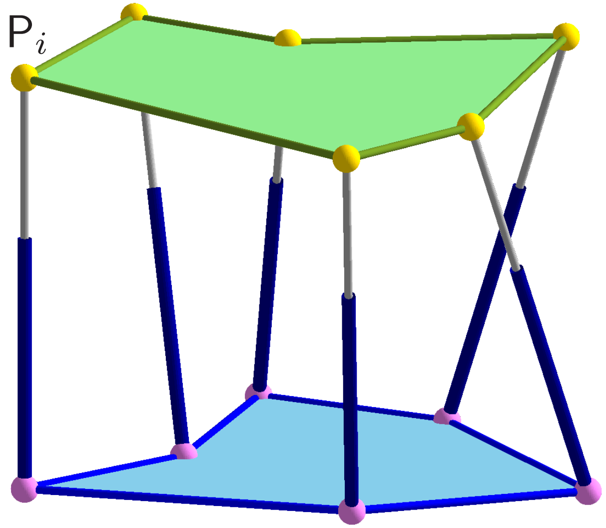
As the obtained distance of the relaxed problem is less or equal the distance of the original problem, it can be used as the radius of a hypersphere, which is guaranteed singularity-free.



3. Singularity Distance

These results motivate the following systematic procedure for defining distance measures for parallel manipulators of SG type.

For hexapods the set of transformations (α belongs to) can be extended step by step from the Euclidean group to



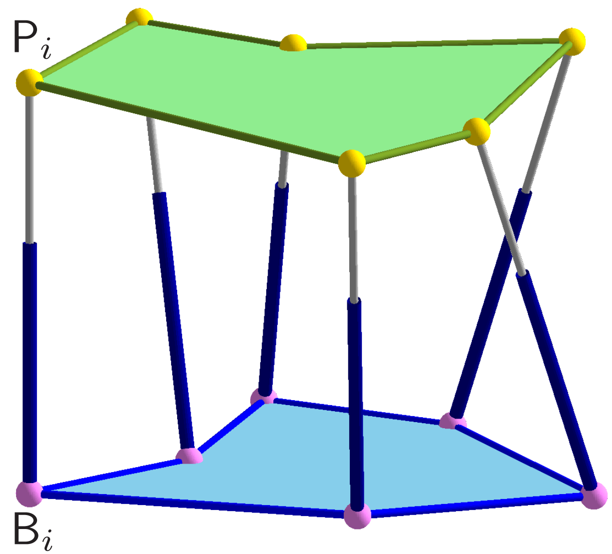
- $\star$ *equiform transformations*
- $\star$ *affine transformations*
- $\star$ *projective transformations*
- $\star$ *general transformations* which denote the mapping $P_i \mapsto P_i^\alpha$ for $i = 1, \ldots, n$.

3. Singularity Distance

The distance measure d_n has the following drawback: Assume we compute the distance p of a given configuration to the closest singularity in the sense of d_n.

Then we change our point of view by considering the platform as fixed and the base as moving part and compute again the distance to the next singularity according to d_n.

We get a second distance b which differs from p in the general case. This circumstance is less satisfactory from the geometric point of view.



Remark: An ad hoc solution of this point of criticism would be $(b+p)/2$. ◇

3. Singularity Distance

Hence we transform base and platform simultaneously and use the distance function

$$D_n := \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\langle \mathsf{P}_i^\alpha - \mathsf{P}_i, \mathsf{P}_i^\alpha - \mathsf{P}_i\rangle + \langle \mathsf{B}_i^\beta - \mathsf{B}_i, \mathsf{B}_i^\beta - \mathsf{B}_i\rangle\right]}$$

where B_i^β denote the transformed base points by the *base transformation* β.

Then the singularity distance equals the *global minimum* of D_n under the side condition that the configuration of n lines $[\mathsf{P}_i^\alpha, \mathsf{B}_i^\beta]$ is singular.

The obtained singularity distance depends on the set (*Euclidean*, *equiform*, *affine*, *projective* or *general transformation*) both transformations α and β belong to.

These singularity distances decrease (or remain unchanged) with respect to every extension step of the transformation set. Therefore all of them can be used as radius of a hypersphere, which is guaranteed singularity-free.

3. Singularity Distance

Let G_n denote the singularity distance (based on D_n), where the platform and the base transformations are both **general ones**. We favor G_n over all others possible singularity distances due to the following physical interpretation:

Theorem 1. If the radial clearance of the $2n$ passive joints is smaller than G_n then the parallel manipulator is guaranteed to be not in a singular configuration.

Analogous considerations/results as for hexapods ($n = 6$) also hold for linear pentapods ($n = 5$) and 3-RPR manipulators ($n = 3$), respectively.

Remark: For linear pentapods the *general transformation* of the base is a projectivity if the base is non-planar.
For 3-RPR manipulators the *general transformation* of the base/platform is an affinity if the base/platform points are not collinear. ◇

4. Results for 3-RPR Manipulators

The coordinates of the 3-RPR manipulator's base/platform points with respect to the fixed/moving frame are:

$$\mathsf{B}_1 = \mathsf{P}_1 = (0,0)^T, \quad \mathsf{B}_2 = (11,0)^T, \quad \mathsf{B}_3 = (5,7)^T, \quad \mathsf{P}_2 = (3,0)^T, \quad \mathsf{P}_3 = (1,2)^T.$$

We consider the following one-parametric motion with parameter $\varphi \in [0, 2\pi[$:

$$\mathsf{P}_i \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \mathsf{P}_i + \frac{1}{2} \begin{pmatrix} 11 - 6\sin\varphi \\ 3 - 3\cos\varphi \end{pmatrix}.$$

We can extend the *planar Euclidean motion group* SE(2) to the *planar equiform motion group* and further to the *group of planar affine transformations*.

4. Results for 3-RPR Manipulators

	Euclidean group	Equiform group	Affine group
d_3	s_3	e_3	g_3
D_3	S_3	E_3	G_3

The constrained optimization problem is solved by the Lagrange approach. If α and β are affine transformations then we set

$$\mathsf{P}_i^\alpha = (x_i, y_i)^T \qquad \mathsf{B}_i^\beta = (X_i, Y_i)^T$$

for $i = 1, 2, 3$. For an equiform transformation we have $i = 1, 2$ and set

$$\mathsf{P}_3^\alpha = \mathsf{P}_1^\alpha + \left(\overrightarrow{\mathsf{P}_1^\alpha \mathsf{P}_2^\alpha} \quad \overrightarrow{\mathsf{P}_1^\alpha \mathsf{P}_2^\alpha}^\perp \right) \frac{\overrightarrow{\mathsf{P}_1 \mathsf{P}_3}}{\overline{\overline{\mathsf{P}_1 \mathsf{P}_2}}} \qquad \mathsf{B}_3^\beta = \mathsf{B}_1^\beta + \left(\overrightarrow{\mathsf{B}_1^\beta \mathsf{B}_2^\beta} \quad \overrightarrow{\mathsf{B}_1^\beta \mathsf{B}_2^\beta}^\perp \right) \frac{\overrightarrow{\mathsf{B}_1 \mathsf{B}_3}}{\overline{\overline{\mathsf{B}_1 \mathsf{B}_2}}} \qquad (\star)$$

where the $\perp$ sign indicates the rotation of the vector by $90°$.

4. Results for 3-RPR Manipulators

The Lagrange function L for the computation of e_3, g_3 and E_3, G_3, respectively, reads as:

$$L: \quad d_3^2 - \lambda V_3 = 0 \qquad\qquad L: \quad D_3^2 - \lambda V_3 = 0$$

where V_3 denotes the algebraic condition that the three legs of the transformed 3-RPR manipulator belong to a pencil of lines. If we add the conditions

$$M: \quad \overline{\mathsf{P}_1^\alpha \mathsf{P}_2^\alpha}^2 - \overline{\mathsf{P}_1 \mathsf{P}_2}^2 = 0 \qquad\qquad N: \quad \overline{\mathsf{B}_1^\beta \mathsf{B}_2^\beta}^2 - \overline{\mathsf{B}_1 \mathsf{B}_2}^2 = 0$$

to the ansatz $(\star)$ we end up with Euclidean displacements. Thus the Lagrange function L for computing s_3 and S_3, respectively, can be formulated as follows:

$$L: \quad d_3^2 - \lambda V_3 - \mu M = 0 \qquad\qquad L: \quad D_3^2 - \lambda V_3 - \mu M - \nu N = 0$$

4. Results for 3-RPR Manipulators

The system of u partial derivatives L_i $(i = 1, \ldots, u)$ of L is solved using the Gröbner base method. For the case of G_3 the pseudo Maple code reads e.g. as:

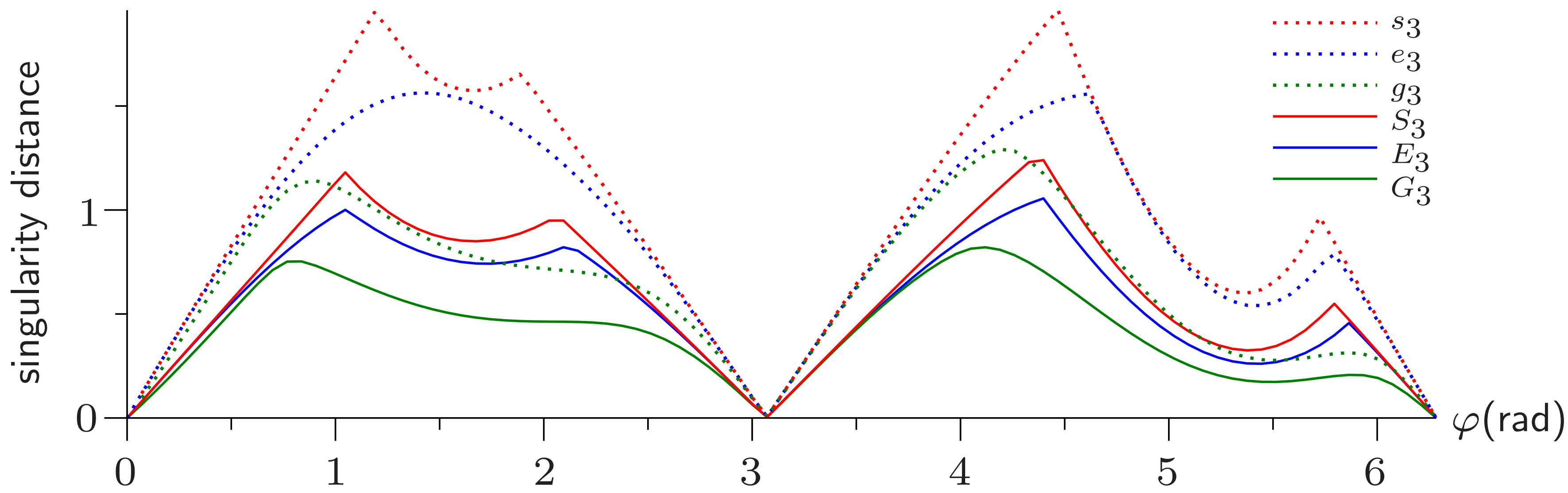
$$[> B := Basis([L_1, \ldots, L_{13}], tdeg(\lambda, x_1, y_1, X_1, Y_1, \ldots x_3, y_3, X_3, Y_3)) :$$

$$[> E := Basis([op(B)], plex(\lambda, x_1, y_1, X_1, Y_1, \ldots x_3, y_3, X_3, Y_3)) :$$

The degree of the univariate polynomial (given by $E[1]$ in the Maple code) equals the number of local extrema over $\mathbb{C}$ listed in the following table:

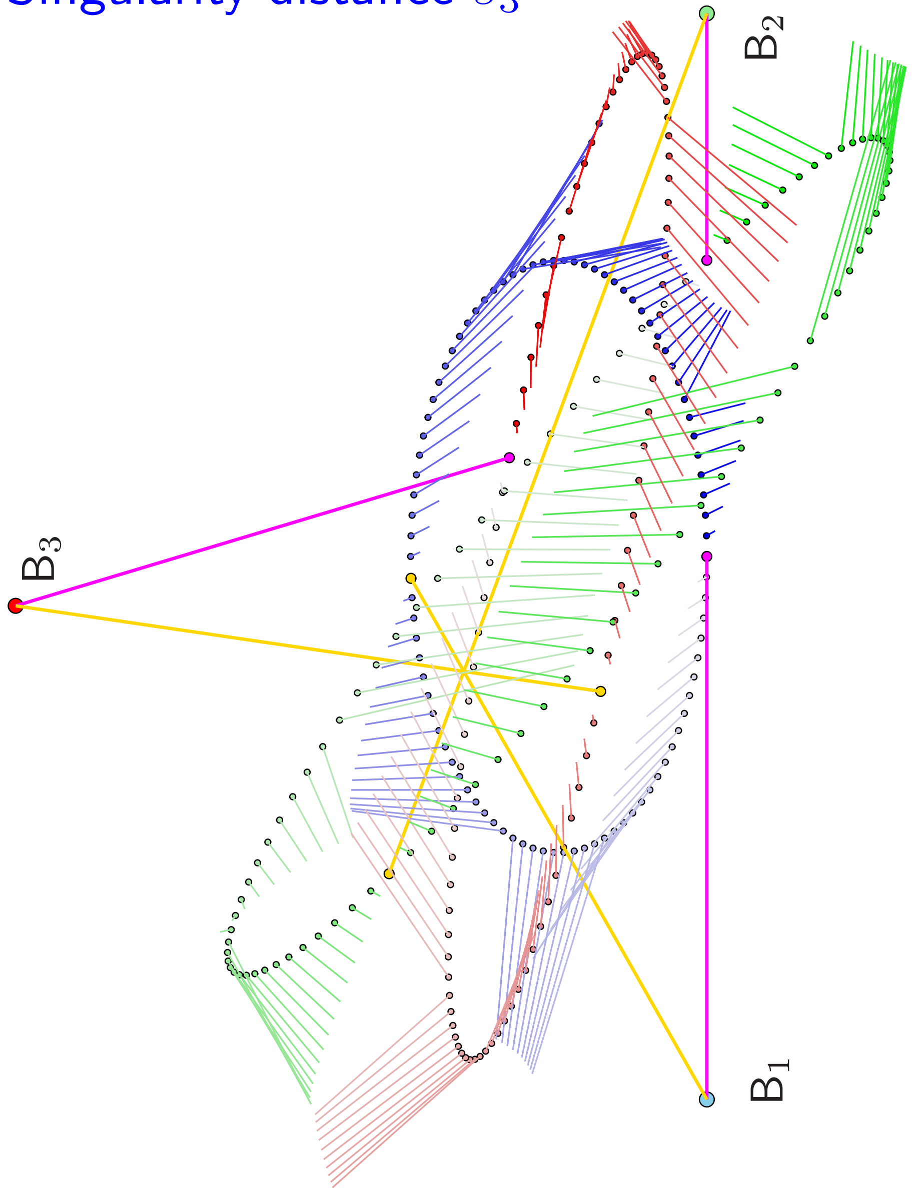
singularity distance	s_3	e_3	g_3	S_3	E_3	G_3
u (# variables in L)	6	5	7	11	9	13
# local extrema	32	19	22	88	34	50

4. Results for 3-RPR Manipulators

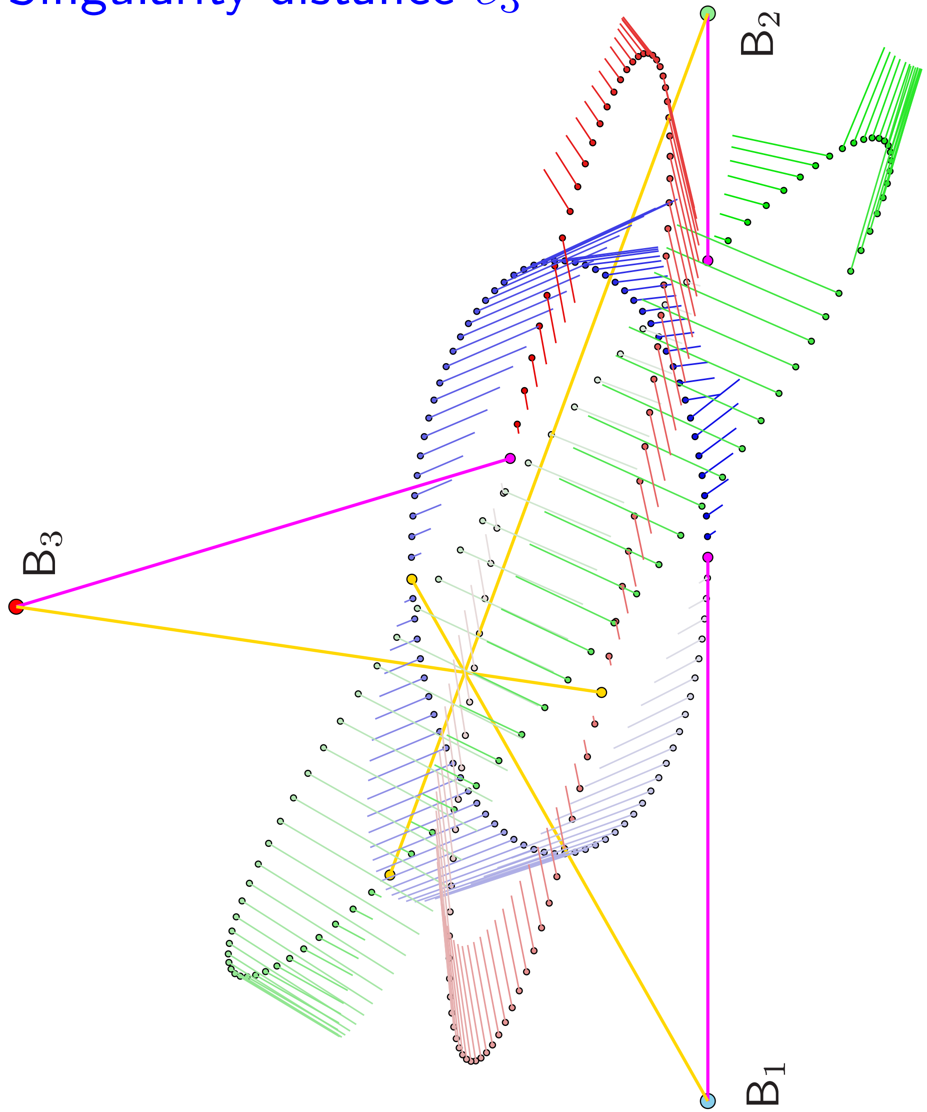


Remark: In the next slide the discontinuity of the *closest singular configuration* is caused by passing through the cut locus of the singularity variety (with respect to the used distance function). ◇

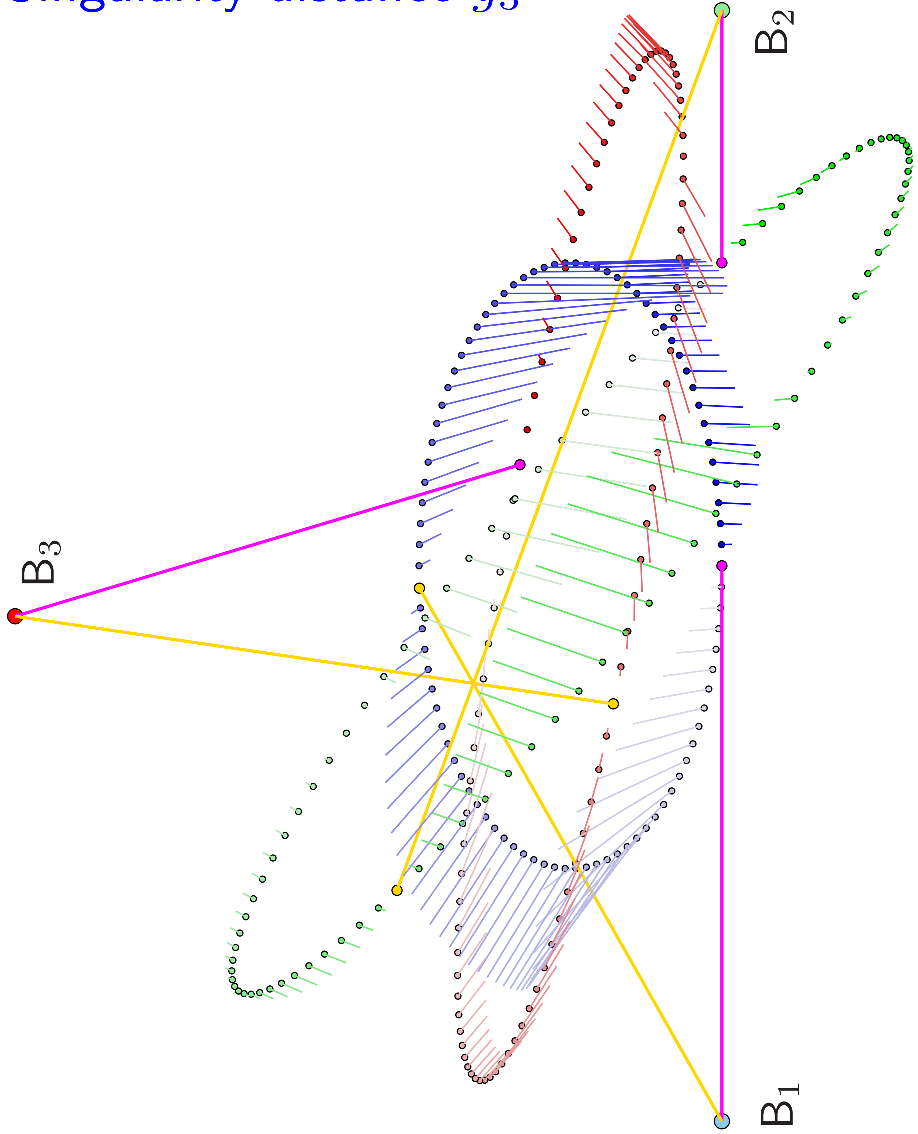
Singularity distance s_3



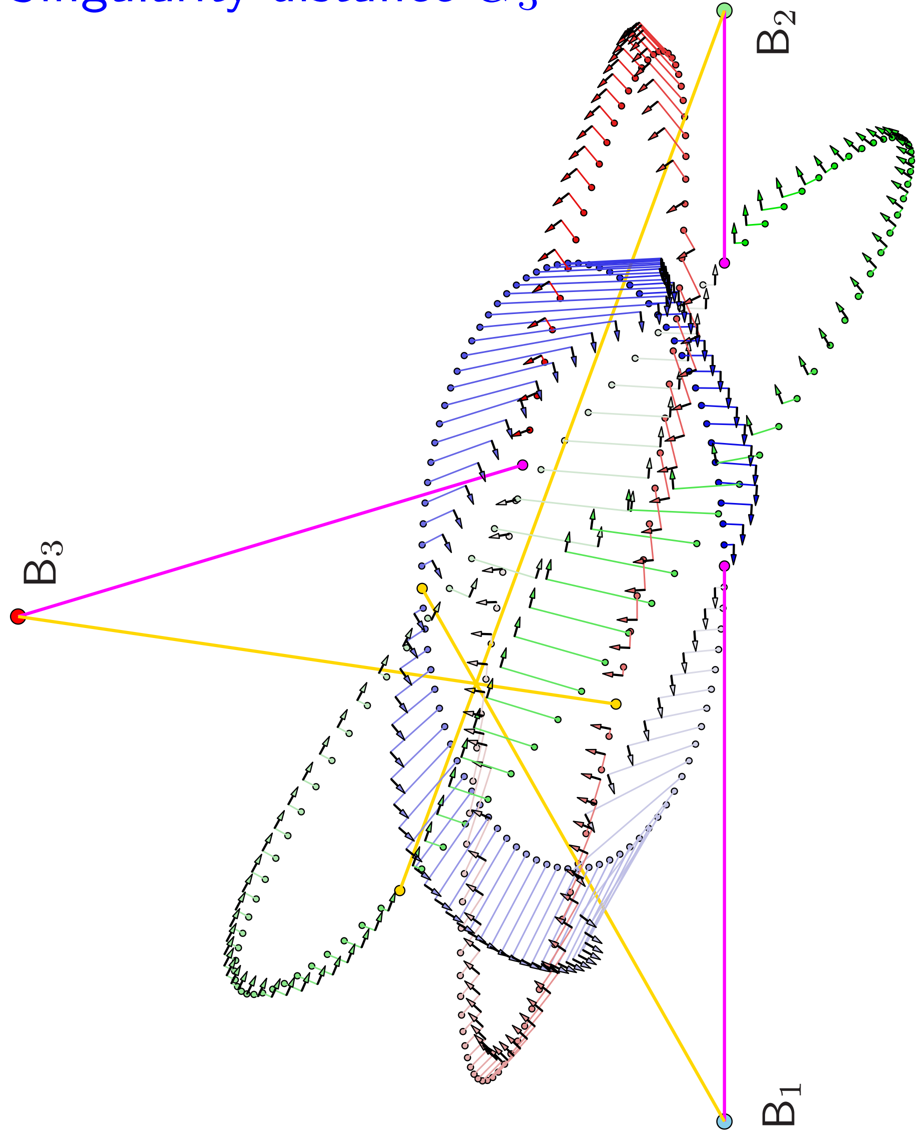
Singularity distance e_3



Singularity distance g_3



Singularity distance G_3



Conclusion & References

We presented measures for evaluating the distance of a parallel manipulator of SG type to the next singularity and computed them for a 3-RPR manipulator (based on the Gröbner base method).

Due to the degree and number of unknowns the computation of the singularity distance for hexapods and linear pentapods has to be based on the homotopy continuation method (e.g. BERTINI [2]), which is ongoing research.

All references refer to the list of publications given in the following paper:

NAWRATIL, G.: Singularity distance for parallel manipulators of Stewart Gough type. Advances in Mechanism and Machine Science – IFToMM WC 2019 (T. Uhl, Ed.), pages 259–268, Springer Nature, 2019