On origami-like quasi-mechanisms with an antiprismatic skeleton

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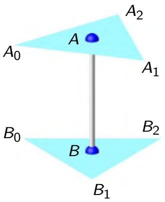
Introduction

Polyhedral structures with an antiprismatic skeleton



We consider polyhedral structures, where the bottom face α and the parallel top face β are regular convex n-gons $A_0, \ldots, A_{n-1}$ (with center A) and $B_0, \ldots, B_{n-1}$ (with center B) and a side length of 1.

Moreover, these two faces can be twisted against each other by a rotation about the axis AB orthogonal to α and β.

This antiprismatic skeleton is covered by a polyhedral belt composed of triangular faces only.

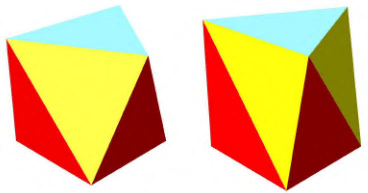
Quasi-mechanisms and realizations

The discussed polyhedra are rigid from mathematical perspective.

But their physical models can flex due to non-destructive elastic deformations of material, i.e. small changes in the intrinsic metric have significant effects on the spatial shape.

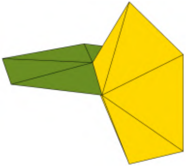
These structures are called *quasi-mechanisms* (or *model flexors*).

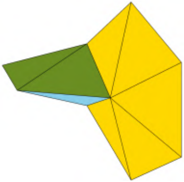
If the inner geometry of the polyhedron is fixed, then the embedding of the polyhedron into the Euclidean 3-space is in general not uniquely determined; i.e. different incongruent realizations exist.

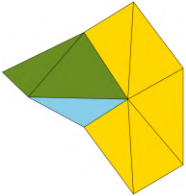


Two kinds of quasi-mechanisms

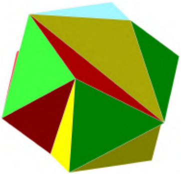
(i) Snapping quasi-mechanism:
The shape variation results from the snap (caused by deformation) of a given realization into another one. The best known example is the Siamese dipyramid, which snaps between three realizations [1,2].







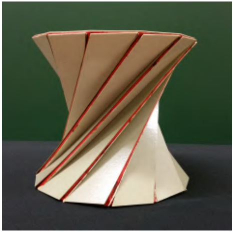
(ii) Shaky quasi-mechanism:
Now the deformed states originate from a given *shaky* (also known as *singular* or *infinitesimal flexible*) realization. The best known example is the Jessen icosahedron [1,3,4].



Review

Quasi-mechanisms with an antiprismatic skeleton

Kresling pattern: A flat strip of congruent triangles which can be folded up and closed to a belt for the skeleton. The resulting antiprismatic structure has a bi-stable behavior [8,9].



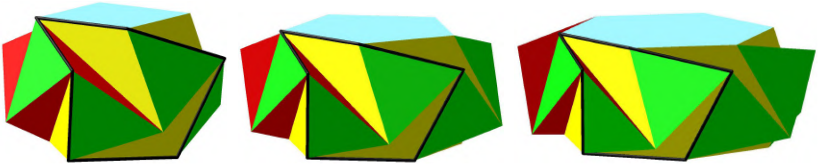
Quasi-mechanisms with an antiprismatic skeleton

Extreme birosettes: Generalizations of the Jessen icosahedron.

The belt consists of $2n$ equilateral triangles (green) with side length of 1 and $2n$ *petals* (yellow/red), which are skew rhombi of side length 1 broken along one of its diagonals of length p.

Moreover, the maximal value of p such that the birosette can be assembled yields the *extreme* birosette [11].

Can be generated from repetitive rotations of a *unit-cell* by $\frac{2\pi}{n}$.

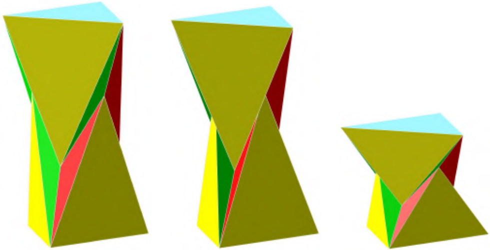


Extreme birosettes ($n = 4, 5, 6$) where the unit-cells are framed in black.

Quasi-mechanisms with an antiprismatic skeleton

Trisymmetric sandglass icosahedra: This snapping/shaky polyhedra of Wunderlich [13] possess the same combinatorial structure as birosettes with $n = 3$.

The belt consists of six congruent isosceles triangles (yellow) with bases of length 1 and the gaps between them are filled by further 12 congruent isosceles triangles (green/red).



Two snapping realizations [13, Fig. 6] and a shaky one [13, Fig. 4].

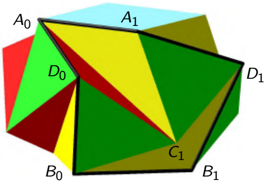
Goal & Outline

Goal: Generalization of the sandglass polyhedra to arbitrary n in analogy to the birosette construction, with the additional feature that the belt is developable as the Kresling pattern.

1. **Preliminaries**
2. **Quasi-mechanisms with an extremal snap**
   (a) Snappability index
3. **Shaky quasi-mechanisms**
   (a) Shakeability index
4. **Open problems & References**

1. Preliminaries: Assumptions

The assumptions on the edge lengths of a bi-rosette can be weaken keeping its symmetry:



$$L_1 := \overline{B_0 D_0} = \overline{A_1 C_1}, \; L_2 := \overline{B_0 C_1} = \overline{A_0 D_0},$$
$$L_3 := \overline{D_0 C_1} = \overline{C_1 D_1}, \; L_4 := \overline{B_0 D_1} = \overline{A_0 C_1}$$

beside the unit length of $\overline{A_0 A_1}$ and $\overline{B_0 B_1}$.

We get a sandglass structure by adding the condition $L_1 = L_4$.

Remark 1: Wunderlich [13] further assumed in the study of the trisymmetric sandglass that $L_2 = L_3$ holds.

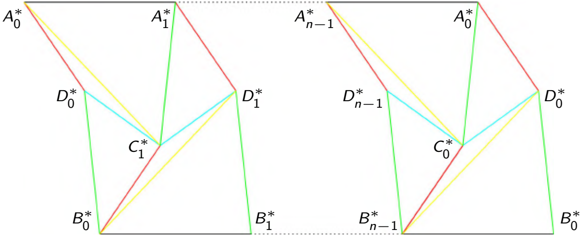
We assume that the skeleton edges are undeformable under the model flexibility in contrast to the other edge lengths $L_1, L_2, L_3 > 0$.

Remark 2: The study [11] on birosettes is more restrictive as only the length p is allowed to change.

1. Preliminaries: Developability condition of the belt

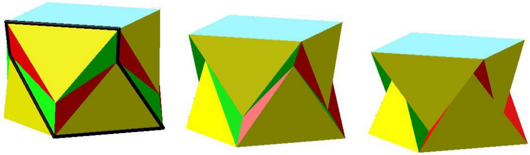
One can compute [15] the condition for the developability of the generalized birosette belt in terms of $L_1, \ldots, L_4$. Under consideration of $L_1 = L_4$ this condition simplifies to:

$$Q_3 = Q_1 + Q_2 - \sqrt{Q_2(4Q_1 - 1)} \quad \text{with} \quad Q_i := L_i^2$$



Sketch for computing the developability condition of a generalized birosette belt, where the upper index $*$ indicates the development.

2. Quasi-mechanisms with an extremal snap



Snap from one realization (left) over the passed shaky configuration (center) to the other realization (right).

One realization is on the boundary of self-intersection (right); i.e. green and red faces touch each other. The self-blocking of the faces increases the structure's load carrying capacity.

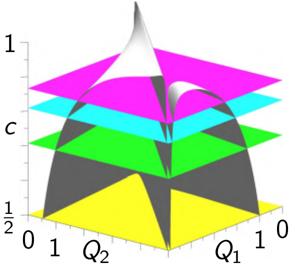
According to [2,14] the snap between two realizations has to pass a shaky configuration at the maximum state of deformation (center).

2. Quasi-mechanisms with an extremal snap

The 1-parametric solution set is characterized by the equation:

$$4cQ_2Q_1 - 2cQ_2^2 - 2Q_1^2 - 28Q_2Q_1 - 2Q_2^2 + Q_1 + 5Q_2 - 2cQ_1^2 +$$
$$(4Q_1-1)^{3/2}\sqrt{Q_2} + 8Q_2^{3/2}\sqrt{4Q_1-1} + 4Q_1\sqrt{Q_2}\sqrt{4Q_1-1} = 0$$

This equation is linear $c := \cos\frac{\pi}{n}$ and can be illustrated as follows:



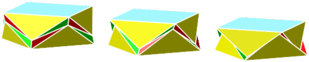
The c-planes for $n = 3, \ldots, 6$ are colored in yellow, green, cyan, magenta.

2. Quasi-mechanisms with an extremal snap

1-parametric family of snapping sandglass structures for $n = 3$.

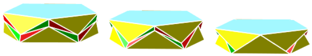
2. Quasi-mechanisms with an extremal snap

1-parametric family of snapping sandglass structures for $n = 4$.



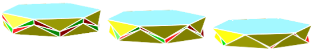
2. Quasi-mechanisms with an extremal snap

1-parametric family of snapping sandglass structures for $n = 5$.

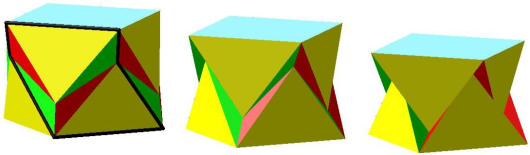


2. Quasi-mechanisms with an extremal snap

1-parametric family of snapping sandglass structures for $n = 6$.



2a. Snappability index

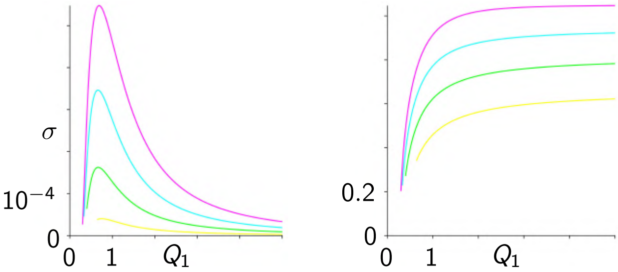


Snap from one realization (left) over the passed shaky configuration (center) to the other realization (right).

The shaky configuration (with squared edge lengths S_1, S_2, S_3) can be used for the evaluation of the snapping capability in terms of the *snappability* [2,14].
By considering the belt as a joint-bar structure this index σ, which is based on its total elastic strain energy density, is given by:

$$\sigma := \left(4n\frac{(Q_1-S_1)^2}{8L_1^3} + 2n\frac{(Q_2-S_2)^2}{8L_2^3} + 2n\frac{(Q_3-S_3)^2}{8L_3^3}\right)/(4nL_1+2nL_2+2nL_3)$$

2a. Snappability index



Graphs for $n = 3, \dots, 6$ are colored in yellow, green, cyan, magenta.
Left: Snappability of the computed snapping sandglass realizations.
Right: Increase of the volume during the snap relative to the volume of the realization at the boundary of self-intersection.

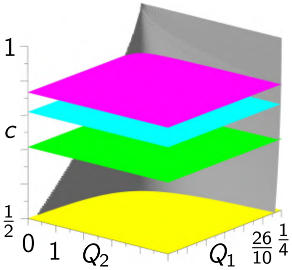
Remark 3: This change in volume confirms that there does not exist a continuous isometric deformation between the two realizations due to the *Bellows conjecture*.

3. Shaky quasi-mechanisms

The 1-parametric solution set is given by an equation of the form:

$$w_4 c^4 + w_3 c^3 + w_2 c^2 + w_1 c + w_0 = 0$$

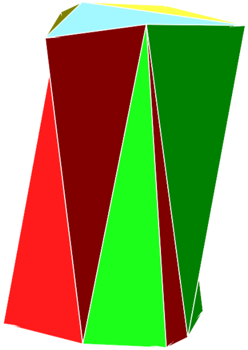
where the coefficients w_i are functions in Q_1 and Q_2 (cf. [15]).



The c-planes for $n = 3, \ldots, 6$ are colored in yellow, green, cyan, magenta.
Feasible values for Q_2 are only within a narrow domain of Q_1,
where $Q_1 = \frac{1}{4}$ is an asymptote for the Q_2 values.

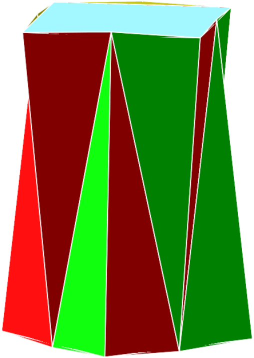
3. Shaky quasi-mechanisms

1-parametric family of shaky sandglass structures for $n = 3$.



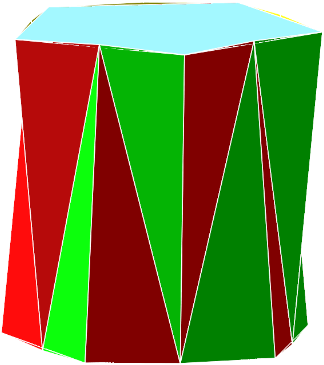
3. Shaky quasi-mechanisms

1-parametric family of shaky sandglass structures for $n = 4$.



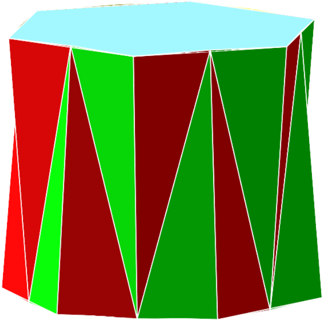
3. Shaky quasi-mechanisms

1-parametric family of shaky sandglass structures for $n = 5$.



3. Shaky quasi-mechanisms

1-parametric family of shaky sandglass structures for $n = 6$.



3a. Shakeability index

The structure's capability to shake can be evaluated by the so-called *shakeability* κ, which is defined as curvature of the snappability function over the space of squared edge lengths in direction associated with the infinitesimal mobility.

As the snappability function is already dimensionless we also have to normalize the velocity vectors in such a way. This can e.g. be achieved by the condition, that the mean of the relative instantaneous changes of the squared edge lengths is equal to 1; i.e.

$$\left(4n\frac{\|\mathbf{v}(B_0)-\mathbf{v}(D_0)\|^2}{Q_1}+2n\frac{\|\mathbf{v}(B_0)-\mathbf{v}(C_1)\|^2}{Q_2}+2n\frac{\|\mathbf{v}(D_0)-\mathbf{v}(C_1)\|^2}{Q_3}\right)/(8n)=1$$

where $\mathbf{v}(X_i)$ denote the velocity vectors of the vertices X_i.

3a. Shakeability index

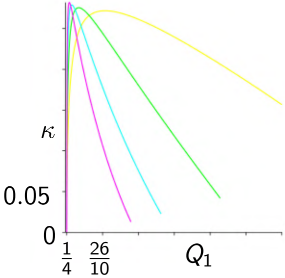
Assumed that this normalization condition holds, we can set

$$S_1 = Q_1 + t\|\mathbf{v}(B_0) - \mathbf{v}(D_0)\|^2, \quad S_2 = Q_2 + t\|\mathbf{v}(B_0) - \mathbf{v}(C_1)\|^2,$$
$$S_3 = Q_3 + t\|\mathbf{v}(D_0) - \mathbf{v}(C_1)\|^2,$$

and plug these expressions into the snappability function σ, which now depends quadratically on t; i.e. $\sigma(t)$.

According to the well-known curvature formula the shakeability κ can then be computed as

$$\kappa := \frac{\sigma''}{(1+\sigma'^2)^{3/2}}\Big|_{t=0} = \sigma''\big|_{t=0}$$



Graphs for $n = 3, \ldots, 6$ are colored in yellow, green, cyan, magenta.

4. Open problems

The presented study can be generalized by:

- omitting the sandglass/developability condition(s),
- using an antifrustum as skeleton (i.e. radii of α and β differ).

References

All references refer to the list of publications given in the presented paper:

Nawratil, G.: On origami-like quasi-mechanisms with an antiprismatic skeleton. Advances in Robot Kinematics 2022 (O. Altuzarra, A. Kecskeméthy, eds.), pages 13–21 (2022)

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