Flexible arrangement of two Bennett tubes

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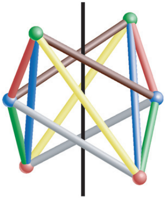
Motivation & Review

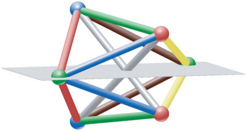
Bricard octahedra

In 1897 Bricard* proved that there are three types of flexible octahedra in the Euclidean 3-space.

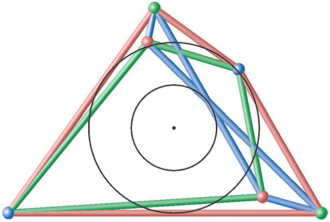
Line-symmetric



Plane-symmetric



Asymmetric



Bricard octahedra can also be seen as flexible bipyramids, where each quadrilateral pyramid corresponds to a spherical 4R-loop.

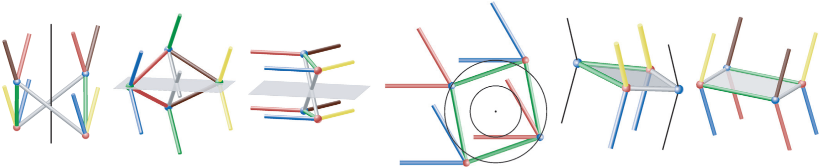
* **Bricard, R.:** Mémoire sur la théorie de l'octaèdre articulé. Journal de Mathématiques pures et appliquées, Liouville **3**:113–148 (1897)

Bricard octahedra with vertices at infinity

As also planar 4R-loops are flexible, we can replace one or both pyramids by quadrilateral prisms.

Flexible arrangements of quadrilateral pyramids and prisms were studied by Nawratil*, where it was shown that only two cases can exist, namely limits of the plane-symmetric type and the asymmetric type.

The full classification of flexible arrangements of two quadrilateral prisms was also given by Nawratil†.

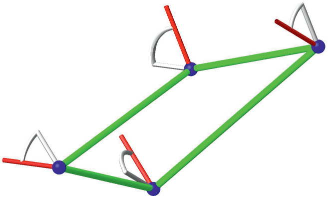


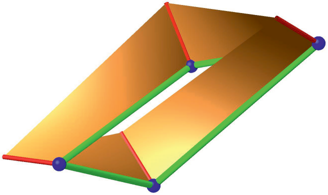
* Self-motions of TSSM manipulators with two parallel rotary axes. ASME Journal of Mechanisms and Robotics **3**(3):031007 (2011)

† Flexible octahedra in the projective extension of the Euclidean 3-space. Journal for Geometry and Graphics **14**(2):147–169 (2010)

Bennett tubes

Beside planar and spherical 4R-loops there also exist spatial ones known as Bennett* mechanisms, which can be realized as so-called Bennett tubes by using skew faces.





Thus one can ask for flexible arrangements of a Bennett tube with a quadrilateral pyramid/prism and of two Bennett tubes, respectively.

Goal: First results on flexible arrangements of two Bennett tubes.

* **Bennett, G.T.:** A new mechanism. Engineering **76**:777–778 (1903)

Basics & Fundamentals

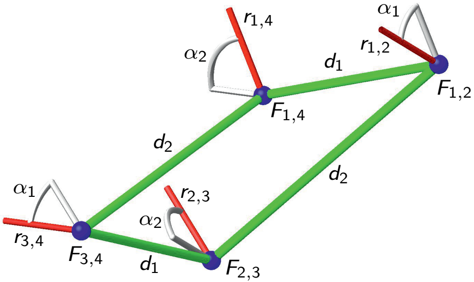
Bennett loop: Definition

We consider a closed serial 4R chain consisting of the systems $\Sigma_1, \ldots, \Sigma_4$, which are linked in a loop by rotary axes $r_{1,4}, r_{1,2}, r_{2,3}, r_{3,4}$, where $r_{i,j}$ denotes the connection between Σ_i and Σ_j with $i < j$.

A Bennett loop is characterized by the following conditions on the Denavit-Hartenberg parameters:

$$d_2 = d_4, \quad d_1 = d_3,$$
$$\alpha_2 = \alpha_4, \quad \alpha_1 = \alpha_3,$$
$$d_1 \sin(\alpha_2) = d_2 \sin(\alpha_1)$$

and zero offsets.



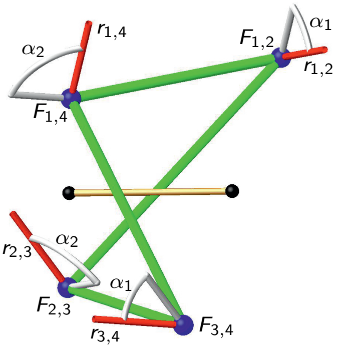
Convention:
Without loss of generality we can assume $d_1, d_2 > 0$, $\alpha_1, \alpha_2 \in (0; \pi)$.

Bennett loop: Properties

Using this convention the rotation angles $\theta_{i-1,i}$ about the axes $r_{i-1,i}$ are related by

$$\theta_{1,2} = -\theta_{3,4}, \quad \theta_{2,3} = -\theta_{1,4},$$

$$\tan\frac{\theta_{1,2}}{2}\tan\frac{\theta_{2,3}}{2} = \frac{\tan\frac{\alpha_1}{2}+\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}-\tan\frac{\alpha_2}{2}}$$



As pointed out by Hon-Cheung* **Bennett loops are line-symmetric** (if neglecting the orientation of the rotation axes). Therefore the relative motion of opposite links is line-symmetric, which was intensively studied by Krames†.

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* **Hon-Cheung, Y.:** The Bennett linkage, its associated tetrahedron and the hyperboloid of its axes. Mechanism and Machine Theory **16**(2):105–114 (1981)

† **Krames, J.:** Zur Geometrie des Bennett'schen Mechanismus. Sitz.ber. Österr. Akad. Wiss. Math.-Nat.wiss. Kl., II, **146**:159–173 (1937)

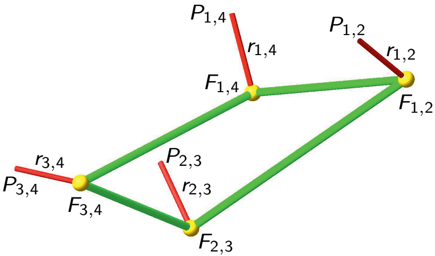
Flexible arrangement of two Bennett tubes
Symmetric Case

Plane-symmetric arrangement

Let us start by parametrizing the points $P_{i,j}$ on the axis $r_{i,j}$ of the Bennett linkage $\mathcal{B}$; i.e.

$$\mathbf{P}_{i,j}(\tau) = \mathbf{F}_{i,j}(\tau) + \mu_{i,j}\mathbf{r}_{i,j}(\tau)$$

with motion parameter τ.



Exist Bennett loops and $\mu_{i,j}$ in such a way that the four points $P_{1,4}, P_{1,2}, P_{2,3}, P_{3,4}$ are coplanar for all $\tau \in \mathbb{R}$?

No $\Longrightarrow$ plane-symmetric arrangements do not exist.

Remark: A point-symmetric arrangement is even more restrictive, as $P_{1,4}, P_{1,2}, P_{2,3}, P_{3,4}$ have to form a parallelogram for all $\tau \in \mathbb{R}$.

Line-symmetric arrangement

According to Cayley* the skew quadrilateral $P_{1,4}, P_{1,2}, P_{2,3}, P_{3,4}$ is line-symmetric if and only if it is a skew isogram; i.e.

$$\|\mathbf{P}_{1,4}-\mathbf{P}_{1,2}\| = \|\mathbf{P}_{2,3}-\mathbf{P}_{3,4}\| \qquad \|\mathbf{P}_{1,4}-\mathbf{P}_{3,4}\| = \|\mathbf{P}_{2,3}-\mathbf{P}_{1,2}\| \quad (1)$$

As these two equations are independent of τ, we only have two conditions on the seven unknowns $k, a_1, a_2, \mu_{1,4}, \mu_{1,2}, \mu_{2,3}, \mu_{3,4}$ with

$$d_i = k\sin(\alpha_i), \quad a_i = \tan\tfrac{\alpha_i}{2} \quad \text{and} \quad k \in \mathbb{R}^+ \setminus \{0\}\,.$$

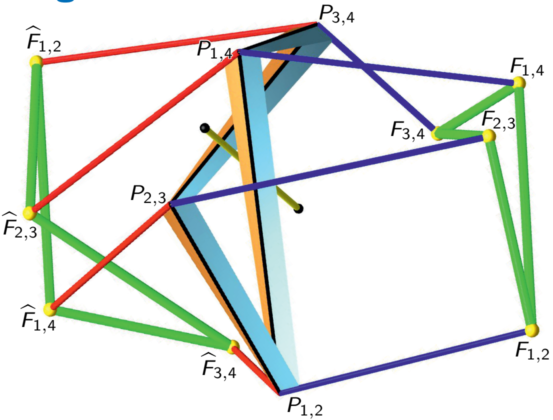
We cancel the factor of similarity by setting $k = 1$ and solve (1) for:

$$a_{\frac{1}{2}} = \sqrt{-\frac{(\mu_{1,4}\mp\mu_{1,2}\pm\mu_{2,3}-\mu_{3,4})(\mu_{1,4}-\mu_{1,2}\mp\mu_{2,3}\pm\mu_{3,4})}{(\mu_{1,4}+\mu_{1,2}+\mu_{2,3}+\mu_{3,4})(\mu_{1,4}\pm\mu_{1,2}-\mu_{2,3}\mp\mu_{3,4})}} \qquad (2)$$

* **Cayley, A.:** Note on the Tetrahedron. Oxford, Cambridge and Dublin Messenger of Mathematics **III**:8–10 (1866)

Line-symmetric arrangement

By a half-turn of $\mathcal{B}$ about the line of symmetry of $P_{1,4}$, $P_{1,2}$, $P_{2,3}$, $P_{3,4}$ we obtain $\widehat{\mathcal{B}}$.

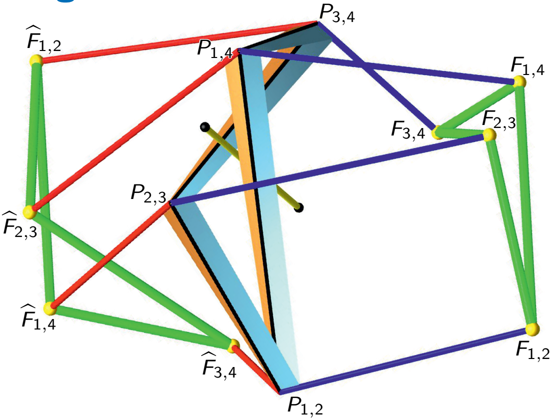


Family (A)

Besides the factor of similarity, Eq. (2) determines a 4-dimensional family of flexible line-symmetric arrangements of two Bennett tubes.

Line-symmetric arrangement

By a half-turn of $\mathcal{B}$ about the line of symmetry of $P_{1,4}$, $P_{1,2}$, $P_{2,3}$, $P_{3,4}$ we obtain $\widehat{\mathcal{B}}$.



Family (A): Property

The spherical indicatrix of each spherical 4R-loop with center $P_{i,j}$ is isogonal (i.e. opposite sides are both equal or supplementary).

Line-symmetric arrangement

There are only two special cases where the equations of Eq. (1) cannot be solved for a_1 and a_2 as given in Eq. (2):

$$\mu_{1,4} = \mu_{2,3}, \qquad \mu_{1,2} = \mu_{3,4} \tag{3}$$

$$\mu_{1,4} = -\mu_{2,3}, \qquad \mu_{1,2} = -\mu_{3,4} \tag{4}$$

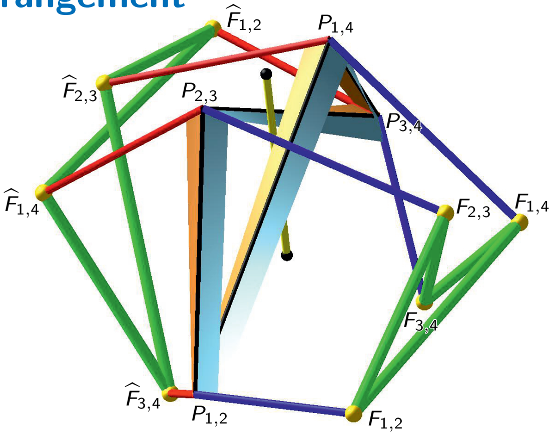
In case of Eq. (4) the line of symmetry of $P_{1,4}, P_{1,2}, P_{2,3}, P_{3,4}$ coincides with the one of $F_{1,4}, F_{1,2}, F_{2,3}, F_{3,4}$. Therefore, $\mathcal{B}$ equals $\widehat{\mathcal{B}}$, which is a trivial solution to our problem.

Family (B)

Besides the factor of similarity, Eq. (3) determines a 4-dimensional family of flexible line-symmetric arrangements of two Bennett tubes.

Line-symmetric arrangement

By a half-turn of $\mathcal{B}$ about the line of symmetry of $P_{1,4}$, $P_{1,2}$, $P_{2,3}$, $P_{3,4}$ we obtain $\widehat{\mathcal{B}}$.



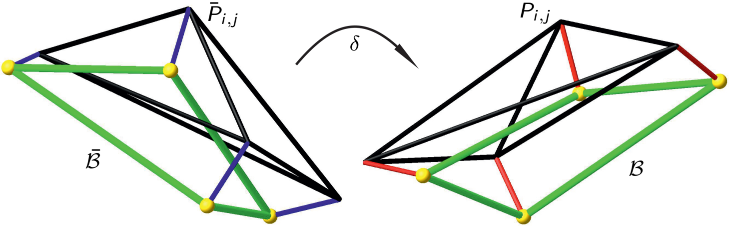
Family (B): Property

The spherical indicatrix of each spherical 4R-loop with center $P_{i,j}$ is deltoidal (symmetry plane is spanned by $P_{i-1,j-1}$ and $P_{i+1,j+1}$).

Flexible arrangement of two Bennett tubes
Non-Symmetric Case

Necessary and sufficient set of algebraic conditions

Beside the Bennett $\mathcal{B}$ we consider a second Bennett $\bar{\mathcal{B}}$ which is parametrized and coordinatized analogously; i.e. the location of the points $\bar{P}_{i,j}$ depends on the motion parameter $\bar{\tau}$ and the design parameters $\bar{a}_1, \bar{a}_2, \bar{k}, \bar{\mu}_{i,j}$.



$\mathcal{B}$ and $\bar{\mathcal{B}}$ can be flexible coupled over the two tetrahedra $P_{1,4}, \ldots, P_{3,4}$ and $\bar{P}_{1,4}, \ldots, \bar{P}_{3,4}$ if for each τ there exits a $\bar{\tau}$ such that the two tetrahedra are related by an isometry δ.

Necessary and sufficient set of algebraic conditions

The algebraic conditions for this read as follows:

$$
\begin{aligned}
\|\mathbf{P}_{1,4} - \mathbf{P}_{1,2}\|^2 - \|\bar{\mathbf{P}}_{1,4} - \bar{\mathbf{P}}_{1,2}\|^2 &= 0 \\
\|\mathbf{P}_{1,2} - \mathbf{P}_{2,3}\|^2 - \|\bar{\mathbf{P}}_{1,2} - \bar{\mathbf{P}}_{2,3}\|^2 &= 0 \\
\|\mathbf{P}_{2,3} - \mathbf{P}_{3,4}\|^2 - \|\bar{\mathbf{P}}_{2,3} - \bar{\mathbf{P}}_{3,4}\|^2 &= 0 \\
\|\mathbf{P}_{3,4} - \mathbf{P}_{1,4}\|^2 - \|\bar{\mathbf{P}}_{3,4} - \bar{\mathbf{P}}_{1,4}\|^2 &= 0
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
\|\mathbf{P}_{1,4} - \mathbf{P}_{2,3}\|^2 - \|\bar{\mathbf{P}}_{1,4} - \bar{\mathbf{P}}_{2,3}\|^2 &= 0 \\
\|\mathbf{P}_{1,2} - \mathbf{P}_{3,4}\|^2 - \|\bar{\mathbf{P}}_{1,2} - \bar{\mathbf{P}}_{3,4}\|^2 &= 0.
\end{aligned} \tag{6}
$$

Only the conditions (6) depend on τ and $\bar{\tau}$. They are of the form:

$$c_{22}\tau^2\bar{\tau}^2 + c_{21}\tau^2\bar{\tau} + c_{12}\tau\bar{\tau}^2 + c_{20}\tau^2 + c_{02}\bar{\tau}^2 + c_{10}\tau + c_{01}\bar{\tau} + c_{00} = 0.$$

Necessary and sufficient set of algebraic conditions

Their resultant with respect to $\bar{\tau}$ yields an equation of degree 8 in τ

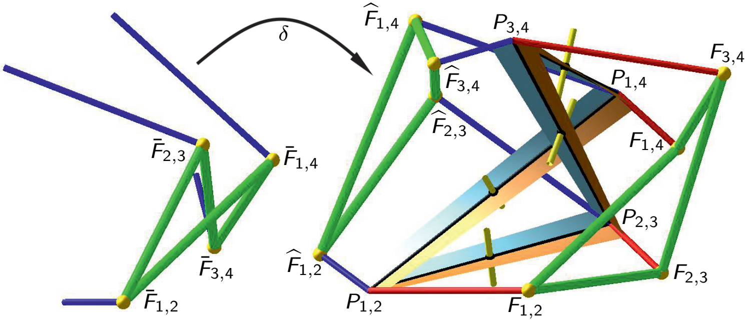
$$c_8\tau^8 + c_7\tau^7 + c_6\tau^6 + c_5\tau^5 + c_4\tau^4 + c_3\tau^3 + c_2\tau^2 + c_1\tau + c_0 = 0$$

where the c_i's depend on the 14 unknowns

$$a_1, a_2, k, \mu_{1,4}, \mu_{1,2}, \mu_{2,3}, \mu_{3,4}, \bar{a}_1, \bar{a}_2, \bar{k}, \bar{\mu}_{1,4}, \bar{\mu}_{1,2}, \bar{\mu}_{2,3}, \bar{\mu}_{3,4}.$$

This equation has to be fulfilled independent of τ implying the nine conditions $c_8 = c_7 = \ldots = c_0 = 0$. Together with the four conditions of Eq. (5) we have 13 equations in 14 unknowns. If we eliminate the scaling factor (e.g. by setting $k = 1$) we get a square system.

We were not able to solve this system due to its complexity. But we succeeded in finding one further family (C).

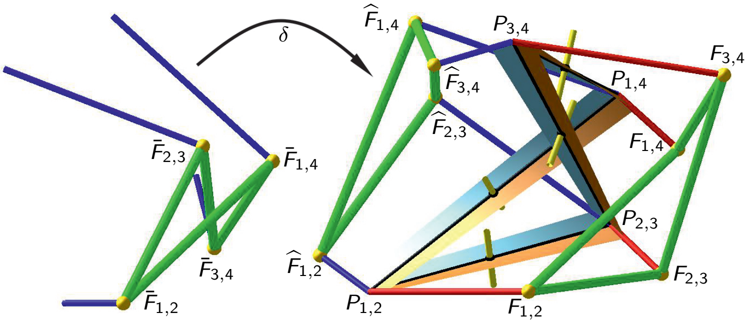


Family (C)

Besides the factor of similarity, a 4-dim family of non-symmetric flexible arrangements is determined by $s \in \{-1, +1\}$ and

$$\bar{a}_1 = a_1, \quad \bar{a}_2 = a_2, \quad \mu_{2,3} = \mu_{1,4}, \quad \mu_{3,4} = \mu_{1,2}, \quad \bar{k} = k$$

$$\bar{\mu}_{1,4} = s\mu_{1,2}, \quad \bar{\mu}_{1,2} = s\mu_{1,4}, \quad \bar{\mu}_{2,3} = s\mu_{1,2}, \quad \bar{\mu}_{3,4} = s\mu_{1,4}.$$



Family (C): Properties

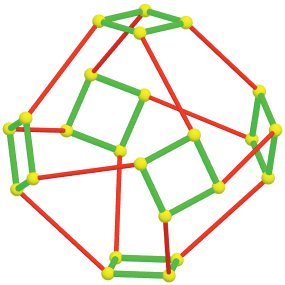
The spherical indicatrices of the spherical 4R-loops around opposite centers $P_{i,j}$ are related by a direct isometry. The spherical indicatrices of the spherical 4R-loops around adjacent centers $P_{i,j}$ correspond to two motion modes of the same spherical 4-bar. Moreover, two adjacent vertices $P_{i,j}$ and $P_{j,k}$ are related by a half-turn ρ with: $P_{i,j} \mapsto P_{j,k}$ $F_{i,j} \mapsto \widehat{F}_{j,k}$ $\widehat{F}_{i,j} \mapsto F_{j,k}$.

Conclusion & Open Problems

Conclusion & Open Problems

We studied flexible arrangements of two Bennett mechanisms and obtained three four-parametric solution families. Families (A) and (B) are globally line-symmetric and family (C) has a local line-symmetric property.

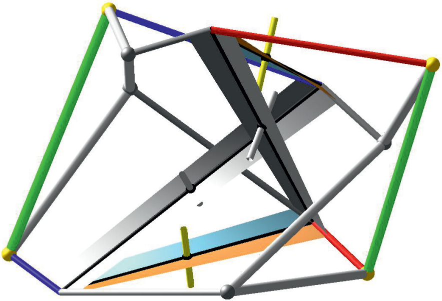
- Complete solution remains open.
- The flexible arrangements of a Bennett tube with a quadrilateral pyramid/prism remains also open.
- Does a flexible truncated octahedron-like structure exists which consists of 8 rigid hexagonal skew faces and 6 Bennett loops?



Conclusion & Open Problems

- One 6R loop is highlighted, which is contained in an flexible arrangement of two Bennett tubes of family (C).

  Are these overconstrained 6R linkages known?



Remark: The corresponding 6R loops in the flexible arrangement of two Bennett tubes of families (A) and (B) are line-symmetric.

Acknowledgment

The research is supported by project F77 (SFB "Advanced Computational Design") of the Austrian Science Fund (FWF).

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