On continuous flexible Kokotsakis belts of the isogonal type and V-hedra with skew faces

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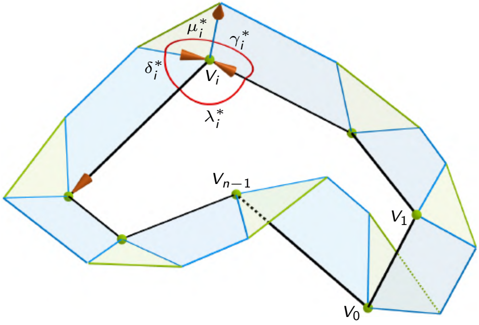
Introduction

Kokotsakis studied the following problem in 1932

Given is a rigid closed polygonal line p **(planar or non-planar), which is surrounded by a polyhedral strip, where at each polygon vertex three faces meet. Determine the geometries of these closed strips with a continuous mobility.**

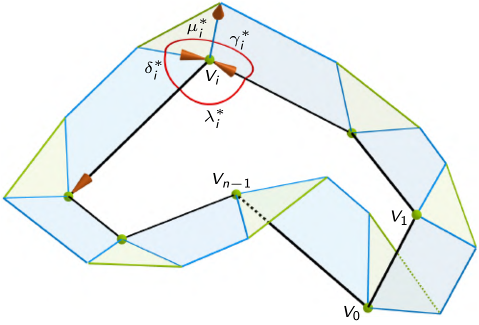
In general these loop structures are rigid, thus continuous flexible ones possess a so-called overconstrained mobility.
Kokotsakis [Kok32] himself only studied flexible belts with planar polygons p.



Review

Kokotsakis [Kok32] only obtained general results (arbitrary n) for the isogonal type; i.e. in every vertex both pairs of opposite angles are (1) equal or (2) supplementary;

$$\begin{aligned} &(1)\quad \lambda_i^* = \mu_i^*, && \delta_i^* = \gamma_i^*, \\ &(2)\quad \lambda_i^* + \mu_i^* = \pi, && \delta_i^* + \gamma_i^* = \pi. \end{aligned}$$



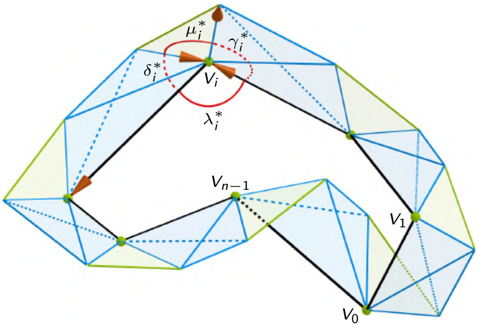
Special cases

n=3: Bricard octahedra of the 3rd type.
n=4: (3×3) building blocks of V-hedra, which are discrete analogs of Voss surfaces.

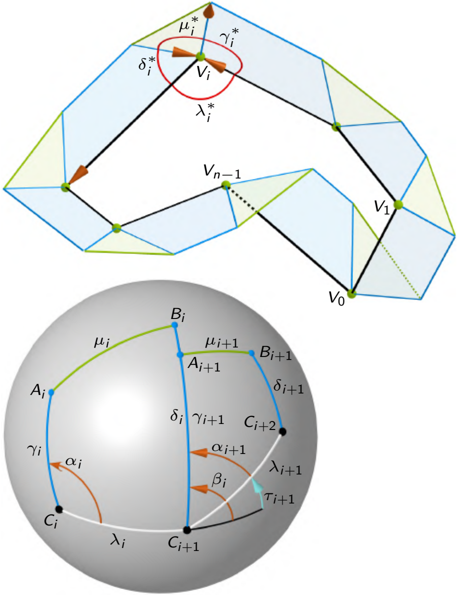
Goal & Outline

Goal: We generalize Kokotsakis' problem by allowing the faces, which are adjacent to polygon line-segments, to be skew. We do not restrict to planar polygons p but to the isogonal type.

1. **Spherical image**
2. **Solving the stated problem**
3. **Continuous flexible skew-quad surfaces (especially V-hedra)**
4. **References**



1. Spherical image of the original Kokotsakis belts

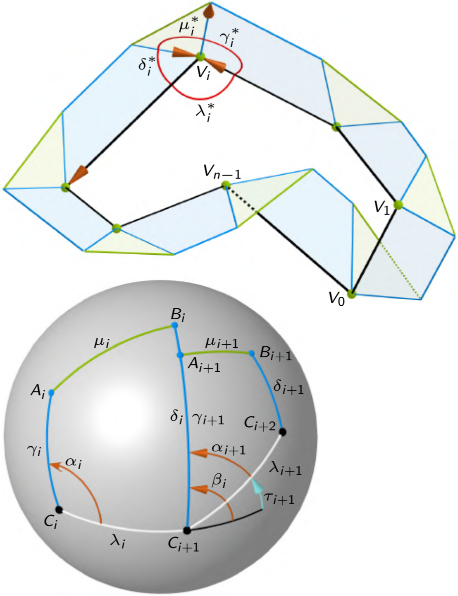


According to Stachel [Sta10] the Kokotsakis belt is continuous flexible if and only if the spherical image has this property.

Taking the orientation of the line-segments into account, the spherical 4-bar mechanism, which corresponds with the arrangement of faces around the vertex V_i, has spherical bar lengths:

$$\delta_i = \pi - \delta_i^*, \qquad \gamma_i = \pi - \gamma_i^*,$$
$$\lambda_i = \pi - \lambda_i^*, \qquad \mu_i = \pi - \mu_i^*.$$

1. Spherical image of the original Kokotsakis belts

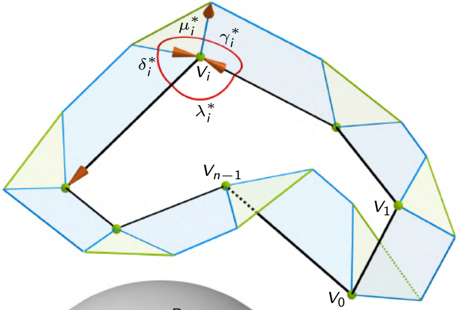


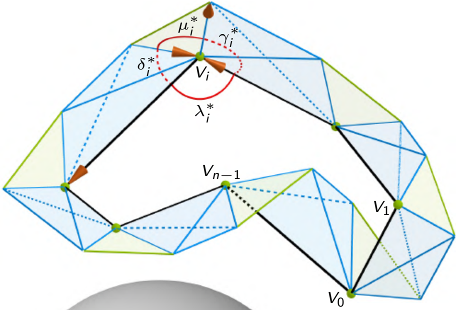
The spherical image of faces around two adjacent vertices V_i and V_{i+1} corresponds to two coupled spherical 4-bar mechanisms.

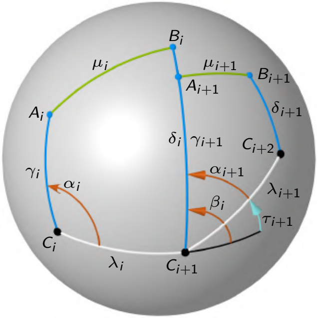
The dihedral angles β_i and α_{i+1} are related by the torsion angle τ_{i+1} of the polygon p.

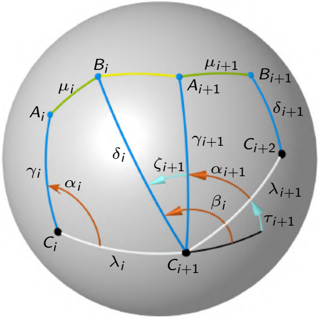
Remark: Note that p is a planar curve if all τ_{i+1} are either zero or π.

1. Spherical image of the generalized Kokotsakis belts









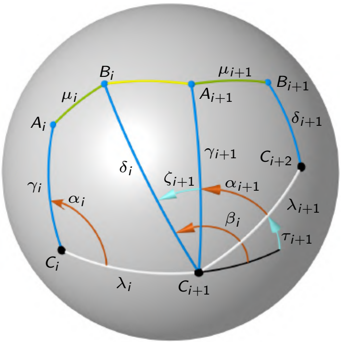
The non-planarity of the faces imply that B_i, C_{i+1}, A_{i+1} are not longer on a great circle.

1. Spherical isogram

In the isogonal case these 4-bar mechanisms are so-called spherical isograms; i.e.

$$\begin{aligned} &(1)\quad \lambda_i = \mu_i, && \delta_i = \gamma_i, \\ &(2)\quad \lambda_i + \mu_i = \pi, && \delta_i + \gamma_i = \pi. \end{aligned}$$

These two types are related by the replacement of one of the vertices of the spherical isogram by its antipodal point. Without loss of generality we can restrict to type (1) by assuming an appropriate choice of orientations.



1. Spherical kinematics

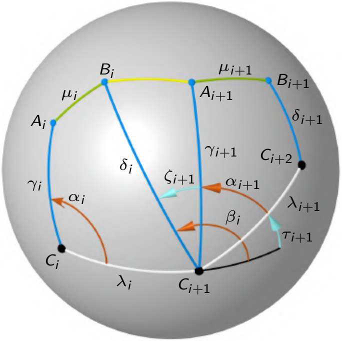
According to Stachel [Sta10] the input angle α_i and the output angle β_i of the i-th spherical isogram of type (1) are related by

$$b_i = f_i a_i \quad \text{with} \quad f_i = \frac{\sin\delta_i \pm \sin\lambda_i}{\sin(\delta_i - \lambda_i)} \neq 0 \qquad (\star)$$

where $a_i = \tan\frac{\alpha_i}{2}$ and $b_i = \tan\frac{\beta_i}{2}$.

The shift between the output angle β_i of the i-th isogram to the input angle α_{i+1} of the $(i+1)$-th isogram is given by the angle ε_{i+1}; i.e.

$$a_{i+1} = \frac{b_i + e_{i+1}}{1 - b_i e_{i+1}} \quad \text{with} \quad e_{i+1} = \tan\frac{\varepsilon_{i+1}}{2}. \qquad (\circ)$$



2. Solving the stated problem

Firstly, we formulate the so-called closure condition

$$a_0 - a_n = 0.$$

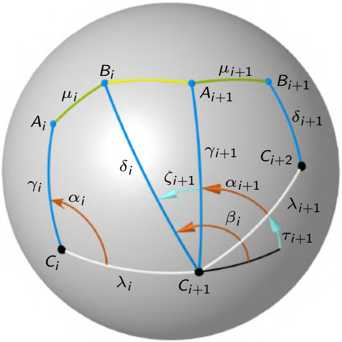
Within this condition we substitute a_n by

$$a_n = \frac{a_{n-1} f_{n-1} + e_n}{1 - a_{n-1} f_{n-1} e_n}$$

which results from ($\circ$) under consideration of ($\star$).
By iterating this kind of substitution we end up with

$$q_2 a_0^2 + q_1 a_0 + q_0 = 0,$$

where q_is are functions in $f_0, \ldots, f_{n-1}, e_0, \ldots, e_{n-1}$.



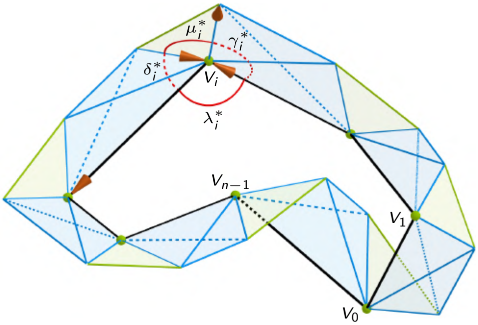
2. Solving the stated problem

Thus the necessary and sufficient conditions for continuous mobility are:

$$q_0 = 0, \quad q_1 = 0, \quad q_2 = 0.$$

Theorem 1.

For a given closed polygon p with n vertices, there exists at least a $(2n-3)$-dimensional set of continuous flexible Kokotsakis belts of the isogonal type over $\mathbb{C}$.



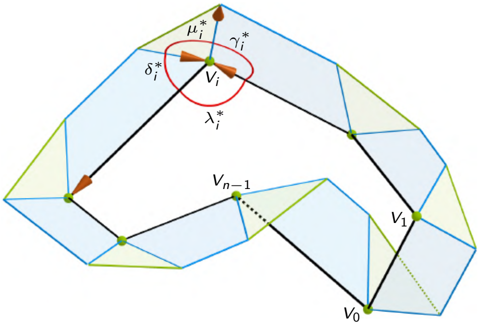
2. Solving the stated problem

Thus the necessary and sufficient conditions for continuous mobility are:

$$q_0 = 0, \quad q_1 = 0, \quad q_2 = 0.$$

Theorem 2.

For a given closed polygon p with $n > 3$ vertices, there exists at least a $(n-3)$-dimensional set of continuous flexible Kokotsakis belts with planar faces of the isogonal type over $\mathbb{C}$. For planar polygons p this dimension raises to $(n-1)$.

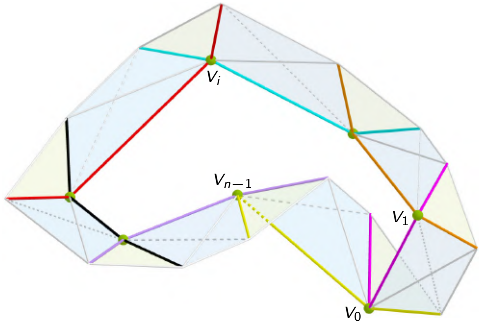


2. Property regarding the rotation angles

Dihedral angles along opposite edges meeting in a vertex V_i have at each time instant the same absolute value of their angular velocities.

Thus the absolute values of the rotation angles around these two edges are the same (measured from an initial configuration).

The same absolute values of the rotation angle can always be assigned to three edges.

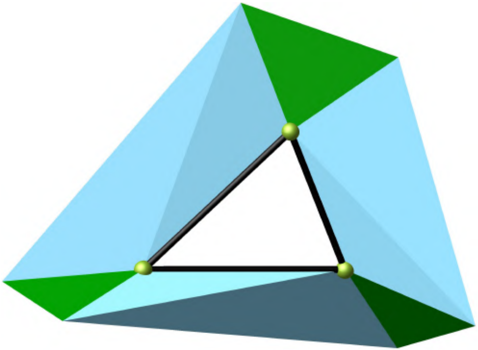


2. Example: n=3

For any choice of δ_i and γ_i with $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$ there exist $e_0, e_1, e_2 \in \mathbb{C}$ such that we get a continuous flexible Kokotsakis belt of the isogonal type.

The resulting structure can be seen as an overconstrained 6R loop, which belongs to the third class of so-called angle-symmetric 6R linkages (cf. Li & Schicho [LS13]).

Remark: Note that for $e_0 = e_1 = e_2 = 0$ we get a Bricard octahedron of the 3rd type.



3. Continuous flexible skew-quad (SQ) surfaces

Vorgegeben sei ein Vierecksnetz mit starren und i.a. nicht-ebenen Vierecksmaschen. Bei der Realisierung etwa durch ein Blechmodell kann man die Vierecke durch irgend welche Flächenstücke, z.B. durch Ausschnitte aus hyperbolischen Paraboloiden, ausfüllen. Wir nehmen an, daß das Vierecksnetz mindestens drei Leitstreifen einer jeden der beiden Scharen, also 3 × 3 Vierecksmaschen, enthält. Ein solches Vierecksnetz ist i.a. *starr*, d.h. es läßt keine Verknickungen durch Drehung benachbarter Maschen um die jeweils gemeinsame Maschenseite zu, ohne daß es zu einer Zerreißung des Netzes kommt. Wir haben aber auch Vierecksnetze kennen gelernt, die eine 1-parametrige Menge von Verknickungen zulassen (*verknickbare Vierecksnetze*), nämlich die V-Netze in § 12 und die T-Netze in § 13. Dabei handelte es sich um ebenflächige Vierecksnetze; ob es auch nicht-ebenflächige verknickbare Vierecksnetze gibt, ist ein ungelöstes Problem. Bei den Verknickungen

On page 168 of Sauer's book [Sau70] the following open problem is mentioned:

Do there exist continuous flexible SQ surfaces?

A key result for the answering of this question is the following generalization of a theorem given by Schief, Bobenko & Hoffmann [SBH08]:

Theorem 3.

A non-degenerate SQ surface is continuous flexible, if and only if this holds true for every (3×3) building block.

3. Building block of a V-hedron with skew quads

$$
\begin{aligned}
q_2 :=& f_0[e_0 f_3(e_1 e_2 f_2 + e_2 e_3 f_1 + e_1 e_3 - f_1 f_2) + e_1 e_2 e_3 f_2 - e_3 f_1 f_2 - e_2 f_1 - e_1], \\
q_1 :=& e_1 e_2 f_0 f_2 f_3 + e_2 e_3 f_0 f_1 f_3 + e_1 e_3 f_0 f_3 - e_1 e_3 f_1 f_2 - f_0 f_1 f_2 f_3 - e_1 e_2 f_1 - e_2 e_3 f_2 + 1 + \\
& e_0(e_1 e_2 e_3 f_1 f_3 - e_1 e_2 e_3 f_0 f_2 - e_1 f_1 f_2 f_3 + e_3 f_0 f_1 f_2 + e_2 f_0 f_1 - e_2 f_2 f_3 + e_1 f_0 - e_3 f_3), \\
q_0 :=& e_0(e_1 e_3 f_1 f_2 + e_1 e_2 f_1 + e_2 e_3 f_2 - 1) + f_3(e_1 e_2 e_3 f_1 - e_1 f_1 f_2 - e_2 f_2 - e_3).
\end{aligned}
$$

We can solve this set of equations explicitly for e_1, e_2, e_3:

$$
\begin{aligned}
e_1 &= \frac{e_0 f_0 f_2 (f_1^2 - 1)(f_3^2 - 1) \pm R_1 R_2}{e_0^2 (f_0 f_2 - f_1 f_3)(f_0 - f_1 f_2 f_3) + (f_0 f_3 - f_1 f_2)(f_0 f_2 f_3 - f_1)}, \\
e_2 &= \frac{\mp R_1 R_2}{e_0^2 (f_0 f_1 - f_2 f_3)(f_0 f_2 - f_1 f_3) + (f_0 f_1 f_3 - f_2)(f_0 f_2 f_3 - f_1)}, \\
e_3 &= \frac{e_0 f_1 f_3 (f_0^2 - 1)(f_2^2 - 1) \pm R_1 R_2}{e_0^2 (f_0 f_2 - f_1 f_3)(f_0 f_1 f_2 - f_3) + (f_0 f_3 - f_1 f_2)(f_0 f_1 f_3 - f_2)}
\end{aligned}
$$

with

$$
\begin{aligned}
R_1 &:= \sqrt{e_0^2 (f_0 f_1 - f_2 f_3)(f_0 f_2 - f_1 f_3) + (f_0 f_1 f_3 - f_2)(f_0 f_2 f_3 - f_1)}\,, \\
R_2 &:= \sqrt{e_0^2 (f_0 f_1 f_2 - f_3)(f_1 f_2 f_3 - f_0) + (f_0 f_1 f_2 f_3 - 1)(f_1 f_2 - f_0 f_3)}\,.
\end{aligned}
$$

3. Building block of a V-hedron with skew quads

$$
\begin{aligned}
q_2 :=& f_0[e_0 f_3(e_1 e_2 f_2 + e_2 e_3 f_1 + e_1 e_3 - f_1 f_2) + e_1 e_2 e_3 f_2 - e_3 f_1 f_2 - e_2 f_1 - e_1],\\
q_1 :=& e_1 e_2 f_0 f_2 f_3 + e_2 e_3 f_0 f_1 f_3 + e_1 e_3 f_0 f_3 - e_1 e_3 f_1 f_2 - f_0 f_1 f_2 f_3 - e_1 e_2 f_1 - e_2 e_3 f_2 + 1 +\\
& e_0(e_1 e_2 e_3 f_1 f_3 - e_1 e_2 e_3 f_0 f_2 - e_1 f_1 f_2 f_3 + e_3 f_0 f_1 f_2 + e_2 f_0 f_1 - e_2 f_2 f_3 + e_1 f_0 - e_3 f_3),\\
q_0 :=& e_0(e_1 e_3 f_1 f_2 + e_1 e_2 f_1 + e_2 e_3 f_2 - 1) + f_3(e_1 e_2 e_3 f_1 - e_1 f_1 f_2 - e_2 f_2 - e_3).
\end{aligned}
$$

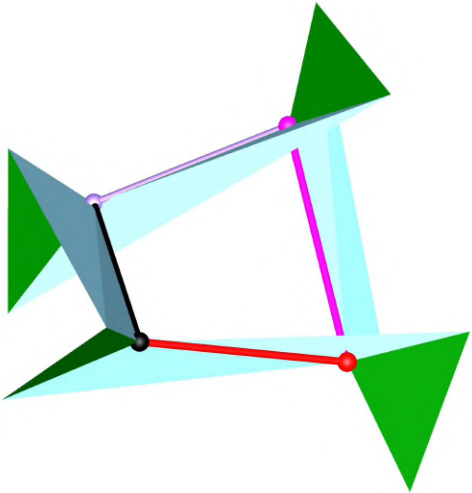
We can solve this set of equations explicitly for f_1, f_2, f_3:

$$
\begin{aligned}
f_1 &= \frac{-(f_0^2+1)e_0 e_1(e_3^2+1)(e_2^2-1)+f_0(e_0^2 e_1^2 e_2^2 - e_0^2 e_1^2 e_3^2 - e_0^2 e_2^2 e_3^2 - e_1^2 e_2^2 e_3^2 + e_0^2 + e_1^2 + e_2^2 - e_3^2) \pm R_3 R_4}{2e_2(e_3^2+1)(e_0 f_0 + e_1)(e_0 e_1 - f_0)},\\
f_2 &= \frac{(f_0^2+1)e_0 e_1(e_3^2+1)(e_2^2+1)-f_0(e_0^2 e_1^2 e_2^2 + e_0^2 e_1^2 e_3^2 - e_0^2 e_2^2 e_3^2 - e_1^2 e_2^2 e_3^2 - e_0^2 - e_1^2 + e_2^2 + e_3^2) \mp R_3 R_4}{2e_2 e_3 f_0(e_1^2+1)(e_0^2+1)},\\
f_3 &= \frac{-(f_0^2+1)e_0 e_1(e_3^2-1)(e_2^2+1)-f_0(e_0^2 e_1^2 e_2^2 - e_0^2 e_1^2 e_3^2 + e_0^2 e_2^2 e_3^2 + e_1^2 e_2^2 e_3^2 - e_0^2 + e_1^2 + e_2^2 - e_3^2) \pm R_3 R_4}{2e_3(e_2^2+1)(e_1 f_0 + e_0)(e_0 e_1 - f_0)}
\end{aligned}
$$

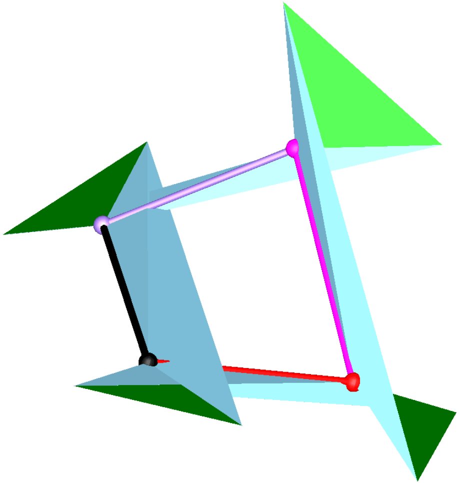
with

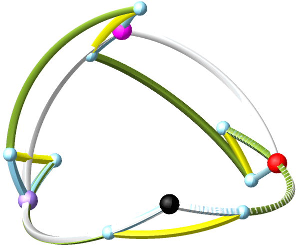
$$
\begin{aligned}
R_{3,4} :=& [f_0(e_0^2 e_1^2 e_2^2 + e_0^2 e_1^2 e_3^2 - e_0^2 e_2^2 e_3^2 - e_1^2 e_2^2 e_3^2 - e_0^2 - e_1^2 + e_2^2 + e_3^2) -\\
& (f_0^2+1)e_0 e_1(e_3^2+1)(e_2^2+1) \pm 2f_0 e_2 e_3(e_1^2+1)(e_0^2+1)]^{\frac{1}{2}}
\end{aligned}
$$

3. Building block of a V-hedron with skew quads



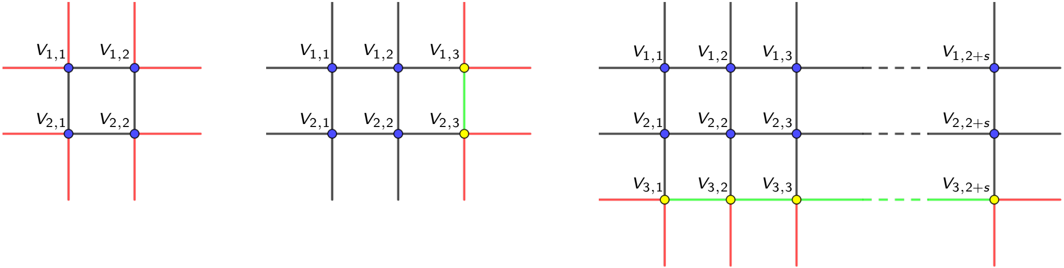
A (3×3) building block of a V-hedron with skew quads (left) and its spherical image (right).





3. Lower bound on the dimension of the design space

This bound q can be obtained by comparing the number q_{par} of free parameters for constructing a $([3+t] \times [3+s])$ skew quad mesh with the number q_{con} of algebraic conditions needed to make the mesh isogonal and continuous flexible.



$$q_{par} = 21 + 10s + 10t + 3st,$$
$$q_{con} = 11 + 7s + 7t + 5st.$$

3. Lower bound on the dimension of the design space

Thus finally we get the lower bound q by:

$$q := q_{par} - q_{con} - 1 = 9 + 3s + 3t - 2st.$$

The subtraction of 1 comes from the fact that the structure has a 1-dimensional mobility.

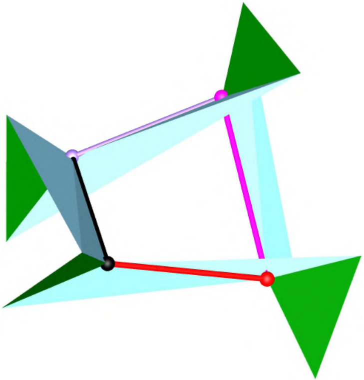
t\s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$i > 15$
0	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	9+3i
1		13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	12+i
2			13	12	11	10	9	8	7	6	5	4	3	2	1	0	< 0
3				9	6	3	0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0
4					1	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0
5						< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0
$\vdots$							$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$

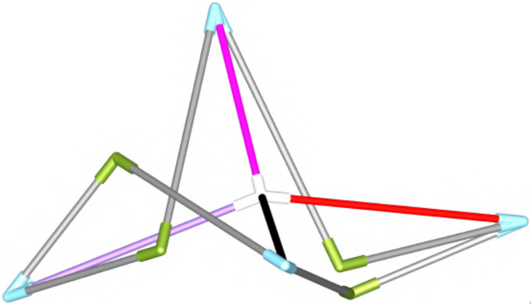
It remains open if a continuous flexible SQ surface of infinite dimension in rows and columns exists.

3. Associated overconstrained mechanism

Definition: Reciprocal-parallel quad meshes $\mathcal{Q}$ and $\mathcal{V}$ (cf. Sauer [Sau70])

- $\mathcal{Q}$ and $\mathcal{V}$ are combinatorial dual; i.e. vertices correspond to faces and vice versa.
- The edges of adjacent faces are mapped to edges between corresponding adjacent vertices and vice versa. Moreover, corresponding edges are parallel.

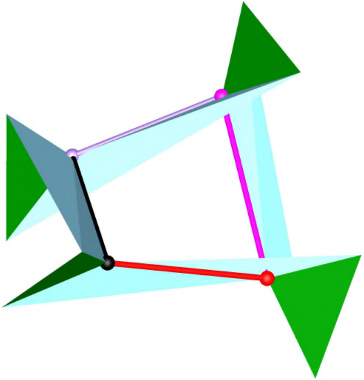


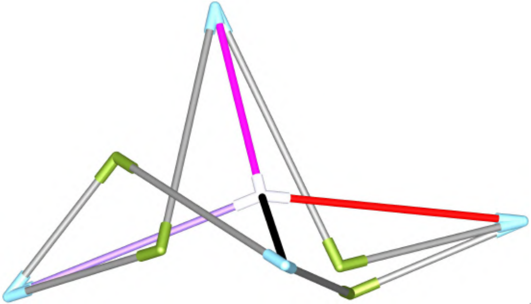


3. Associated overconstrained mechanism

Sauer [Sau70] showed that every infinitesimal flexible quad surface $\mathcal{Q}$ possesses in general a unique (up to scaling) reciprocal-parallel quad mesh $\mathcal{V}$.

The corresponding deformation of $\mathcal{V}$ during the continuous flexion of $\mathcal{Q}$ has to be a conformal transformation, as the vertex stars are rigid.





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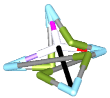

The corresponding overconstrained mechanism consists of rigid vertex stars linked by cylindrical joints and one rotational joint.

4. References

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