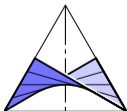


# Generalizing continuous flexible Kokotsakis belts of the isogonal type

Georg Nawratil<sup>1,2</sup>

<sup>1</sup>Institute of Discrete Mathematics and Geometry, TU Wien  
[www.dmg.tuwien.ac.at/nawratil/](http://www.dmg.tuwien.ac.at/nawratil/)

<sup>2</sup>Center for Geometry and Computational Design, TU Wien

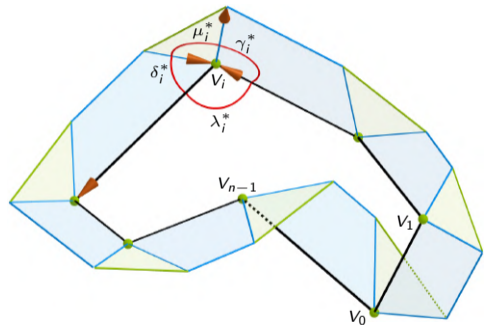


# Introduction

## Kokotsakis studied the following problem in 1932

Given is a rigid closed polygonal line  $p$  (planar or non-planar), which is surrounded by a polyhedral strip, where at each polygon vertex three faces meet. Determine the geometries of these closed strips with a continuous mobility.

In general these loop structures are rigid, thus continuous flexible ones possess a so-called overconstrained mobility. Kokotsakis himself only studied flexible belts with planar polygons  $p$  in [1].



# Review

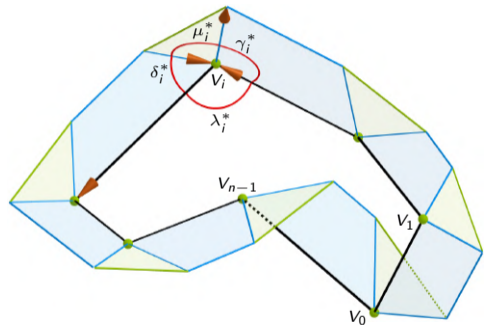
Moreover, Kokotsakis only obtained general results (arbitrary  $n$ ) for the isogonal type; i.e. in every vertex both pairs of opposite angles are (1) equal or (2) supplementary;

$$\begin{aligned} (1) \quad & \lambda_i^* = \mu_i^*, & \delta_i^* &= \gamma_i^*, \\ (2) \quad & \lambda_i^* + \mu_i^* = \pi, & \delta_i^* + \gamma_i^* &= \pi. \end{aligned}$$

## Special cases

**n=3:** Bricard octahedra of the 3rd type.

**n=4:**  $(3 \times 3)$  building blocks of V-hedra, which are discrete analogs of Voss surfaces.

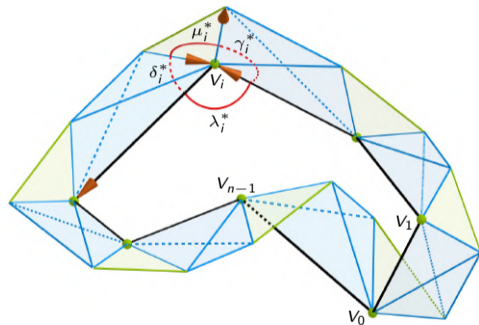




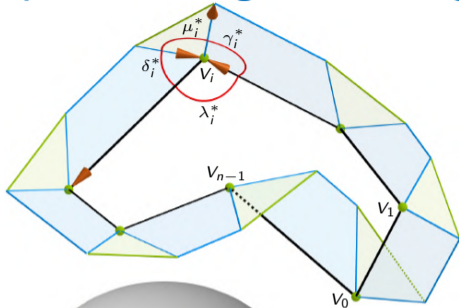
# Goal & Outline

**Goal:** We generalize Kokotsakis' problem by allowing the faces, which are adjacent to polygon line-segments, to be skew. We do not restrict to planar polygons  $p$  but to the isogonal type.

1. Spherical image
2. Solving the stated problem
3. Continuous flexible skew-quad surfaces
4. Open problems & References

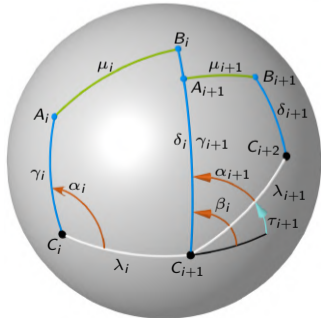


# 1. Spherical image of the original Kokotsakis belts



According to [6] the Kokotsakis belt is continuous flexible if and only if the spherical image has this property.

Taking the orientation of the line-segments into account, the spherical 4-bar mechanism, which corresponds with the arrangement of faces around the vertex  $V_i$ , has spherical bar lengths:



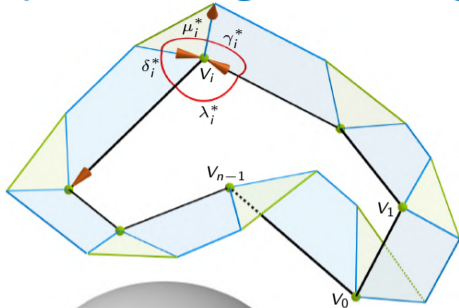
$$\delta_i = \pi - \delta_i^*,$$

$$\gamma_i = \pi - \gamma_i^*,$$

$$\lambda_i = \pi - \lambda_i^*,$$

$$\mu_i = \pi - \mu_i^*.$$

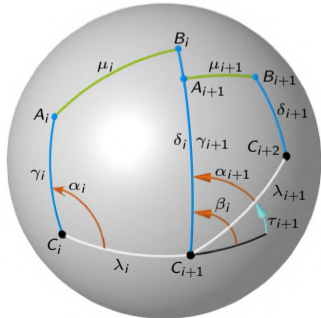
# 1. Spherical image of the original Kokotsakis belts



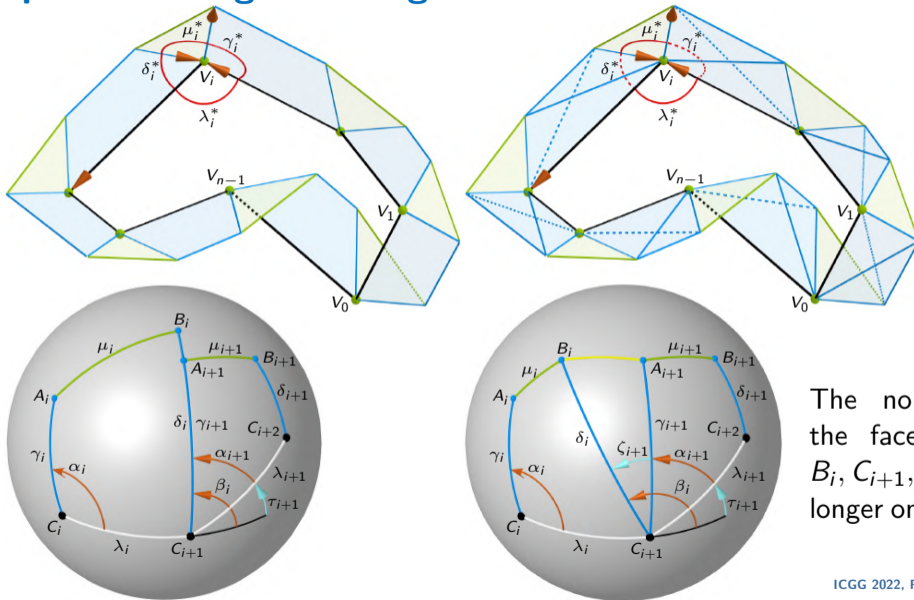
The spherical image of faces around two adjacent vertices  $V_i$  and  $V_{i+1}$  corresponds to two coupled spherical 4-bar mechanisms.

The dihedral angles  $\beta_i$  and  $\alpha_{i+1}$  are related by the torsion angle  $\tau_{i+1}$  of the polygon  $p$ .

**Remark:** Note that  $p$  is a planar curve if all  $\tau_{i+1}$  are either zero or  $\pi$ .



# 1. Spherical image of the generalized Kokotsakis belts



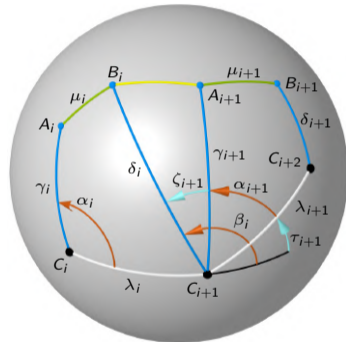
The non-planarity of the faces imply that  $B_i, C_{i+1}, A_{i+1}$  are not longer on a great circle.

# 1. Spherical isogram

In the isogonal case these 4-bar mechanisms are so-called spherical isograms; i.e.

$$\begin{aligned} (1) \quad & \lambda_i = \mu_i, & \delta_i &= \gamma_i, \\ (2) \quad & \lambda_i + \mu_i = \pi, & \delta_i + \gamma_i &= \pi. \end{aligned}$$

These two types are related by the replacement of one of the vertices of the spherical isogram by its antipodal point. Without loss of generality we can restrict to type (1) by assuming an appropriate choice of orientations.



# 1. Spherical kinematics

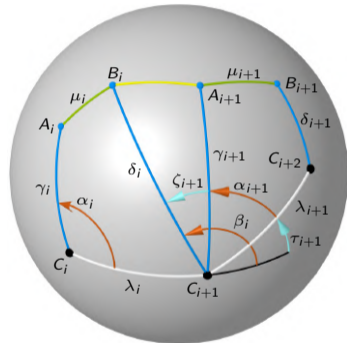
The input angle  $\alpha_i$  and the output angle  $\beta_i$  of the  $i$ -th spherical isogram of type (1) are related by

$$b_i = f_i a_i \quad \text{with} \quad f_i = \frac{\sin \delta_i \pm \sin \lambda_i}{\sin(\delta_i - \lambda_i)} \neq 0 \quad (\star)$$

where  $a_i = \tan \frac{\alpha_i}{2}$  and  $b_i = \tan \frac{\beta_i}{2}$ .

The shift between the output angle  $\beta_i$  of the  $i$ -th isogram to the input angle  $\alpha_{i+1}$  of the  $(i+1)$ -th isogram is given by the angle  $\varepsilon_{i+1}$ ; i.e.

$$a_{i+1} = \frac{b_i + e_{i+1}}{1 - b_i e_{i+1}} \quad \text{with} \quad e_{i+1} = \tan \frac{\varepsilon_{i+1}}{2}. \quad (\circ)$$



## 2. Solving the stated problem

Firstly, we formulate the so-called closure condition

$$a_0 - a_n = 0.$$

Within this condition we substitute  $a_n$  by

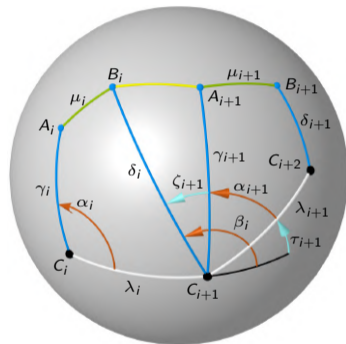
$$a_n = \frac{a_{n-1}f_{n-1} + e_n}{1 - a_{n-1}f_{n-1}e_n}$$

which results from (o) under consideration of (\*).

By iterating this kind of substitution we end up with

$$q_2 a_0^2 + q_1 a_0 + q_0 = 0,$$

where  $q_i$ s are functions in  $f_0, \dots, f_{n-1}, e_0, \dots, e_{n-1}$ .



## 2. Solving the stated problem

Thus the necessary and sufficient conditions for continuous mobility are:

$$q_0 = 0, \quad q_1 = 0, \quad q_2 = 0.$$

### Theorem 1.

For a given closed polygon  $p$  with  $n$  vertices, there exists at least a  $(2n - 3)$ -dimensional set of continuous flexible Kokotsakis belts of the isogonal type over  $\mathbb{C}$ .

### Theorem 2.

For a given closed polygon  $p$  with  $n > 3$  vertices, there exists at least a  $(n - 3)$ -dimensional set of continuous flexible Kokotsakis belts with planar faces of the isogonal type over  $\mathbb{C}$ . For planar polygons  $p$  this dimension raises to  $(n - 1)$ .

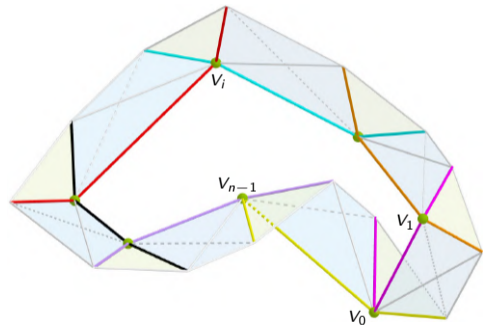


## 2. Property regarding the rotation angles

Dihedral angles along opposite edges meeting in a vertex  $V_i$  have at each time instant the same absolute value of their angular velocities.

Thus the absolute values of the rotation angles around these two edges are the same (measured from an initial configuration).

The same absolute values of the rotation angle can always be assigned to three edges.

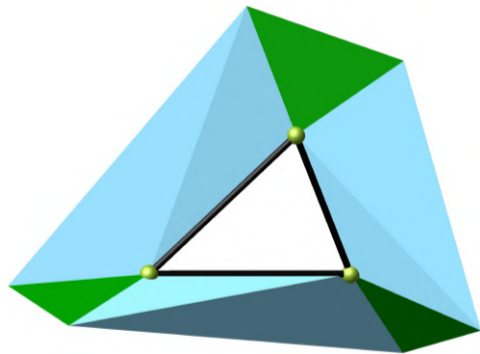


## 2. Example: $n=3$

For any choice of  $\delta_i$  and  $\gamma_i$  with  $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$  there exist  $e_0, e_1, e_2 \in \mathbb{C}$  such that we get a continuous flexible Kokotsakis belt of the isogonal type.

The resulting structure can be seen as an overconstrained 6R loop, which belongs to the third class of so-called angle-symmetric 6R linkages [23].

**Remark:** Note that for  $e_0 = e_1 = e_2 = 0$  we get a Bricard octahedron of the 3rd type.



### 3. Continuous flexible skew-quad (SQ) surfaces

Vorgegeben sei ein Vierecksnetz mit starren und i.a. nicht-ebenen Vierecksmaschen. Bei der Realisierung etwa durch ein Blechmodell kann man die Vierecke durch irgend welche Flächenstücke, z.B. durch Ausschnitte aus hyperbolischen Paraboloiden, ausfüllen. Wir nehmen an, daß das Vierecksnetz mindestens drei Leitstreifen einer jeden der beiden Scharen, also  $3 \times 3$  Vierecksmaschen, enthält. Ein solches Vierecksnetz ist i.a. starr, d.h. es läßt keine Verknickungen durch Drehung benachbarter Maschen um die jeweils gemeinsame Maschenseite zu, ohne daß es zu einer Zerreiung des Netzes kommt. Wir haben aber auch Vierecksnetze kennen gelernt, die eine 1-parametrische Menge von Verknickungen zulassen (verknickbare Vierecksnetze), nmlich die V-Netze in § 12 und die T-Netze in § 13. Dabei handelte es sich um ebenflchige Vierecksnetze; ob es auch nicht-ebenflchige verknickbare Vierecksnetze gibt, ist ein ungelstes Problem. Bei den Verknickungen

On page 168 of Sauer's book [15] the following open problem is mentioned:

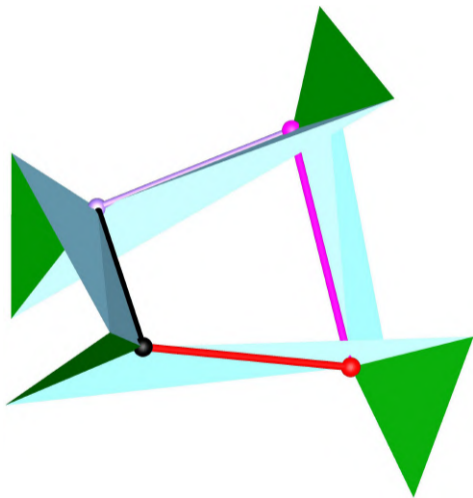
*Do there exist continuous flexible SQ surfaces?*

The restriction to  $(3 \times 3)$  building blocks is sufficient as the following theorem holds:

#### Theorem 3.

A non-degenerate SQ surface is continuous flexible, if and only if this holds true for every  $(3 \times 3)$  building block.

### 3. Building block of a V-hedron with SQ



A  $(3 \times 3)$  building block of a V-hedron with skew quads (left) and its spherical image (right).





### 3. Associated overconstrained mechanism

An overconstrained mechanism which results from the  $(3 \times 3)$  complex of a V-hedron with SQ by reciprocal-parallelism.

It consists of rigid vertex stars linked by cylindrical joints and one rotational joint (black).

## 4. Open problems

Open questions regard the smooth analogs of continuous flexible

- Kokotsakis belts of the isogonal type,
- V-hedra with skew quads.

### References

All references refer to the list of publications given in the presented paper:

Nawratil, G.: *Generalizing continuous flexible Kokotsakis belts of the isogonal type*. Proc. of the 20th International Conference on Geometry and Graphics (Liang-Yee Cheng ed.), Springer (2022)

### Acknowledgment

The research is supported by project F77 (SFB “Advanced Computational Design”, SP7) of the Austrian Science Fund.

