Basic result on type II DM self-motions of planar Stewart Gough platforms

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Abstract. In a recent publication [10] the author showed that self-motions of general planar Stewart Gough platforms can be classified into two so-called Darboux Mannheim (DM) types (I and II). Moreover, in [10] the author was able to compute the set of equations yielding a type II DM self-motion explicitly. Based on these equations we present a basic result for this class of self-motions.

Key words: Self-motion, Stewart Gough platform, Borel Bricard problem

1 Introduction

The geometry of a planar Stewart Gough (SG) platform is given by the six base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, 0)^T$ with respect to the fixed system Σ_0 and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, 0)^T$ with respect to the moving system Σ. By using Study parameters $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3)$ to parametrize Euclidean displacements, the coordinates $\mathbf{m}_i'$ of the platform anchor points with respect to Σ_0 can be written as $K\mathbf{m}_i' = \mathbf{R}\,\mathbf{m}_i + (t_1, t_2, t_3)^T$ with

$$
\begin{aligned}
t_1 &= 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \qquad t_2 = 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3),\\
t_3 &= 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1), \qquad K = e_0^2 + e_1^2 + e_2^2 + e_3^2 \neq 0 \quad \text{and}\\
\mathbf{R} = (r_{ij}) &= \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}.
\end{aligned}
$$

Now all points of $P_{\mathbb{R}}^7$ which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the subspace $e_0 = \ldots = e_3 = 0$ of Ψ, as these points cannot fulfill the normalizing condition $K = 1$.

If the geometry of the manipulator is given as well as the six leg lengths, then the SG platform is in general rigid, but it can even be the case that the manipulator can perform an n-parametric motion ($n > 0$), which is called self-motion. Note that such motions are also solutions to the famous Borel Bricard problem (cf. [1, 3, 4, 11]).

2 Types of self-motions

In section 2 and 3 we give a very short review of the results and ideas stated in [10], where also more details and examples can be found.

It is known that architecturally singular SG platforms, which are well studied, possess self-motions in each pose. Therefore, we are only interested in the computation of self-motions of non-architecturally singular SG platforms. A detailed review of self-motions of this type was given by the author in [10].

Moreover, it is known that if a planar SG platform with anchor points $\mathsf{m}_1,\ldots,\mathsf{M}_6$ is not architecturally singular, then at least a one-parametric set of legs exists, which can be attached to the given manipulator without changing the forward kinematics [5, 9] and the singularity set [2] of the manipulator. Moreover, it was shown that in general the base anchor points M_i as well as the corresponding platform anchor points m_i are located on planar cubic curves C and c, respectively.

Assumption 1. *We assume that there exist such cubics* c *and* C *(which can also be reducible) in the Euclidean domain of the platform and the base, respectively.*

We consider the complex projective extension $P^3_{\mathbb{C}}$ of the Euclidean 3-space with $(a_i,b_i,0)\mapsto(w_i:x_i:y_i:0)$, $(A_i,B_i,0)\mapsto(W_i:X_i:Y_i:0)$ and $w_i,x_i,y_i,W_i,X_i,Y_i\in\mathbb{C}$. Note that ideal points are characterized by $w_i=0$ and $W_i=0$, respectively.

Moreover, we consider the correspondence between the points of C and c, which is determined by the geometry of the manipulator $\mathsf{m}_1,\ldots,\mathsf{M}_6$ (cf. [2, 5, 9]). As this correspondence has not to be a bijection, a point $\in P^3_{\mathbb{C}}$ of c resp. C is in general mapped to a non-empty set of points $\in P^3_{\mathbb{C}}$ of C resp. c. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{\,\}$.

In $P^3_{\mathbb{C}}$ the cubic C has three ideal points $\mathsf{U}_1,\mathsf{U}_2,\mathsf{U}_3$, where at least one of these points (e.g. U_1) is real. The remaining points U_2 and U_3 are real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{u}_1\},\{\mathsf{u}_2\},\{\mathsf{u}_3\}$ of c ($\Rightarrow\{\mathsf{u}_1\}$ contains real points). We denote the ideal points of c by $\mathsf{u}_4,\mathsf{u}_5,\mathsf{u}_6$, where again one (e.g. u_4) has to be real. The remaining points u_5 and u_6 are again real or conjugate complex. Then we compute the corresponding locations $\{\mathsf{U}_4\},\{\mathsf{U}_5\},\{\mathsf{U}_6\}$ of C ($\Rightarrow$ $\{\mathsf{U}_4\}$ contains real points).

Assumption 2. *For guaranteeing a general case, we assume that each of the corresponding locations* $\{\mathsf{u}_1\}$, $\{\mathsf{u}_2\}$, $\{\mathsf{u}_3\}$, $\{\mathsf{U}_4\}$, $\{\mathsf{U}_5\}$, $\{\mathsf{U}_6\}$ *consists of a single point. Moreover, we assume that no 4 collinear platform anchor points* u_j *or base anchor points* U_j *(*$j=1,\ldots,6$*) exist.*

Under consideration of Assumption 1 and 2, following theorem was proven [10]:

Theorem 1. *The resulting manipulator* $\mathsf{u}_1,\ldots,\mathsf{U}_6$ *is architecturally singular.*

Moreover, it was proven in [10] that there only exist type I and type II Darboux Mannheim (DM) self-motions, where the definition of types reads as follows:

Definition 1. Assume $\mathscr{M}$ is a one-parametric self-motion of a non-architecturally singular SG platform $\mathsf{m}_1,\ldots,\mathsf{M}_6$. Then $\mathscr{M}$ is of the type n DM if the corresponding architecturally singular manipulator $\mathsf{u}_1,\ldots,\mathsf{U}_6$ has an n-parametric self-motion.

3 Computation of type II DM self-motions

The only examples of type II DM self-motions known to the author are those constructed by Karger in [7, 8], which are characterized by $e_0 = 0$.

The computation of type II DM self-motions in [10] was based on Darboux and Mannheim constraints, which are repeated next. With this approach it seems for the first time possible to give a complete classification of type II DM self-motions:

Darboux constraint: The constraint that the platform anchor point u_i $(i = 1,2,3)$ moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i can be written as (cf. [10])

$$\Omega_i: \ \overline{X}_i(a_i r_{11} + b_i r_{12} + t_1) + \overline{Y}_i(a_i r_{21} + b_i r_{22} + t_2) + L_i K = 0,$$

with $X_i, Y_i, a_i, b_i, L_i \in \mathbb{C}$. This is a homogeneous quadratic equation in the Study parameters where $\overline{X}_i$ and $\overline{Y}_i$ denote the conjugate complex of X_i and Y_i, respectively.

Mannheim constraint: The constraint that the plane orthogonal to u_i $(i = 4,5,6)$ through the platform point $(g_i, h_i, 0)$ slides through the point U_i of the fixed system can be written as (cf. [10])

$$\begin{aligned}\Pi_i : \overline{x}_i[A_i r_{11} + B_i r_{21} - g_i K - 2(e_0 f_1 - e_1 f_0 - e_2 f_3 + e_3 f_2)] + \\ \overline{y}_i[A_i r_{12} + B_i r_{22} - h_i K - 2(e_0 f_2 + e_1 f_3 - e_2 f_0 - e_3 f_1)] = 0,\end{aligned}$$

with $x_i, y_i, A_i, B_i, g_i, h_i \in \mathbb{C}$. This is again a homogeneous quadratic equation in the Study parameters where $\overline{x}_i$ and $\overline{y}_i$ denote the conjugate complex of x_i and y_i.

The content of the following lemma was also proven in [10]:

Lemma 1. *Without loss of generality (w.l.o.g.) we can assume that the variety of the two-parametric self-motion of* $\mathsf{u}_1, \ldots, \mathsf{U}_6$ *is spanned by* $\Psi, \Omega_1, \Omega_2, \Omega_3, \Pi_4, \Pi_5$. *Moreover, we can choose following special coordinate systems in* Σ_0 *and* Σ *w.l.o.g.:* $X_1 = Y_2 = Y_3 = x_4 = y_5 = 1$, $a_1 = b_1 = y_4 = A_4 = B_4 = Y_1 = h_4 = g_5 = 0$ *and* $X_2(X_2 - X_3)x_5 \neq 0$.

We solve the linear system of equations $\Psi, \Omega_1, \Omega_2, \Pi_4$ for $f_0, \ldots, f_3$ and plug the obtained expressions in the remaining two equations.[1] This yields in general two homogeneous polynomials $\Omega[40]$ and $\Pi[96]$ in the Euler parameters of degree 2 and 4, respectively. The number in the square brackets gives the number of terms.

Finally, we compute the resultant of Ω and Π with respect to one of the Euler parameters. Here we choose[2] e_0. This yields a homogeneous polynomial $\Gamma[117\,652]$ of degree 8 in e_1, e_2, e_3. In the following we denote the coefficients of e_1^i, e_2^j, e_3^k of Γ by Γ_{ijk}. We get a set $\mathscr{E}$ of 24 equations $\Gamma_{ijk} = 0$ in the 14 unknowns $a_2, b_2, a_3, b_3, A_5, B_5, X_2, X_3, x_5, L_1, L_2, L_3, g_4, h_5$.

Moreover, it should be noted that we denote the coefficients of $e_0^i e_1^j, e_2^k, e_3^l$ of Ω and Π by Ω_{ijkl} and Π_{ijkl}, respectively.

[1] For $e_0 e_2 - e_1 e_3 \neq 0$ this can be done w.l.o.g., as this factor belongs to the denominator of f_i.

[2] Therefore we are looking for a common factor of Ω and Π, which depends on e_0.

4 The basic result

An important step in direction of a complete classification of type II DM self-motions is done by the basic result given in Theorem 2. As preparatory work for the formulation of this theorem we have to define the following two special cases: It can easily be seen, that Ω does not depend on e_0 and e_3 (upper signs) or e_1 and e_2 (lower signs) if the following three equations are fulfilled:

$$L_1(\overline{X}_2-\overline{X}_3)-L_2+L_3=0, \quad (1)$$
$$a_2(\overline{X}_2-\overline{X}_3)\pm\overline{X}_3(\overline{X}_2b_2-\overline{X}_3b_3)\pm b_2\mp b_3=0, \quad (2)$$
$$a_3(\overline{X}_2-\overline{X}_3)\pm\overline{X}_2(\overline{X}_2b_2-\overline{X}_3b_3)\pm b_2\mp b_3=0. \quad (3)$$

Theorem 2. *With exception of the above mentioned two special cases, the corresponding manipulator* $\mathsf{u}_1,\dots,\mathsf{U}_6$ *of a planar SG platform (fulfilling Assumptions 1, 2 and Lemma 1) with a type II DM self-motion, has to have further 3 collinear anchor points in the base or in the platform beside the points* $\mathsf{U}_1,\mathsf{U}_2,\mathsf{U}_3$ *and* $\mathsf{u}_4,\mathsf{u}_5,\mathsf{u}_6$.

The proof is done by contradiction, i.e. we stop the case study if 3 anchor points beside $\mathsf{U}_1,\mathsf{U}_2,\mathsf{U}_3$ and $\mathsf{u}_4,\mathsf{u}_5,\mathsf{u}_6$ are collinear or if we get one of the 2 special cases.

Proof for the general case $\Omega_{2000}\Pi_{3000}\neq 0$

We assume $\Omega_{200}\Pi_{3000}\neq 0$, as only those solutions of $\mathscr{E}$ correspond to type II self-motions, which do not cause a vanishing of the coefficient of the highest power of e_0 in Ω and Π, respectively.

Γ_{800} can only vanish without contradiction (w.c.) for $L_1=g_4$ or if $F=\overline{X}_2(L_1-a_2)-\overline{X}_3(L_1-a_3)-L_2+L_3+b_2-b_3$ is fulfilled identically. We distinguish 3 parts:

Part [A] Assuming $L_1\neq g_4$: Now $F=0$ has to hold. W.l.o.g. we express L_1 from $F=0$. Then $\Gamma_{710}=0$ implies $a_2=a_3-\overline{X}_2b_2+\overline{X}_3b_3$. Now Γ_{620} cannot vanish w.c..

Part [B] $L_1=g_4$ and $F=0$: We express L_1 from $F=0$. W.l.o.g. we can compute h_5 from the only non-contradicting factor of Γ_{602}. Now Γ_{530} can vanish w.c. for:

1. $L_3=\overline{X}_3(L_2-b_2)/\overline{X}_2+\overline{X}_3(a_2-a_3)+b_3$: W.l.o.g. we can express A_5 from the only non-contradicting factor of Γ_{422}. Again we distinguish two cases:
   a. $\overline{X}_2b_2-\overline{X}_3b_3+a_2-a_3\neq 0$: Now Γ_{350} has only one non-contradicting factor, which can be solved for L_2 w.l.o.g.. Then we can solve the only non-contradicting factor of Γ_{314} for $\overline{x}_5$ w.l.o.g.. Now the resultant of the only non-contradicting factors of Γ_{206} and Γ_{242} with respect to B_5 cannot vanish w.c..
   b. $a_3=\overline{X}_2b_2-\overline{X}_3b_3+a_2$: Then $\Gamma_{314}=0$ implies $L_2=2\overline{X}_2^2b_2+\overline{X}_2a_2+b_2$.
      i. $\overline{X}_3(\overline{X}_2b_2-\overline{X}_3b_3)+a_2(\overline{X}_3-\overline{X}_2)+\overline{x}_5^2(b_3-b_2)\neq 0$: Under this assumption we can express B_5 from the only non-contradicting factor of Γ_{242}. Then Γ_{224} can only vanish w.c. for $\overline{X}_i=-\overline{x}_5$ with $i=2$ or $i=3$. As for $\overline{x}_5b_j+\overline{X}_jb_i=0$ with $i\neq j$ and $i,j\in\{2,3\}$ the expression Γ_{080} cannot vanish w.c., we can assume $\overline{x}_5b_j+\overline{X}_jb_i\neq 0$. Under this assumption we can compute a_2 from $\Gamma_{080}=0$ w.l.o.g.. Then the linear-combination $\Gamma_{044}-\Gamma_{026}-\Gamma_{062}$ equals $b_2^2b_3^2(\overline{X}_j+\overline{x}_5)^2(b_2-b_3)$, a contradiction.

ii. $\overline{X}_3(\overline{X}_2b_2-\overline{X}_3b_3)+a_2(\overline{X}_3-\overline{X}_2)+\overline{x}_5^2(b_3-b_2)=0$: W.l.o.g. we can solve this equation for a_2. Then Γ_{242} can only vanish w.c. for $\overline{X}_i=-\overline{x}_5$ and $\Gamma_{080}=0$ implies $\overline{X}_j=\overline{x}_5$ with $i\neq j$ and $i,j\in\{2,3\}$. Now $\overline{x}_5^2\Gamma_{206}-\Gamma_{026}$ equals $b_ib_j^2B_5\overline{x}_5^6(b_2-b_3)$, a contradiction.

2. $a_2=\overline{X}_3b_3-\overline{X}_2b_2+a_3$ and $\overline{X}_2\overline{X}_3(a_2-a_3)+\overline{X}_2(b_3-L_3)-\overline{X}_3(b_2-L_2)\neq 0$: In this case $\Gamma_{440}=0$ already yields the contradiction.

Part [C] Assuming $F\neq 0$: Now $L_1=g_4$ has to hold. Then Γ_{080} factors into $G[8]H[18]^2$. We distinguish 3 cases:

1. $G[8]=0$: W.l.o.g. we can express L_1 from $G[8]=0$. Now Γ_{170} can only vanish w.c. for $(\overline{X}_3b_3-\overline{X}_2b_2-a_2+a_3)V[18]=0$:

   a. $a_2=\overline{X}_3b_3-\overline{X}_2b_2+a_3$: We can solve the only non-contradicting factor of Γ_{620} for h_5. Now we can express L_3 from the only non-contradicting factor of Γ_{602}. Moreover, we can compute A_5 from the only non-contradicting factor of Γ_{260}. Then we can solve the only non-contradicting factor of Γ_{062} for L_2.
      i. $\overline{x}_5^2(b_2-b_3)+2\overline{x}_5(\overline{X}_2b_2-\overline{X}_3b_3)+a_3(\overline{X}_3-\overline{X}_2)+\overline{X}_2(\overline{X}_2b_2-\overline{X}_3b_3)\neq 0$: Now we can compute B_5 from the only non-contradicting factor of Γ_{404} w.l.o.g.. We distinguish two cases:
         ⋆ $\overline{X}_2b_3-\overline{X}_3b_2\neq 0$: Under this assumption we can express a_3 from the only non-contradicting factor of Γ_{026}. Then Γ_{206} cannot vanish w.c..
         ⋆ $\overline{X}_2=\overline{X}_3b_2/b_3$: Now $\Gamma_{026}=0$ implies $b_2=-b_3$. Finally $\Gamma_{206}=0$ yields the contradiction.
      ii. $\overline{x}_5^2(b_2-b_3)+2\overline{x}_5(\overline{X}_2b_2-\overline{X}_3b_3)+a_3(\overline{X}_3-\overline{X}_2)+\overline{X}_2(\overline{X}_2b_2-\overline{X}_3b_3)=0$: W.l.o.g. we can solve this equation for a_3. Then Γ_{206} can only vanish w.c. for the following two cases:
         ⋆ $\overline{X}_3=-\overline{x}_5$: Now $\Gamma_{404}=0$ implies $b_2=-b_3$ and from $\Gamma_{422}=0$ we get $B_5=-b_3$. Then $\Gamma_{440}-\Gamma_{242}=0$ yields the contradiction.
         ⋆ $\overline{X}_2=-\overline{x}_5$: $\Gamma_{404}=0$ implies $b_2=-b_3$ and Γ_{422} cannot vanish w.c..

   b. $V[18]=0$, $\overline{X}_3b_3-\overline{X}_2b_2-a_2+a_3\neq 0$: W.l.o.g. we can solve this equation for A_5. We can solve the only non-contradicting factor of Γ_{620} for h_5. Then we can express L_3 from the only non-contradicting factor of Γ_{602}. Moreover, we can solve the only non-contradicting factor of Γ_{062} for L_2.
      i. $\overline{x}_5^2(b_3-b_2)+\overline{x}_5(a_2-a_3+\overline{X}_3b_3-\overline{X}_2b_2)+\overline{X}_2a_2-\overline{X}_3a_3\neq 0$: Under this assumption we can solve $\Gamma_{404}=0$ for B_5. Then $\Gamma_{026}=0$ implies $a_2=\overline{X}_3a_3b_2/(\overline{X}_2b_3)$ and $\Gamma_{206}=0$ yields the contradiction.
      ii. $\overline{x}_5^2(b_3-b_2)+\overline{x}_5(a_2-a_3+\overline{X}_3b_3-\overline{X}_2b_2)+\overline{X}_2a_2-\overline{X}_3a_3=0$:
         ⋆ $a_2-\overline{x}_5b_2\neq 0$: In this case we can express $\overline{X}_2$ from the above equation. Now $\Gamma_{206}=0$ implies $a_3=\overline{x}_5b_3$. From $\Gamma_{404}=0$ we get $a_2=-\overline{X}_3b_2$ and $\Gamma_{422}=0$ yields $B_5=b_3$. Then $\Gamma_{440}-\Gamma_{242}$ cannot vanish w.c..
         ⋆ $a_2=\overline{x}_5b_2$: Now $\Gamma_{026}=0$ implies $b_3=\overline{X}_3a_3/(\overline{X}_2\overline{x}_5)$ and from $\Gamma_{422}=0$ we get $\overline{X}_3=-\overline{x}_5$. Then $\Gamma_{440}-\Gamma_{242}=0$ yields the contradiction.

2. $H[18]=0$, $G[8]\neq 0$ and $\overline{X}_2a_2-\overline{X}_3a_3\neq 0$: Under this assumption we can compute h_5 from $H[18]=0$. Then we can express B_5 from the only non-contradicting

factor of Γ_{620}. Moreover, from the only non-contradicting factor of Γ_{602} we can compute L_2. Then we consider the only non-contradicting factor $I[14]$ of Γ_{260}.

a. $\bar{x}_5^2(b_2-b_3)+\bar{x}_5(\overline{X}_3b_3-\overline{X}_2b_2-a_2+a_3)+\overline{X}_2a_2-\overline{X}_3a_3\neq 0$: Under this assumption we can express A_5 from $I[14]=0$.
   i. $a_3-\bar{x}_5b_3\neq 0$: Under this assumption we can express L_3 from the only non-contradicting factor of Γ_{062}. Then $\Gamma_{044}=0$ implies $a_2=\overline{X}_2b_2-\overline{X}_3b_3+a_3$. Then the resultant of Γ_{404} and Γ_{422} with respect to $\bar{x}_5$ can only vanish w.c. for $J[12](\overline{X}_2b_3-a_3)(\overline{X}_3b_2+\overline{X}_3b_3-\overline{X}_2b_2-a_3)=0$:
      ⋆ $J[12]=0$: W.l.o.g. we can solve this equation for a_3. Then $\Gamma_{404}=0$ implies $\overline{X}_3=\bar{x}_5$. Now Γ_{026} cannot vanish w.c..
      ⋆ $a_3=\overline{X}_3b_2+\overline{X}_3b_3-\overline{X}_2b_2$: Then Γ_{404} cannot vanish w.c..
      ⋆ $a_3=\overline{X}_2b_3$: Now $\Gamma_{404}=0$ implies $\overline{X}_3=\bar{x}_5$ and Γ_{206} cannot vanish w.c..
   ii. $a_3=\bar{x}_5b_3$: From $\Gamma_{062}=0$ we get $L_1=0$. Then $\Gamma_{404}=0$ implies $a_2=\overline{X}_2b_2-\overline{X}_3b_2+\bar{x}_5b_2$. Now Γ_{422} cannot vanish w.c..

b. $\bar{x}_5^2(b_2-b_3)+\bar{x}_5(\overline{X}_3b_3-\overline{X}_2b_2-a_2+a_3)+\overline{X}_2a_2-\overline{X}_3a_3=0$:
   i. $\overline{X}_3\neq\bar{x}_5$: Under this assumption we can express a_3 from the above equation. Then $I=0$ implies $\overline{X}_2=\bar{x}_5$. Now we can solve the only non-contradicting factor of Γ_{404} for A_5 w.l.o.g.. Then $\Gamma_{026}=0$ implies $a_2=\overline{X}_3b_2$. Now the difference of the only non-contradicting factors of Γ_{062} and Γ_{422} can only vanish w.c. for:
      ⋆ $\overline{X}_3=1/\bar{x}_5$: Then we can solve $\Gamma_{062}=0$ for L_3 w.l.o.g.. Finally $\Gamma_{044}=0$ yields the contradiction.
      ⋆ $L_1=4b_2b_3(\bar{x}_5-\overline{X}_3)/(b_2-b_3)$: Now Γ_{062} cannot vanish w.c..
   ii. $\overline{X}_3=\bar{x}_5$: The equation of item b can only vanish w.c. for $a_2=\bar{x}_5b_2$. Then we can express A_5 w.l.o.g. from the only non-contradicting factor of Γ_{404}. Now $\Gamma_{026}=0$ implies $a_3=\overline{X}_2b_3$. Then the difference of the only non-contradicting factors of Γ_{062} and Γ_{422} can only vanish w.c. for:
      ⋆ $\overline{X}_2=1/\bar{x}_5$: Now we can solve $\Gamma_{062}=0$ for L_3 w.l.o.g.. Then $\Gamma_{044}=0$ implies $b_2=-b_3$. Finally $\Gamma_{026}=0$ yields the contradiction.
      ⋆ $L_1=4b_2b_3(\overline{X}_2-\bar{x}_5)/(b_2-b_3)$: Now Γ_{062} cannot vanish w.c..

3. $H[18]=0$, $G[8]\neq 0$ and $a_2=\overline{X}_3a_3/\overline{X}_2$: Now $H=0$ implies $A_5=-\overline{X}_3a_3/\bar{x}_5$. W.l.o.g. we can express h_5 from the only non-contradicting factor of Γ_{620}. Moreover, the only non-contradicting factor of Γ_{602} can be solved w.l.o.g. for L_2. Then we consider the only non-contradicting factor $E[12]$ of Γ_{260}.

   a. $\overline{X}_2\bar{x}_5(b_3-b_2)-\overline{X}_2(\overline{X}_3b_3-\overline{X}_2b_2)-a_3(\overline{X}_2-\overline{X}_3)\neq 0$: Under this assumption we can solve $E[12]=0$ for B_5. Then $\Gamma_{062}=0$ implies $a_3=\bar{x}_5b_3$. Now we can express $\bar{x}_5$ w.l.o.g. from the only non-contradicting factor of Γ_{404}. Then $\Gamma_{026}=0$ implies $\overline{X}_3=0$. Finally $\Gamma_{044}=0$ yields the contradiction.
   b. $\overline{X}_2\bar{x}_5(b_3-b_2)-\overline{X}_2(\overline{X}_3b_3-\overline{X}_2b_2)-a_3(\overline{X}_2-\overline{X}_3)=0$: W.l.o.g. we can express a_3 from this equation. Then E can only vanish w.c. for:
      i. $\overline{X}_3=\bar{x}_5$: Now $\Gamma_{062}=0$ already yields the contradiction.
      ii. $\overline{X}_2=\bar{x}_5$: Now $\Gamma_{062}=0$ implies $B_5=-b_3$. Then we can solve the only non-contradicting factor of Γ_{404} for $\bar{x}_5$. $\Gamma_{026}=0$ yields the contradiction.

Proof for the special case $\Omega_{2000}\Pi_{3000} = 0$

If we set e_i equal to zero for any $i \in \{0,\ldots,3\}$, then Ω and Π have to be fulfilled identically. It can be seen immediately that the conditions implied by $\Omega = 0$ already yield a contradiction. Therefore we can assume $e_0e_1e_2e_3 \neq 0$ w.l.o.g. for this section of the proof.

Part [A] $\Omega_{2000} = 0$, $\Omega_{1000}\Pi_{3000} \neq 0$: From $\Omega_{2000} = 0$ we can express L_1 w.l.o.g.. Moreover, we can compute e_0 from $\Omega = 0$ and plug the resulting expression into Π which yields in the numerator a homogeneous polynomial $\Gamma[10058]$ of degree 7 in e_1, e_2, e_3. From $\Gamma_{700} = 0$ we can compute g_4. Then $\Gamma_{610} = 0$ yields the contradiction.

Part [B] $\Omega_{2000} = \Pi_{3000} = 0$, $\Omega_{1000}\Pi_{2000} \neq 0$: Again we express L_1 from $\Omega_{2000} = 0$. It can be seen immediately from $\Omega = 0$ that all coefficients of $\Pi_{3000} = 0$ with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can compute g_4 and h_5 from $\Pi_{3100} = 0$ and $\Pi_{3010} = 0$, respectively. We solve $\Omega = 0$ for e_0 and plug it into Π which yields in the numerator a homogeneous polynomial $\Gamma[1666]$ of degree 5 in e_1, e_2, e_3.

W.l.o.g. we can compute L_3 from $\Gamma_{500} = 0$. Then we can solve the only non-contradicting factor of Γ_{410} for L_2. Moreover, the only non-contradicting factor of Γ_{320} can be solved for A_5. Now $\Gamma_{302} = 0$ has only one non-contradicting factor which can be solved for $\bar{x}_5$. Then the difference of the only non-contradicting factors of Γ_{230} and Γ_{104} can only vanish w.c. for $a_2 = \overline{X}_3a_3b_2/(\overline{X}_2b_3)$.

1. $\overline{X}_2^2b_3^2(\overline{X}_2B_5 - a_3) + \overline{X}_3a_3(\overline{X}_2b_3^2 - B_5a_3) \neq 0$: Now we can express b_2 from the only non-contradicting factor of Γ_{230}. Then Γ_{014} cannot vanish w.c..
2. $\overline{X}_2^2b_3^2(\overline{X}_2B_5 - a_3) + \overline{X}_3a_3(\overline{X}_2b_3^2 - B_5a_3) = 0$:
    a. $a_3B_5 - \overline{X}_2b_3^2 \neq 0$: Under this assumption we can compute $\overline{X}_3$ from the above equation. Now $\Gamma_{230} = 0$ implies $a_3 = \overline{X}_2B_5$. Then Γ_{014} cannot vanish w.c..
    b. $a_3 = \overline{X}_2b_3^2/B_5$: Now the equation of item 2 can only vanish w.c. for $B_5 = \pm b_3$. In both cases $\Gamma_{212} = 0$ yields the contradiction.

Part [C] $\Omega_{2000} = \Pi_{3000} = \Pi_{2000} = 0$, $\Omega_{1000}\Pi_{1000} \neq 0$: Again we express L_1 from $\Omega_{2000} = 0$. It can be seen immediately from $\Omega = 0$ that all coefficients of $\Pi_{i000} = 0$ (for $i = 2,3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we compute g_4 and h_5 from $\Pi_{3100} = 0$ and $\Pi_{3010} = 0$, respectively. Moreover, we can solve $\Pi_{2101} = 0$ and $\Pi_{2011} = 0$ for L_3 and L_2 w.l.o.g.. We solve $\Omega = 0$ for e_0 and plug it into Π which yields in the numerator a homogeneous polynomial $\Gamma[191]$ of degree 5 in e_1, e_2, e_3.

Now $\Gamma_{410} = 0$ implies $b_2 = -B_5$. Then we can solve the only non-contradicting factor of Γ_{320} for A_5 w.l.o.g.. Now $\Gamma_{230} = 0$ implies $a_2 = -B_5\bar{x}_5$. Then we can solve the only non-contradicting factor of Γ_{302} for a_3 w.l.o.g.. We get $\overline{X}_3 = -\bar{x}_5$ from $\Gamma_{104} = 0$. Finally $\Gamma_{212} = 0$ yields the contradiction.

Part [D] $\Pi_{3000} = 0$, $\Omega_{2000}\Pi_{2000} \neq 0$: It can be seen immediately from $\Omega = 0$ that all coefficients of $\Pi_{3000} = 0$ with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express L_1 and h_5 from $\Pi_{3100} = 0$ and $\Pi_{3010} = 0$, respectively. Then we compute the resultant of $\Omega[40]$

and $\mathit{\Pi}[44]$ with respect to e_0 which yields a homogeneous polynomial $\mathit{\Gamma}[15153]$ of degree 8 in e_1, e_2, e_3. $\mathit{\Gamma}_{080}$ can only vanish w.c. in the following 2 cases:

1. $L_2 = g_4(\overline{X}_2 - \overline{X}_3) + L_3 + \overline{X}_2 a_2 - \overline{X}_3 a_3 - b_2 + b_3$: Then we can solve $\mathit{\Gamma}_{602} = 0$ for L_3 w.l.o.g.. Now $\mathit{\Gamma}_{620}$ can only vanish w.c. in the following 2 cases:
   a. $B_5 = -b_2$: Due to $\mathit{\Gamma}_{170} = 0$ we have to distinguish again 2 cases:
      i. $a_2 = -A_5 \overline{x}_5 / \overline{X}_2$: Then $\mathit{\Gamma}_{440}$ can only vanish w.c. for:
         ⋆ $\overline{X}_2 = \overline{x}_5$: Then $\mathit{\Gamma}_{062} = 0$ implies $g_4 = 0$ and from $\mathit{\Gamma}_{026} = 0$ we get $A_5 = -\overline{x}_5 b_2$. Now $\mathit{\Gamma}_{242} = 0$ implies $b_3 = \overline{X}_3 a_3 / \overline{x}_5^2$. Finally $\mathit{\Gamma}_{422} = 0$ yields the contradiction.
         ⋆ $A_5 = -\overline{x}_5 b_2$: Again $\mathit{\Gamma}_{062} = 0$ implies $g_4 = 0$. Moreover, we get $b_3 = \overline{X}_3 a_3 / (\overline{X}_2 \overline{x}_5)$ from $\mathit{\Gamma}_{242} = 0$ and $\overline{X}_3 = -\overline{x}_5$ from $\mathit{\Gamma}_{206} = 0$. Finally $\mathit{\Gamma}_{422} = 0$ yields the contradiction.
         ⋆ $b_2 = b_3 - \overline{X}_3 a_3 - A_5 \overline{x}_5$: Again $\mathit{\Gamma}_{062} = 0$ yields $g_4 = 0$. Then $\mathit{\Gamma}_{026} = 0$ implies $\overline{X}_2 = -\overline{x}_5$. We can express b_3 from the only non-contradicting factor of $\mathit{\Gamma}_{350} = 0$. Then $\mathit{\Gamma}_{206}$ can only vanish w.c. for $ST = 0$ with

$$\begin{aligned} S &= A_5 \overline{x}_5 (\overline{x}_5^2 + 1) + \overline{X}_3 a_3 (\overline{x}_5^2 - 1) - 2 \overline{x}_5 a_3, \\ T &= A_5 \overline{x}_5 (\overline{X}_3^2 - 1) + \overline{X}_3 a_3 (\overline{X}_3^2 + 1) - 2 \overline{X}_3 A_5. \end{aligned} \tag{4}$$

         α) $S = 0$: As for $\overline{x}_5 = \pm i$ the equation $S = 0$ cannot vanish w.c. we can assume w.l.o.g. $1 + \overline{x}_5^2 \neq 0$. Now we can express A_5 from $S = 0$. Then $\mathit{\Gamma}_{224} = 0$ implies $\overline{X}_3 = 1/\overline{x}_5$. Finally $\mathit{\Gamma}_{242} = 0$ yields the contradiction.
         β) $T = 0, \overline{x}_5(\overline{X}_3^2 - 1) - 2\overline{X}_3 \neq 0$: Under this assumption we can express A_5 from $T = 0$. Then $\mathit{\Gamma}_{224} = 0$ implies $\overline{X}_3 = 1/\overline{x}_5$. Now $\mathit{\Gamma}_{242} = 0$ yields the contradiction.
         γ) $T = 0$, $\overline{x}_5(\overline{X}_3^2 - 1) - 2\overline{X}_3 = 0$: W.l.o.g. we can express $\overline{x}_5$ from the last equation. Then $T = 0$ implies $a_3 = 0$. Now $\mathit{\Gamma}_{224}$ cannot vanish w.c..
      ii. $a_2 = a_3 + \overline{X}_3 b_3 - \overline{X}_2 b_2$, $\overline{X}_2 a_2 + A_5 \overline{x}_5 \neq 0$: Then $\mathit{\Gamma}_{260}$ cannot vanish w.c..
   b. $b_2 = \overline{X}_2 a_2 - \overline{X}_3 a_3 + b_3$, $B_5 \neq -b_2$: Now $\mathit{\Gamma}_{530}$ cannot vanish w.c..

2. $a_2 = -A_5 \overline{x}_5 / \overline{X}_2$, $g_4(\overline{X}_2 - \overline{X}_3) - L_2 + L_3 + \overline{X}_2 a_2 - \overline{X}_3 a_3 - b_2 + b_3 \neq 0$: W.l.o.g. we can express b_2 from the only non-contradicting factor of $\mathit{\Gamma}_{260}$. Then $\mathit{\Gamma}_{440}$ can only vanish w.c. for:
   a. $\overline{X}_2 = \overline{x}_5$: Then $\mathit{\Gamma}_{602} = 0$ implies an expression for L_3. Now $\mathit{\Gamma}_{404}$ can only vanish w.c in the following 2 cases:
      i. $a_3 = \overline{X}_3 b_3 + B_5 \overline{x}_5 - A_5$: We distinguish again 2 cases:
         ⋆ $\overline{X}_3 B_5 + \overline{x}_5 b_3 \neq 0$: Under this assumption we can compute A_5 from the only non-contradicting factor of $\mathit{\Gamma}_{206}$. Then we can solve the only non-contradicting factor of $\mathit{\Gamma}_{062}$ for g_4 w.l.o.g.. Now $\mathit{\Gamma}_{242} = 0$ implies $\overline{X}_3 = -\overline{x}_5$ and $\mathit{\Gamma}_{422} = 0$ yields the contradiction.
         ⋆ $\overline{X}_3 = -\overline{x}_5 b_3 / B_5$: Now $\mathit{\Gamma}_{206} = 0$ implies $B_5 = b_3$ and from $\mathit{\Gamma}_{062} = 0$ we get an expression for g_4. Then we get $A_5 = \overline{x}_5 b_3$ from $\mathit{\Gamma}_{026} = 0$ and $\mathit{\Gamma}_{422} = 0$ yields the contradiction.

ii. $L_2 = \bar{x}_5(A_5+g_4) - B_5, \overline{X}_3 b_3 + B_5\bar{x}_5 - A_5 - a_3 \neq 0$: Then $\Gamma_{422}=0$ can be solved w.l.o.g. for g_4. From the only non-contradicting factor of Γ_{314} we can express a_3 w.l.o.g.. We distinguish 2 cases:
   - ⋆ $\overline{X}_3 B_5 + \bar{x}_5 b_3 \neq 0$: Under this assumption we can solve $\Gamma_{242}=0$ for A_5. Then $\Gamma_{206}=0$ implies $\overline{X}_3=0$. Finally $\Gamma_{062}=0$ yields the contradiction.
   - ⋆ $\overline{X}_3 = -\bar{x}_5 b_3/B_5$: Now $\Gamma_{242}=0$ cannot vanish w.c..

b. $A_5 = \overline{X}_2 B_5$, $\overline{X}_2 \neq \bar{x}_5$: W.l.o.g. we can solve $\Gamma_{602}=0$ for L_2. Then Γ_{062} can only vanish w.c. for:
   i. $g_4=0$: Then $\Gamma_{422}=0$ implies $a_3 = b_3(\overline{X}_2 - \overline{X}_3 + \bar{x}_5)$. Now $\Gamma_{242}=0$ yields $\overline{X}_3 = \bar{x}_5$. Finally Γ_{224} cannot vanish w.c..
   ii. $B_5 = -\overline{X}_3 a_3/(\overline{X}_2\bar{x}_5)$, $g_4 \neq 0$: Now the only non-contradicting factor of Γ_{044} can be solved for b_3. Then $\Gamma_{026}=0$ yields the contradiction.

Part [E] $\Pi_{3000} = \Pi_{2000} = 0$, $\Omega_{2000}\Pi_{1000} \neq 0$: It can be seen immediately from $\Omega = 0$ that all coefficients of $\Pi_{i000}=0$ (for $i=2,3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express L_1 and h_5 from $\Pi_{3100}=0$ and $\Pi_{3010}=0$, respectively. Moreover, we can compute L_2 and g_4 from $\Pi_{2101}=0$ and $\Pi_{2011}=0$, respectively. Then we solve $\Pi=0$ for e_0 and plug it into Ω which yields in the numerator a homogeneous polynomial $\Gamma[2408]$ of degree 8 in e_1,e_2,e_3.

Now Γ_{602} can only vanish w.c. for $B_5=b_2$. Then we can solve the only non-contradicting factor of Γ_{620} for L_3. Then $\Gamma_{530}=0$ implies $a_2 = \overline{X}_3 b_3 - \overline{X}_2 b_2 + a_3$. Finally Γ_{440} cannot vanish w.c..

Part [F] $\Omega_{2000} = \Omega_{1000} = 0$: W.l.o.g. we can express L_1 and a_2 from $\Omega_{2000}=0$ and $\Omega_{1001}=0$, respectively. As Ω_{0002} cannot vanish w.c. we proceed as follows:

1. $\Pi_{0003} \neq 0$: Now we compute the resultant of Ω and Π with respect to e_3 which yields a homogeneous polynomial $\Gamma[87839]$ of degree 8 in e_0,e_1,e_2. In the following we denote the coefficients of e_1^i, e_2^j, e_0^k of Γ by Γ_{ijk}. Now Γ_{080} equals $(b_2-b_3)C[4]D[10]$.

   a. $C[4]=0$: W.l.o.g. we can express a_3 from $C=0$. Moreover, we can solve the only non-contradicting factor of Γ_{800} for A_5 w.l.o.g.. Then we can compute L_2 from the only non-contradicting factor of Γ_{170}. Now $\Gamma_{206}=0$ implies $g_4=0$ and from $\Gamma_{404}=0$ we get $h_5 = B_5 - b_3 - L_3$. Moreover, we can express L_3 from the only non-contradicting factor of Γ_{062}. Then $\Gamma_{026}=0$ implies $\overline{X}_3=0$ and finally $\Gamma_{620}=0$ yields the contradiction.
   b. $D[10]=0$, $C[4]\neq 0$: W.l.o.g. we can solve $D=0$ for L_2. Then $\Gamma_{206}=0$ implies $g_4=0$. Now we can solve the only non-contradicting factor of Γ_{026} for h_5. Then $\Gamma_{404}=0$ implies $L_3 = -b_3 - \overline{X}_3 a_3$. W.l.o.g. we can solve the only non-contradicting factor of Γ_{602} for a_3. Now $\Gamma_{044}=0$ yields $A_5 = B_5\bar{x}_5$. The difference of the only non-contradicting factors of Γ_{260} and Γ_{062} implies $\overline{X}_2 = -\overline{X}_3$. Then Γ_{260} can only vanish w.c. for:
      i. $\overline{X}_3 = \bar{x}_5$: $\Gamma_{620}=0$ implies $B_5=b_3$ and $\Gamma_{242}=0$ yields the contradiction.
      ii. $\overline{X}_3 = -\bar{x}_5$: Then $\Gamma_{620}=0$ already yields the contradiction.

2. $\Pi_{0003}=0$, $\Pi_{0002}\neq 0$: It can be seen immediately from $\Omega=0$ that all coefficients of $\Pi_{0003}=0$ with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express h_5 and L_2 from $\Pi_{0103}=0$ and $\Pi_{0013}=0$, respectively. Now we compute the resultant of Ω and Π with respect to e_3 which yields a homogeneous polynomial $\Gamma[7821]$ of degree 8 in e_0,e_1,e_2. In the following we denote the coefficients of e_1^i,e_2^j,e_0^k of Γ by Γ_{ijk}.
W.l.o.g. we can solve the only non-contradicting factor of Γ_{800} for a_3. Moreover, we can compute b_2 from the only non-contradicting factor of Γ_{620} w.l.o.g.. Then $\Gamma_{206}=0$ implies $g_4=0$. Now $\Gamma_{602}=0$ yields $A_5=B_5\overline{X}_2$. Then we solve the only non-contradicting factor of Γ_{026} for L_3.
   a. $\overline{X}_3B_5-\overline{X}_2b_3\neq 0$: Under this assumption we can express $\bar{x}_5$ from the only non-contradicting factor of Γ_{062}. Then $\Gamma_{224}=0$ implies $\overline{X}_2=-\overline{X}_3$. Finally $\Gamma_{242}=0$ yields the contradiction.
   b. $b_3=\overline{X}_3B_5/\overline{X}_2$: Then $\Gamma_{062}=0$ implies $\overline{X}_2=-\overline{X}_3$. From $\Gamma_{224}=0$ we get $\overline{X}_3=-\bar{x}_5$. Finally $\Gamma_{242}=0$ yields the contradiction.
3. $\Pi_{0003}=\Pi_{0002}=0$, $\Pi_{0001}\neq 0$: It can be seen immediately from $\Omega=0$ that all coefficients of $\Pi_{000i}=0$ (for $i=2,3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express h_5 and L_2 from $\Pi_{0103}=0$ and $\Pi_{0013}=0$, respectively. Moreover, we can compute g_4 and L_3 from $\Pi_{1102}=0$ and $\Pi_{1012}=0$, respectively. Then we solve Π for e_3 and plug it into Ω which yields in the numerator a homogeneous polynomial $\Gamma[1766]$ of degree 8 in e_0,e_1,e_2. In the following we denote the coefficients of e_1^i,e_2^j,e_0^k of Γ by Γ_{ijk}.
W.l.o.g. we can solve the only non-contradicting factor of Γ_{800} for a_3. Moreover, we can compute b_2 from the only non-contradicting factor of Γ_{620} w.l.o.g.. Now $\Gamma_{206}=0$ yields $A_5=B_5\overline{X}_2$. Finally $\Gamma_{062}=0$ yields the contradiction.
4. $\Pi_{0003}=\Pi_{0002}=\Pi_{0001}=0$, $\Pi_{0300}\neq 0$: It can be seen immediately from $\Omega=0$ that all coefficients of $\Pi_{000i}=0$ (for $i=1,2,3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can express h_5 and L_2 from $\Pi_{0103}=0$ and $\Pi_{0013}=0$, respectively. Moreover, we can compute g_4 and L_3 from $\Pi_{1102}=0$ and $\Pi_{1012}=0$, respectively. $\Pi_{0121}=0$ implies $B_5=b_2$. From $\Pi_{2011}=0$ we get $A_5=a_3+\bar{x}_5b_2-\overline{X}_3b_3$. Now Π_{0211} can only vanish w.c. for $\overline{X}_2=\bar{x}_5$.
As Ω_{0200} cannot vanish w.c. and due to our assumption $\Pi_{0300}\neq 0$ we can compute the resultant of Ω and Π with respect to e_1 which yields a homogeneous polynomial $\Gamma[1013]$ of degree 6 in e_0,e_2,e_3. Now the coefficient of e_3^6 of Γ cannot vanish w.c..
5. $\Pi_{0003}=\Pi_{0002}=\Pi_{0001}=\Pi_{0300}=0$: Now we proceed analogously to the first and second paragraph of the last case but with the extra condition $\Pi_{0300}=0$ which implies $a_3=\overline{X}_3b_3$. Now Π_{0200} and Ω_{0200} cannot vanish w.c. and therefore we can compute the resultant of Ω and Π with respect to e_1 which yields a homogeneous polynomial $\Gamma[87]$ of degree 6 in e_0,e_2,e_3. Again the coefficient of e_3^6 of Γ cannot vanish w.c..

Part [G] $\Pi_{3000} = \Pi_{2000} = \Pi_{1000} = 0$: In contrast to $\Pi_{1000} = 0$, it can be seen immediately from $\Omega = 0$ that all coefficients of $\Pi_{i000} = 0$ (for $i = 2,3$) with respect to the remaining Study parameters have to vanish in order to get no contradiction. Therefore we can solve $\Pi_{3100} = 0$ for L_1, $\Pi_{3010} = 0$ for h_5, $\Pi_{2101} = 0$ for L_2 and $\Pi_{2011} = 0$ for g_4 w.l.o.g..

Part [G1] $\Pi_{1000} = 0$ does not vanish identically for all e_1, e_2, e_3: If the coefficient Z of e_3^2 of Π_{1000} vanishes, then $\Pi_{1000} = 0$ only depends on e_1, e_2 and this already yields together with $\Omega = 0$ the contradiction. Therefore we can assume w.l.o.g. $Z \neq 0$. Now we have to distinguish the following cases:

1. $\Omega_{0002}\Pi_{0003} \neq 0$: We can compute the resultant of Π_{1000} and Ω resp. Π with respect to e_3 w.l.o.g. which yields R_Ω and R_Π, respectively. Now R_Ω and R_Π have to vanish independently of e_0, e_1, e_2, whereby R_Π spits up into $e_2 P[8] Q[38]^2$. It can easily be seen that the coefficients of the quadratic homogeneous polynomial $P = 0$ in the unknowns e_1, e_2 cannot vanish w.c.. Therefore we set $Q = 0$ which is a quartic polynomial in e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Now Q_{04} can only vanish w.c. for:
   a. $b_2 = -B_5$: $Q_{40} = 0$ implies an expression for a_2 and Q_{31} cannot vanish w.c..
   b. $A_5 = a_2 + B_5\bar{x}_5 - \overline{X}_2 b_2$: Then Q_{40} can vanish w.c. for:
      i. $b_2 = B_5$: Then $Q_{31} = 0$ implies an expression for a_2. Finally Q_{13} cannot vanish w.c..
      ii. $a_2 = \overline{X}_2 b_2$: Now Q_{31} can only vanish w.c. for $\bar{x}_5 = \pm\overline{X}_2$. As for $\bar{x}_5 = -\overline{X}_2$ the expression Q_{22} cannot vanish w.c. we set $\bar{x}_5 = \overline{X}_2$. Then Q is fulfilled identically. Now it is not difficult to verify that the coefficients of R_Ω cannot vanish w.c. (proof is left to the reader).
2. $\Pi_{0003} = 0$, $\Omega_{0002} \neq 0$: W.l.o.g. we can compute B_5 from $\Pi_{0103} = 0$.
   a. $\overline{X}_2 \neq \bar{x}_5$: Under this assumption we can express b_2 from $\Pi_{0013} = 0$. Now it can easily be seen that Π_{0002} cannot vanish w.c.. Therefore we can compute the resultant of Π_{1000} and Ω resp. Π with respect to e_3 w.l.o.g. which yields R_Ω and R_Π, respectively. Then R_Π splits up and can only vanish w.c. for $P[6] = 0$ or $Q[14]$. It can easily be seen that the coefficients of the quadratic homogeneous polynomial $P = 0$ in the unknowns e_1, e_2 cannot vanish w.c.. Therefore we set $Q = 0$ which is also a quadratic polynomial in the unknowns e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Now $Q_{02} = 0$ implies $A_5 = -\overline{X}_2 a_2/\bar{x}_5$. Then we get $\bar{x}_5 = -1/\overline{X}_2$ from $Q_{11} = 0$. Then Q is fulfilled identically. Now $\Pi_{1000} = 0$ cannot vanish w.c..
   b. $\overline{X}_2 = \bar{x}_5$: In this case Π_{0013} can only vanish w.c. for:
      i. $A_5 = a_2$: Now Π_{1000} is a factor of Π. Therefore we can only compute the resultant of Π_{1000} and Ω with respect to e_3 w.l.o.g. which yields a homogeneous polynomial $R_\Omega[674]$ of degree 6 in e_0, e_1, e_2. It is again not difficult to verify that the coefficients of R_Ω cannot vanish w.c. (proof is left to the reader).
      ii. $\bar{x}_5 = \pm i$, $A_5 \neq a_2$: Now it can easily be seen that Π_{0002} cannot vanish w.c.. Therefore we can compute the resultant of Π_{1000} and Ω resp. Π

with respect to e_3 w.l.o.g. which yields R_Ω and R_Π, respectively. Then R_Π splits up and can only vanish w.c. for $P[5]=0$. It can easily be seen that the coefficients of the linear homogeneous polynomial $P=0$ in the unknowns e_1,e_2 cannot vanish w.c..

3. $\Omega_{0002}=0$, $\Omega_{0001}\Pi_{0003}\neq 0$: W.l.o.g. we can express L_3 from $\Omega_{0002}=0$. Now the remaining discussion of this case can exactly be done as in item 1 of part [G1].
4. $\Omega_{0002}=\Omega_{0001}=0$, $\Pi_{0003}\neq 0$: W.l.o.g. we can express L_3 from $\Omega_{0002}=0$ and a_2 from $\Omega_{1001}=0$. Then we compute the resultant of Π_{1000} and Π with respect to e_3 w.l.o.g. which yields R_Π. Now R_Π splits up and can only vanish w.c. for $P[11]=0$ or $Q[49]=0$.

   a. $P[11]=0$: Now the coefficients of this equation can only vanish w.c. for:
   $$b_2=-B_5,\quad A_5=\overline{X}_2B_5,\quad a_3=B_5(\overline{X}_2-\overline{x}_5)+\overline{X}_3b_3.$$
   But then Π_{1000} equals $e_1e_3^2B_5(\overline{X}_2-\overline{x}_5)$ which yields the contradiction.

   b. $Q[49]=0$: This is a quartic polynomial in the unknowns e_1,e_2. We denote the coefficients of $e_1^ie_2^j$ of Q by Q_{ij}. Then Q_{04} can only vanish w.c. for:
      i. $b_2=-B_5$: $Q_{40}=0$ implies an expression for a_3. Q_{31} cannot vanish w.c..
      ii. $A_5=a_3+B_5\overline{x}_5-\overline{X}_3b_3$: Then Q_{40} can vanish w.c. for:
         ⋆ $b_2=B_5$: Then $Q_{31}=0$ implies an expression for a_3. Finally Q_{13} cannot vanish w.c..
         ⋆ $a_3=\overline{X}_3b_3$: Now Q_{31} can only vanish w.c. for $\overline{x}_5=\pm\overline{X}_2$. As for $\overline{x}_5=-\overline{X}_2$ the expression Q_{22} cannot vanish w.c. we set $\overline{x}_5=\overline{X}_2$. Then Q is fulfilled identically. As now the coefficient of the highest exponent of e_2 in Π_{1000} and $\Omega=0$ cannot vanish w.c. we can compute the resultant of $\Pi_{1000}=0$ and Ω with respect to e_2 w.l.o.g. which yields $R_\Omega[87]$. Now it is not difficult to verify that the coefficients of R_Ω cannot vanish w.c. (proof is left to the reader).

5. $\Omega_{0002}=\Pi_{0003}=0$, $\Omega_{0001}\neq 0$: We can express L_3 from $\Omega_{0002}=0$ and $\Pi_{0103}=0$ implies $b_2=\overline{x}_5A_5+B_5-\overline{X}_2a_2$.

   a. $\overline{X}_2\neq\overline{x}_5$: Under this assumption we can express B_5 from $\Pi_{0013}=0$. As Π_{0002} cannot vanish w.c. we can compute the resultant of Π_{1000} and Π with respect to e_3, which yields R_Π. Now R_Π splits up and can only vanish w.c. for $P[6]=0$ or $Q[14]=0$. As it can easily be seen, that the coefficients of $P[6]=0$ cannot vanish w.c., we set $Q[14]$ equal to zero, which is a quadratic polynomial in the unknowns e_1,e_2. We denote the coefficients of $e_1^ie_2^j$ of Q by Q_{ij}. Then $Q_{02}=0$ implies $a_2=-\overline{x}_5A_5/\overline{X}_2$. Now $Q_{11}=0$ can only vanish w.c. for:
      i. $\overline{X}_2=-\overline{x}_5$: Then Q_{20} can only vanish for $x_5=\pm 1$. In both cases $\Pi_{1000}=0$ yields the contradiction.
      ii. $\overline{x}_5=-1/\overline{X}_2$, $\overline{X}_2+\overline{x}_5\neq 0$: Again, $\Pi_{1000}=0$ yields the contradiction.

   b. $\overline{X}_2=\overline{x}_5$: Now Π_{0013} can only vanish w.c. for:
      i. $a_2=A_5$: Now Π_{1000} is a factor of Π. Therefore we compute the resultant of Π_{1000} and Ω with respect to e_3, which yields $R_\Omega[244]$. Moreover,

$R_\Omega = 0$ is a homogeneous equation of degree 5 in e_0, e_1, e_2. Now it is not difficult to verify that the coefficients of R_Ω cannot vanish w.c. (proof is left to the reader).

ii. $x_5 = \pm i$, $a_2 \neq A_5$: In this case we can compute the resultant of Ω and the only non-contradicting factor of Π_{1000} with respect to e_3, which yields $R_\Omega[207]$. Moreover, $R_\Omega = 0$ is a homogeneous equation of degree 4 in e_0, e_1, e_2. Then the coefficient of e_2^3 of R_Ω yields the contradiction.

6. $\Omega_{0002} = \Omega_{0001} = \Pi_{0003} = 0$: We can express L_3 from $\Omega_{0002} = 0$ and $\Pi_{0001} = 0$ implies $a_2 = a_3 + \overline{X}_2 b_2 - \overline{X}_3 b_3$. Moreover we get $a_3 = A_5 \overline{X}_3 b_3 - \bar{x}_5 B_5$ from $\Pi_{0013} = 0$.

a. $\overline{X}_2 \neq \bar{x}_5$: Under this assumption we can express A_5 from $\Pi_{0103} = 0$. As Π_{0002} cannot vanish w.c. we can compute the resultant of Π_{1000} and Π with respect to e_3, which yields R_Π. Now R_Π splits up and can only vanish w.c. for $P[6] = 0$ or $Q[14] = 0$. As it can easily be seen, that the coefficients of $P[6] = 0$ cannot vanish w.c., we set $Q[14]$ equal to zero, which is a quadratic polynomial in the unknowns e_1, e_2. We denote the coefficients of $e_1^i e_2^j$ of Q by Q_{ij}. Then $Q_{20} = 0$ implies $B_5 = -b_2$. Now $Q_{02} = 0$ can only vanish w.c. for:
  i. $\overline{X}_2 = -\bar{x}_5$: Then $Q_{11} = 0$ yields the contradiction.
  ii. $\bar{x}_5 = -1/\overline{X}_2$, $\overline{X}_2 + \bar{x}_5 \neq 0$: Now $\Pi_{1000} = 0$ yields the contradiction.

b. $\overline{X}_2 = \bar{x}_5$: Now Π_{0103} can only vanish w.c. for:
  i. $b_2 = B_5$: Now Π_{1000} is a factor of Π. Moreover, Ω_{0020} cannot vanish w.c..
    ⋆ $A_5 \neq 0$: Under this assumption we can compute the resultant of Π_{1000} and Ω with respect to e_2, which yields $R_\Omega[792]$. The coefficient of e_0^6 of R_Ω cannot vanish w.c..
    ⋆ $A_5 = 0$: Now the coefficient of e_2^2 of Π_{1000} cannot vanish w.c., and therefore we can compute the resultant of Π_{1000} and Ω with respect to e_2, which yields $R_\Omega[80]$. Then the coefficient of e_0^4 of R_Ω yields the contradiction.
  ii. $X_2 = \pm i$, $b_2 \neq B_5$: We distinguish two cases:
    ⋆ $2A_5 \pm b_2 i \mp B_5 i \neq 0$: Under this assumption the highest exponent of e_2 in Π and Π_{1000} cannot vanish w.c.. Therefore we can compute the resultant of the only non-contradicting factors of Π_{1000} and Π with respect to e_2, which yields R_Π. It can immediately be seen, that R_Π cannot vanish w.c..
    ⋆ $A_5 = (\pm B_5 i \mp b_2 i)/2$: Now Π_{1000} can only vanish w.c. for $e_1 = \mp e_3^2 i/e_2$. Then it can immediately be seen, that $\Pi = 0$ yields the contradiction.

Part [G2] $\Pi_{1000} = 0$ vanishes identically for all e_1, e_2, e_3: Now $\Pi_{1210} = 0$ implies $b_2 = -B_5$ and from $\Pi_{1012} = 0$ we get $a_2 = -A_5 \bar{x}_5/\overline{X}_2$. Then $\Pi_{1120} = 0$ and $\Pi_{1102} = 0$ can only vanish w.c. for $\overline{X}_2 = \bar{x}_5$. Now we distinguish the following cases:

1. $\Omega_{0200} \neq 0$: As Π_{0300} cannot vanish w.c. and due to our assumption $\Omega_{0200} \neq 0$ we can compute the resultant of Ω and Π with respect to e_1 which yields a homogeneous polynomial $\Gamma[3200]$ of degree 6 in e_0, e_2, e_3. In the following we denote the coefficients of e_0^i, e_2^j, e_3^k of Γ by Γ_{ijk}.

W.l.o.g. we can solve $\Gamma_{600}=0$ for L_3. Then $\Gamma_{303}=0$ implies $a_3=\overline{X}_3b_3+B_5\overline{x}_5-A_5$. Now Γ_{006} can only vanish w.c. for $\overline{X}_5(A_5B_5-\overline{X}_3B_5b_3-B_5^2\overline{x}_5)+A_5\overline{x}_5b_3=0$. Now we have to distinguish again two cases:

a. $\overline{X}_3B_5+\overline{x}_5b_3\neq 0$: Under this assumption we can compute A_5 from $N=0$.
   i. $\overline{X}_3-2\overline{X}_3^2\overline{x}_5-\overline{x}_5\neq 0$: Now we can solve the only non-contradicting factor of $\Gamma_{024}=0$ for b_3. Then $\Gamma_{042}=0$ yields the contradiction.
   ii. $\overline{X}_3-2\overline{X}_3^2\overline{x}_5-\overline{x}_5=0$: W.l.o.g. we can solve this expression for $\overline{x}_5$. Now $\Gamma_{024}=0$ yields the contradiction.

b. $\overline{X}_3=-\overline{x}_5b_3/B_5$: Now $N=0$ implies $B_5=b_3$. Then Γ_{024} can only vanish w.c. for $b_3=-A_5(1+2\overline{x}_5^2)/\overline{x}_5$. Finally $\Gamma_{060}=0$ yields the contradiction.

2. $\Omega_{0200}=0$, $\Omega_{0100}\neq 0$: W.l.o.g. we can express L_3 from $\Omega_{0200}=0$. Then we solve Ω for e_1 and plug it into Π which yields in the numerator a homogeneous polynomial $\Gamma[621]$ of degree 6 in e_0,e_2,e_3. Then the coefficient of e_0^6 of Γ already yields the contradiction.
3. $\Omega_{0200}=\Omega_{0100}=0$, $\Pi_{0030}\neq 0$: W.l.o.g. we can express L_3 from $\Omega_{0200}=0$ and a_3 from $\Omega_{0110}=0$. Now it can easily be seen that Ω_{0020} cannot vanish w.c.. Due to this fact an the assumption $\Pi_{0020}\neq 0$ we can compute the resultant of Ω and Π with respect to e_2 which yields a homogeneous polynomial $\Gamma[838]$ of degree 6 in e_0,e_1,e_3. Again we get the contradiction from the coefficient of e_0^6 of Γ.
4. $\Omega_{0200}=\Omega_{0100}=\Pi_{0030}=0$: W.l.o.g. we can express L_3 from $\Omega_{0200}=0$ and a_3 from $\Omega_{0110}=0$. Now $\Pi_{0030}=0$ implies $A_5=B_5\overline{x}_5$.
   Now it can easily be seen that Ω_{0020} as well as Π_{0020} cannot vanish w.c.. Therefore we can compute the resultant of Ω and Π with respect to e_2 w.l.o.g. which yields a homogeneous polynomial $\Gamma[100]$ of degree 4 in e_0,e_1,e_3. Finally we get the contradiction from the coefficient of e_0^4 of Γ.

Due to the structure[3] of Ω it can easily be seen, that Ω and Π can only have a common factor, which does not depend on e_0 (cf. footnote 2) if $\Omega=0$ has this property too. As this case was already treated in part [F] we remain with the discussion of those cases excluded by the assumption $e_0e_2-e_1e_3\neq 0$ (cf. footnote 1).

Proof for the case $e_0e_2-e_1e_3=0$

We split up this section of the proof into three parts.

Part [A] As $e_0=e_1=e_2=e_3=0$ does not correspond with an Euclidean motion, we start by discussing the following 4 cases:

$$e_0=e_1=e_2=0,\quad e_0=e_1=e_3=0,\quad e_0=e_2=e_3=0,\quad e_1=e_2=e_3=0.$$

We only discuss the case $e_0=e_1=e_2=0$ in more detail because the other 3 cases can be done analogously. Now $\Psi=0$ implies $f_3=0$. Then $\Omega_1=0$ yields an expression for f_2 and from $\Omega_2=0$ we get an expression for f_1. This cannot yield a 2-parametric self-motion as only the homogeneous parameters e_3 and f_0 are free.

[3] $\Omega:\sum_{i=0}^{3}c_ie_i^2+c_4e_0e_3+c_5e_1e_2$ where $c_0,\ldots,c_5$ only depend on the geometry of the SG platform.

Part [B] In this part we discuss the following four special cases:

1. $e_0 = e_1 = 0$: Due to part [A] we can assume w.l.o.g. $e_2e_3 \neq 0$. We can compute f_2 w.l.o.g. from $\Psi = 0$. Then Ω_1 implies $f_3 = -L_1e_2/2$. Then Π_4 can only vanish w.c. for $g_4 = -L_1$. Moreover, we can express f_1 from Π_5 w.l.o.g.. Finally the coefficients of e_2f_0 of Ω_2 and Ω_3 cannot vanish w.c..
2. $e_2 = e_3 = 0$: This case can be done analogously to the last one.
3. $e_0 = e_3 = 0$: Due to part [A] we can assume $e_1e_2 \neq 0$. We can compute f_1 from $\Psi = 0$. Then we can express f_0 from $\Pi_4 = 0$. Moreover, we can compute f_3 from $\Pi_5 = 0$. Now Ω_1, Ω_2 and Ω_3 have to vanish independently of the choice of the unknowns e_1, e_2, f_2.
   The coefficient of e_1^4 of Ω_2 implies an expression for h_5. Then we get L_2 from the coefficient of $e_1^1e_2^3$ of Ω_2 and L_3 from the coefficient of e_1^4 of Ω_3. Then the coefficients of e_1^4 and e_2^4 of Ω_1 imply $L_1 = g_4 = 0$. Now we can compute a_2 from the coefficient of $e_1^1e_2^3$ of Ω_1. Moreover, the coefficient of $e_1^3e_2^1$ of Ω_1 implies $B_5 = \overline{x}_5A_5$ and from the coefficient of $e_1^1e_2^3$ of Ω_3 we get $a_3 = A_5(1+\overline{x}_5^2) - \overline{X}_3b_3$. Then the coefficient of $e_1^2e_2^2$ of Ω_1 can only vanish w.c. for $x_5 = \mp i$. Then the coefficient of e_2^4 of Ω_2 implies $X_2 = \pm i$. Finally, the coefficient of e_2^4 of Ω_3 yields the contradiction.
4. $e_1 = e_2 = 0$: This case can be done analogously to the last one.

Part [C] Due to the discussion of the special cases in part [A] and part [B] we can assume w.l.o.g. $e_0e_1e_2e_3 \neq 0$. Therefore we can solve $e_0e_2 - e_1e_3 = 0$ w.l.o.g. for e_2. Moreover, we can solve $\Psi, \Omega_1, \Pi_4, \Pi_5$ w.l.o.g. for f_0, f_1, f_2, f_3.

Now Ω_2 and Ω_3 have to vanish independently of the choice of the unknowns e_0, e_1, e_3. Therefore the coefficient of e_0^6 of Ω_2 implies $L_1 = g_4$. Then the coefficient of $e_0^5e_3$ of Ω_2 yields an expression for L_2. Now we get $g_4 = 2a_2 - 2\overline{X}_2b_2$ from the coefficient of $e_0^4e_3^2$ of Ω_2. Moreover, we get $a_2 = \overline{X}_2b_2$ from the coefficient of $e_1^2e_3^4$ of Ω_2. Finally the coefficient of $e_0e_1^2e_3^3$ of Ω_2 cannot vanish w.c..

This finishes the proof of Theorem 2. □

5 Conclusion and future research

In this paper we presented the basic result (cf. Theorem 2) on type II Darboux Mannheim (DM) self-motions of planar SG platforms. Due to Lemma 2 of [6] and Theorem 1 we can replace the word “or” in Theorem 2 by the word “and”; i.e. with exception of the two special cases there always exist three collinear platform points $\mathsf{u}_i, \mathsf{u}_j, \mathsf{u}_k$ and three collinear base points $\mathsf{U}_l, \mathsf{U}_m, \mathsf{U}_n$ beside the points $\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ and $\mathsf{u}_4, \mathsf{u}_5, \mathsf{u}_6$ where (i, j, k, l, m, n) consists of all indices from 1 to 6.

The presented basic result raises the hope of giving a complete classification of type II DM self-motions in the future, which would be an important step in solving the famous Borel Bricard problem. On base of Theorem 2 the work towards this goal is in progress.

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