

Review and recent results on Stewart Gough platforms with self-motions

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Abstract. In Section 1, we give a review on Stewart Gough (SG) platforms with self-motions and the related Borel Bricard problem. Moreover, in Section 2, we report about recent results achieved by the author on this topic (SG platforms with type II DM self-motions). In context of these results, we also present two new theorems in Section 3, which open the way for addressed future work. In Section 4, we give some final remarks on planar SG platforms with type I DM self-motions and formulate a central conjecture.

1 Introduction

A Stewart Gough (SG) platform is a parallel manipulator consisting of a moving platform, which is connected via six spherical-prismatic-universal legs with the base, where the spherical and universal joints are passive and the prismatic joints are active. Therefore, the geometry of a SG platform is given by the six base anchor points M_i and by the six platform anchor points m_i for $i = 1, \dots, 6$. A SG platform m_1, \dots, m_6 is called planar if m_1, \dots, m_6 as well as M_1, \dots, M_6 are coplanar.

If the geometry of the SG platform and the six leg lengths are given, then the manipulator is in general rigid in one of its 40 possible assembly modes. But, under particular conditions the manipulator can have $n > 0$ degrees of freedom. The resulting n -parametric motion is called self-motion. Clearly, in each pose of a self-motion the SG platform is singular; i.e. the carrier lines of the prismatic legs belong at least to a linear line complex (cf. [1]).

1.1 Historical background. All self-motions of SG manipulators are solutions to the still unsolved problem posed by the French Academy of Science for the *Prix Vaillant* of the year 1904, which reads as follows: "Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."

The papers of Borel [2] and Bricard [3] were awarded prizes, but both authors only presented partial solutions as they assumed that there are linear relations between the direction cosines of the motion (for more details see [4,5]). Therefore this problem is also known as *Borel Bricard (BB) problem*. Moreover, displacements where distinct points have spherical paths are also called *BB motions*.

But, it should be noted that some results on BB motions were already known before the year 1904:

- (a) Chasles [6] recognized within a static context, that if the points of two conic sections are in a projective correspondence, then there exists a spatial motion, which keeps the corresponding points at fixed distance.
- (b) The only non-trivial motion, where all points of the moving system have spherical paths, was presented¹ by Bricard [7] in 1896.
- (c) Moreover, in 1897 Bricard [8] proved the existence of three types of flexible octahedra in the Euclidean 3-space.
- (d) In 1898, Duporcq [9] stated his famous theorem (for more details see [10]).

¹ The uniqueness of this motion was later proven by Bricard in [3].

- (e) The motions where all points of a straight line have spherical trajectories were studied by Duporcq [11], Mannheim [12] and by Darboux in his note to the book of Kœnigs [13].

It was not until 1937 that a further important contribution to the solution of the BB problem was given by Krames study [14,15] of line-symmetric motions (cf. [16]). All later investigations of BB motions were done in the context of self-motions of SG platforms, which are reviewed next.

1.2 Stewart Gough platforms with self-motions. It is well known, that manipulators, which are singular in every possible configuration, possess self-motions in each pose because they are redundant. These SG platforms are called architecturally singular [17] and their designs are characterized in [18,19,20,21] for the planar case and in [22,23] for the non-planar one.² Therefore, we are only interested in non-architecturally singular SG platforms with self-motions. Until now only few examples of this type are known, as their computation is a very complicated task:

The first paper mentioning self-motions, was written by Husty and Zsombor-Murray [28], where SG platforms with Schönflies self-motion of item (b)³ were reported. Beside these manipulators, further SG platforms with a line-symmetric self-motion (which was already known to Borel [2], Bricard [3] and Krames [15]) was given in Zsombor-Murray et al. [30].

Moreover, Husty and Karger [31] proved that the list of Schönflies Borel Bricard motions given by Borel [2] is complete.

Karger studied also planar parallel manipulators of SG type with affinely equivalent platform and base [32] (for the special cases of equiform and congruent platforms see [33] and [34], respectively) with respect to their self-motions.

The self-motions of TSSM designs (and 6-3 SG platforms, respectively) with intersecting rotary axes correspond to Bricard's flexible octahedra [8]. Recently, a complete list of self-motions of TSSM designs (and 6-3 SG platforms, respectively) was given by the author in [35,36] as by-product of the determination of all flexible octahedra in the projective extension of the Euclidean 3-space [37]. Moreover, the author presented in Theorem 8 of [38] a result on the self-motional behaviour of a remarkable set of planar Schönflies-singular SG platforms.

The classification of all self-motions of the original SG platform was done by Karger and Husty [39]. Based on this study, Karger [40,41] presented a method for designing planar SG platforms with self-motions of the type $e_0 = 0$, where e_0 denotes an Euler parameter.

Further SG platforms, where the self-motions have a planar spherical image, can be computed with a pure algebraic method presented by Geiß and Schreyer [42]. This publication also shows the limitation of the proposed method and that geometric-kinematic considerations are essential for the solution of the BB problem.

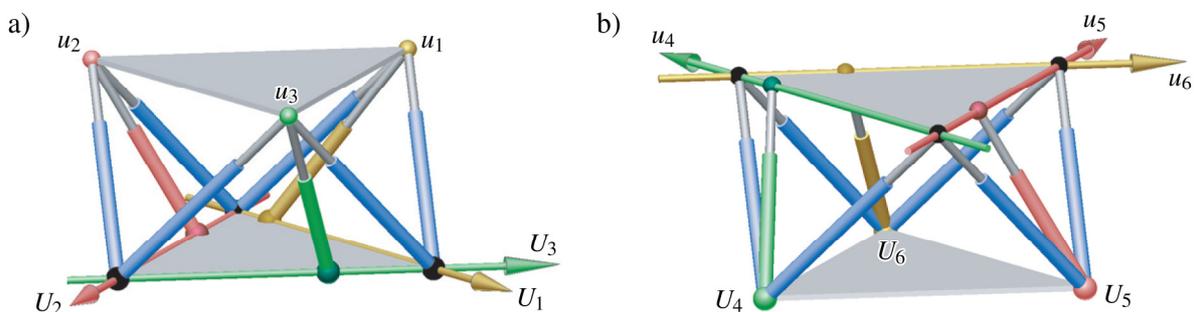


Figure 1. Octahedral manipulator: (a) Cubic C of base anchor points. (b) Cubic c of platform anchor points. Moreover, u_i and U_i ($i = 1, \dots, 6$) are displayed as well as some additional legs of Λ .

² Nevertheless, architecturally singular designs are well studied, less is known about their self-motions (cf. [24,25,26]). As demonstrated by Sommese and Wampler (e.g. [27]) the method of numerical continuation is a powerful tool for the numeric computation of these self-motions.

³ Planar SG platforms of this type are also called *polygon platforms* (cf. [29]).

2 Recent results on SG platforms with self-motions

In recent publications [43,44,45,46] of the author, it was shown, that based on the geometric-kinematic approach of redundancy, amazing new results can be achieved in the field of self-motions of SG platforms. In this section we give a short overview of these results.

Mielczarek et al. [47] investigated redundant SG platforms with a maximal forward kinematics solution set. They showed that the set Λ of additional legs, which can be attached to a given planar SG platform m_1, \dots, M_6 without restricting the forward kinematics, is determined by a linear system of equations given in Eq. (30) of [47]. As the solvability condition of this system is equivalent with the criterion given in Eq. (12) of [48], also the singularity surface of the manipulator does not change by adding legs of Λ . Moreover, it was shown in [47], that in the general case Λ is one-parametric and that the base anchor points as well as the corresponding platform anchor points of Λ are located on planar cubic curves C and c (see Fig. 1). Note that C and c have to do with the cubic correspondence of point paths and the centers of their osculating spheres (cf. page 142 of [49]).

Assumption 1. *We assume that there exist such cubics c and C (which can also be reducible) in the Euclidean domain of the platform and the base, respectively.*

The correspondence between the points of C and c in the complex projective extension P^3 of the Euclidean 3-space is determined by the geometry of the manipulator m_1, \dots, M_6 and can be computed according to [47,48]. As this correspondence has not to be a bijection (see Fig. 1), a point $\in P^3$ of c or C , respectively, is in general mapped to a non-empty set of points $\in P^3$ of C or c , respectively. We denote this set by the term *corresponding location* and indicate this fact by the usage of brackets $\{ \}$. In P^3 the cubic C has three ideal points U_1, U_2, U_3 and we can compute the corresponding locations $\{u_1\}, \{u_2\}, \{u_3\}$ of c . We denote the ideal points of c by u_4, u_5, u_6 and compute the corresponding locations $\{U_4\}, \{U_5\}, \{U_6\}$ of C (see also Fig. 1).

Assumption 2. *For guaranteeing a general case, we assume that each of the corresponding locations $\{u_1\}, \{u_2\}, \{u_3\}, \{U_4\}, \{U_5\}, \{U_6\}$ consists of a single point. Moreover, we assume that no four collinear platform anchor points u_i or base anchor points U_i ($i = 1, \dots, 6$) exist.*

Now the basic idea can simply be expressed by attaching the special legs $\overline{u_i U_i}$ of Λ with $i = 1, \dots, 6$ to the manipulator m_1, \dots, M_6 . The attachment of the leg $\overline{u_i U_i}$ for $i \in \{1, 2, 3\}$ corresponds with the so-called *Darboux constraint* (cf. [50] and Darboux's note to the book of Kœnigs [13]), that the u_i moves in a plane of the fixed system orthogonal to the direction of the ideal point U_i (see Fig. 2a). The attachment of the leg $\overline{u_i U_i}$ for $i \in \{4, 5, 6\}$ corresponds with the so-called *Mannheim constraint* (cf. [12,51]), that a plane of the moving system orthogonal to u_i slides through U_i (see Fig. 2b).

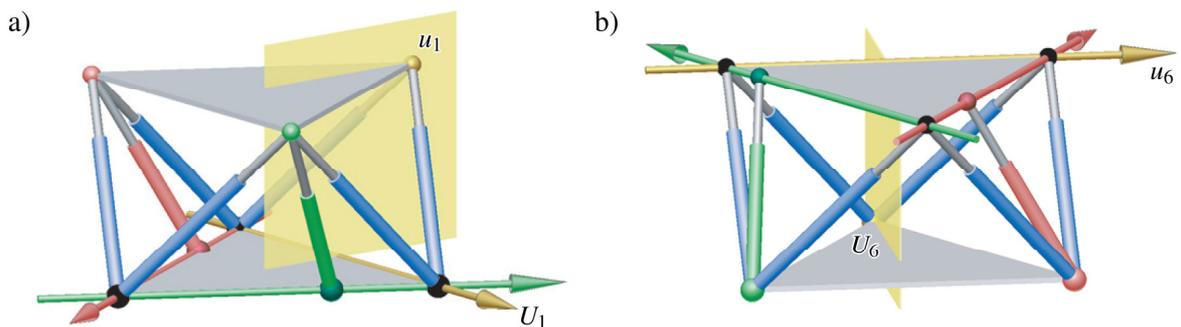


Figure 2. Octahedral manipulator: (a) Darboux constraint ($i = 1$). (b) Mannheim constraint ($i = 6$).

By removing the originally six legs $\overline{m_i M_i}$ with $i=1,\dots,6$ we remain with the manipulator u_1,\dots,U_6 , which is uniquely determined due to Assumption 1 and 2. Moreover, under consideration of these two assumptions, the following statement holds (cf. [43]):

Theorem 1. *The manipulator u_1,\dots,U_6 is redundant and therefore architecturally singular. Moreover, all anchor points of the platform u_1,\dots,u_6 and of the base U_1,\dots,U_6 are distinct.*

It was also proven in [43] that there only exist type I and type II Darboux Mannheim (DM) self-motions, where the definition of types reads as follows:

Definition 1. *Assume Ψ is a one-parametric self-motion of a non-architecturally singular SG platform m_1,\dots,M_6 . Then Ψ is of type n DM if the corresponding architecturally singular manipulator u_1,\dots,U_6 has a n -parametric self-motion.*

In [44,45] the author proved the necessity of three conditions for obtaining a type II DM self-motion. Moreover, a geometric interpretation of these conditions was given, which also identifies a new property of type 1 Bricard octahedra (see Fig. 3). Based on these necessary conditions, the author [46] was able to determine all planar SG platforms (fulfilling Assumption 1 and 2) with a type II DM self-motion. These manipulators are either generalizations of type 1 Bricard octahedra (12-dimensional solution set) or special polygon platforms (cf. footnote 3; 11-dimensional solution set). Moreover, it was shown that the type II DM self-motions of all these SG platforms are line-symmetric and octahedral, where the latter property is defined as follows:

Definition 2. *A DM self-motion is called octahedral⁴, if the following triples of points are collinear for $i \neq j \neq k \neq i$ and $i, j, k \in \{1,2,3\}$:*

$$(u_i, u_j, u_6), (u_i, u_k, u_5), (u_j, u_k, u_4), (U_4, U_5, U_k), (U_5, U_6, U_i), (U_4, U_6, U_j). \quad (1)$$

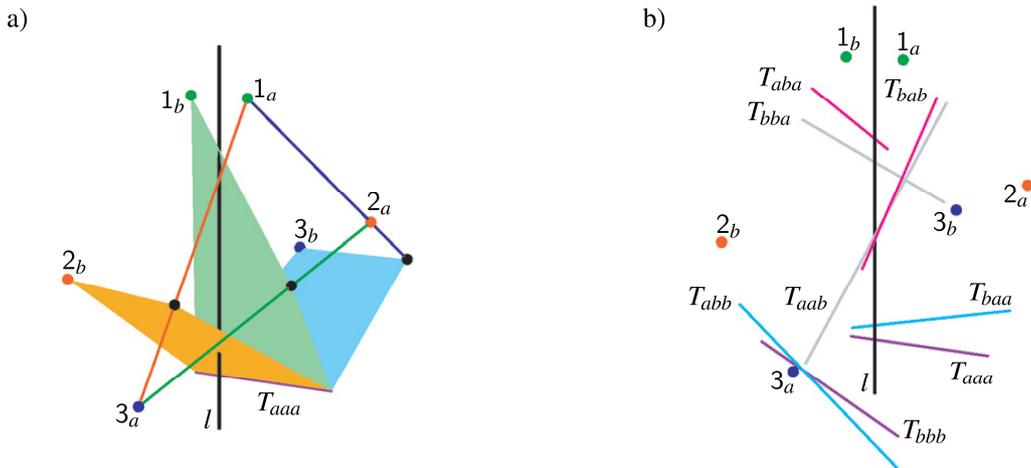


Figure 3. The vertices v_a and v_b of a type 1 Bricard octahedron $1_a, 1_b, 2_a, 2_b, 3_a, 3_b$ are symmetric with respect to the line l for $v \in \{1,2,3\}$. The following three planes have a common line T_{ijk} : (i) plane orthogonal to $[1_i, 2_j]$ though $3_{k'}$, (ii) plane orthogonal to $[2_j, 3_k]$ though $1_{i'}$, (iii) plane orthogonal to $[3_k, 1_i]$ though $2_{j'}$, with $i \neq i', j \neq j', k \neq k'$ and $i, j, k, i', j', k' \in \{a, b\}$. This is illustrated for $i = j = k = a$ in (a). In (b) all eight possible axes T_{ijk} are displayed.

⁴ The reason for this nomenclature, which was introduced by the author in [43], is that all octahedral manipulators have this point-configuration. For $i=1, j=2$ and $k=3$ this can be seen in Fig. 1.

3 New results in the context of Assumption 1 and 2

In the following we present the basic result for the determination of all planar SG platforms with a type II DM self-motion, where Assumption 1 holds in contrast to Assumption 2. For the study of this case, which is dedicated to future research, we can always assume without loss of generality (w.l.o.g.) that four platform anchor points u_i or four base anchor points U_i are collinear, due to the following theorem:

Theorem 2. *Given is a planar SG platform fulfilling Assumption 1. If one cannot choose anchor points within $\{u_1\}$, $\{u_2\}$, $\{u_3\}$, $\{U_4\}$, $\{U_5\}$, $\{U_6\}$ such that four platform anchor points u_i or base anchor points U_i are collinear, then each of these corresponding locations has to consist of a single point.*

Proof: We choose an arbitrary manipulator u_1, \dots, U_6 within the set of all possible manipulators determined by the corresponding locations $\{u_1\}$, $\{u_2\}$, $\{u_3\}$, $\{U_4\}$, $\{U_5\}$, $\{U_6\}$. For the same reasons as given in the proof of Theorem 1 of [43], this manipulator is redundant and therefore architecturally singular. Moreover, due to [18,20] we know that architecture shakiness of a planar manipulator does not change under even different projectivities of the platform and the base. This property is important for the following argumentation:

Due to Lemma 3 of [18], there always exist four pairs of anchor points such that the anchor points in the platform and the corresponding ones in the base form quadrilaterals. After permutation of indices, we can assume w.l.o.g. that these four pairs equal (u_i, U_i) for $i=1, \dots, 4$. Then we apply projectivities to the platform and the base such that the anchor points have the following projective coordinates in the platform and the base, respectively:

$$u_1 = U_1 = (1:0:0), u_2 = U_2 = (0:1:0), u_3 = U_3 = (0:0:1), u_4 = U_4 = (1:1:1). \quad (2)$$

Now we have exactly the same situation as in Section 3 of [20], where Röschel and Mick considered the regular projectivity κ determined by $u_i \mapsto U_i$ for $i=1, \dots, 4$. They demonstrated that there only exist more than six pairs of anchor points if $U_5 = u_5 \kappa$ holds. In this case all corresponding point pairs on the conic sections k and $k \kappa$ determined by u_1, \dots, u_5 and U_1, \dots, U_5 , respectively, are possible anchor points. Note that also $U_6 = u_6 \kappa$ holds and that this case corresponds with item (a) of Section 1.1. As any regular projectivity κ is a bijection, each of the corresponding locations has to consist of single point, which finishes the proof. \square

Under consideration that Γ denotes the set of planar architecturally singular manipulators with no four points collinear, which do not belong to item (a) of Section 1.1, we formulate the following theorem:

Theorem 3. *To any planar SG platform m_1, \dots, M_6 with exception of the set Γ , at least a one-parametric set of legs A can be attached without changing the forward kinematics and singularity surface.*

Proof: For the proof we consider the linear system of equations given in Eq. (30) of [47] (cf. second paragraph of Section 2):

$$\mathbf{D} (\lambda_1, \dots, \lambda_6)^T = \delta_7^T \text{ with } \mathbf{D} = (\delta_1^T, \dots, \delta_6^T) \text{ and } \delta_i = (1, a_i, b_i, A_i, B_i, a_i A_i, a_i B_i, b_i A_i, b_i B_i), \quad (3)$$

where (a_i, b_i) and (A_i, B_i) are the coordinates of the platform and base points, respectively. Moreover, the points with index 7 represent the anchor points of the additional legs of Λ . In the following, we have to distinguish two cases:

- For $\text{rank}(\mathbf{D}) < 6$ the manipulator m_1, \dots, M_6 is architecturally singular (cf. Remark 1 of [20]). Due to the considerations given in the proof of Theorem 2, we only have to investigate the 12 cases of architecturally singular SG platforms with four points on a line, which were listed by Karger in Theorem 3 of [22]. Due to the geometric interpretation given in Section 5.1 of [22] for the cases 1–10 and in Section 3 of [19] for the cases 11–12, it is clear that we can attach in each case at least a one-parametric set of legs without changing the forward kinematics and singularity surface.
- If the rank of \mathbf{D} is maximal, i.e. $\text{rank}(\mathbf{D}) = 6$, then we can solve six from the nine equations of Eq. (3) for $\lambda_1, \dots, \lambda_6$. Plugging the resulting expressions in the remaining equations yield three conditions in four unknowns (a_7, b_7, A_7, B_7) . Therefore, we get at least a one-parametric solution set, which finishes the proof. \square

Beside the result of Theorem 3, only the following non-trivial exceptional case of Assumption 1 is known to the author⁵: To a planar SG platform, where the platform and base anchor points are related by a projectivity (so-called planar projective SG platform), one can add a two-parametric set of legs without changing the direct kinematics and singularity surface, whereas the correspondence of anchor points is given by the projectivity itself (cf. [52]). In this context it should be noted, that based on this redundancy property, the author [52] was recently able to prove the following result:

Theorem 4. *Non-architecturally singular planar projective SG platforms can only have elliptic self-motions (cf. Definition 1 of [52]) or pure translational self-motions. In the latter case the projectivity has to be a regular affinity $\mathbf{t} + \mathbf{T}\mathbf{x}$ (\Rightarrow planar affine SG platform), where the singular values s_1 and s_2 of the 2×2 transformation matrix \mathbf{T} with $0 < s_1 \leq s_2$ fulfill $s_1 \leq 1 \leq s_2$.*

For more details on these self-motions⁶ and further properties of planar projective (affine) SG platforms, we refer to [52]. But, it should be mentioned, that all planar affine SG platforms, with exception of those given in Theorem 4, only have "ordinary" singularities causing a local shakiness of the manipulator but no self-motion itself. Note that a self-motion is extremely dangerous in practice, as the manipulator collapse under the force of gravity in an uncontrollable manner, which can damage or even destroy the manipulator and endangers the operators. Therefore such classes of manipulators, which are free of self-motions, are also of interest for designers.

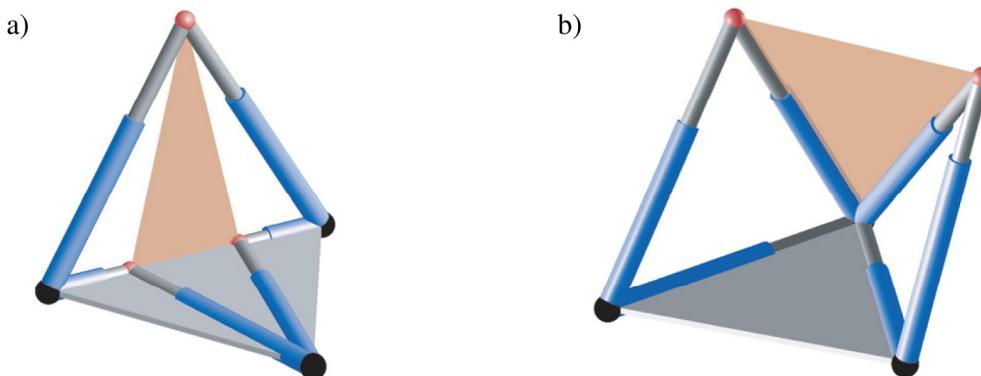


Figure 4. Octahedral manipulator: (a) Butterfly motion. (b) Spherical four-bar motion.

⁵ A completion of this list of special cases is dedicated to future research.

⁶ Until now it is open whether planar projective SG platforms with elliptic self-motions even exist, as no example is known to the author so far. The study of this problem is also dedicated to future research.

4 Final remarks and central conjecture

Finally, we should also devote some words to type I DM self-motions under consideration of the following definition, which refines the proposed classification scheme (cf. Definition 1 and 2) of SG platforms with self-motions:

Definition 3. *Assume a planar SG platform is given, where one can add the set Λ of legs⁷ without changing the direct kinematics and singularity surface. A self-motion of this manipulator is called degenerated, if one can add further legs beside Λ to the planar manipulator without restricting the self-motion.*

Until now the following type I DM self-motions are known to the author, which can be classified as follows:

- i. The *butterfly motion* (see Fig. 4a) and the *spherical four-bar motion* (see Fig. 4b) of the octahedral manipulator [53] and the corresponding motions of the original SG platform [39]. Note that all these self-motions are degenerated and that they are pure spherical motions. Moreover, it can easily be seen that any SG platform with four collinear anchor points possesses a butterfly self-motion.
- ii. In general⁸, a polygon platform (cf. footnote 3) possesses a degenerated type I DM self-motion, which is not a pure spherical motion but a Schönflies motion.
- iii. Non-degenerated type I DM self-motions are the self-motions of Bricard octahedra of type 2 and type 3 (cf. [8]).

In view of all SG platforms with self-motions, known to the author (cf. Section 1, 2, 3 and 4), we have good reasons to close this paper with the following central conjecture:

Conjecture 1. *All non-degenerated one-parametric self-motions of non-architecturally singular planar SG platforms, fulfilling Assumption 1 and 2, are octahedral.*

5 Summary

In Section 1 we reviewed literature with respect to SG platforms with self-motions and the related historical background in form of the Borel Bricard problem. To the best knowledge of the author the presented list of relevant references given in the paper is as complete as possible.

Moreover, in Section 2 we reported about recent results on SG platforms with type II DM self-motions, which were achieved by the author within the last year.

In Section 3, we presented two new theorems, which open the way for addressed future work on SG platforms with self-motions. In this context, we also described a set of SG platforms which is free of self-motions.

Moreover, in Section 4 we listed all planar SG platforms with type I DM self-motions known to the author. Finally, we closed the paper by a central conjecture, which has to be proved or disproved in the future.

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⁷ Note that Λ can also be empty \leftrightarrow planar SG platform belongs to the set Γ .

⁸ Only for the special polygon platforms mentioned in the paragraph before Definition 2, we get degenerated type II DM self-motions, which are also Schönflies motions.

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