Isometrically deformable cones and cylinders carrying planar curves

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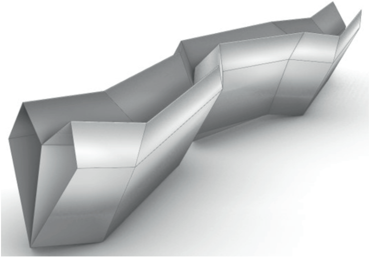
Zipper T-Tubes



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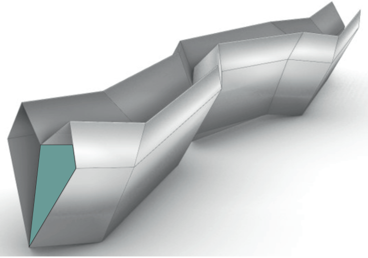

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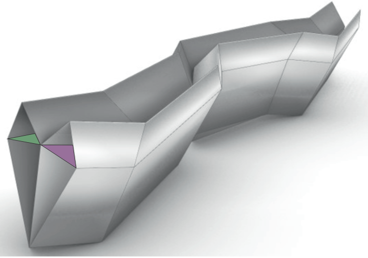

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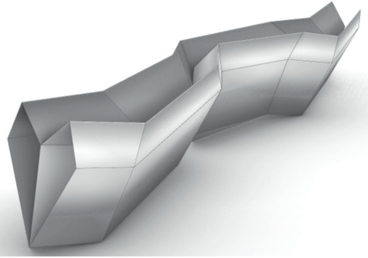

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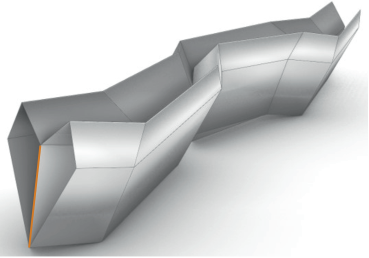

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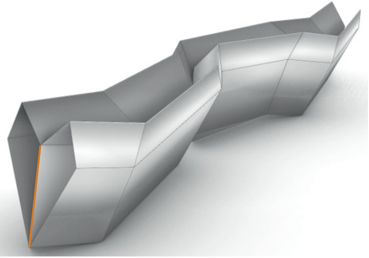

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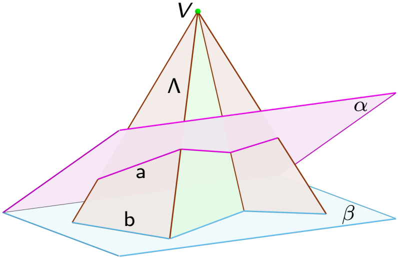
The profile curve of the zip row has to be a straight line-segment according to [4].

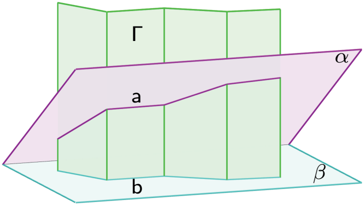
Due to a result of Sauer and Graf [5,6] the zip row can be assumed to be cylindrical or conical.

Construction of Zipper T-Tubes

Reduction to the following geometric/kinematic problem:

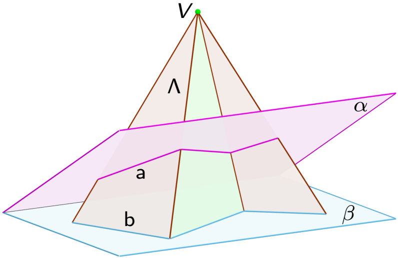
Determine all discrete/smooth cones Λ and cylinders Γ in $\mathbb{R}^3$ allowing a 1-parametric isometric deformation ι in a way that at least two planar curves a and b exist on Λ and Γ, respectively, which remain planar under ι. Moreover, we assume that the two carrier planes α and β of a and b, respectively, are not parallel.

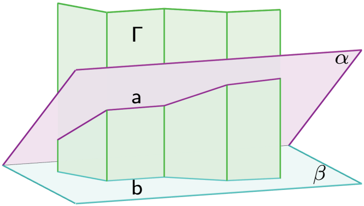




If a solution to this problem exists then

1. planes parallel to α (resp. β) also carry curves of Λ and Γ, respectively, which remain planar during ι.





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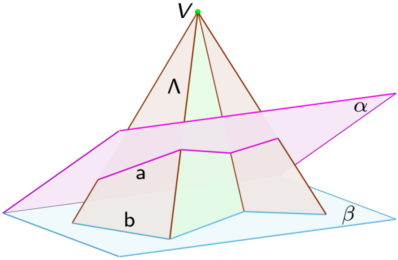
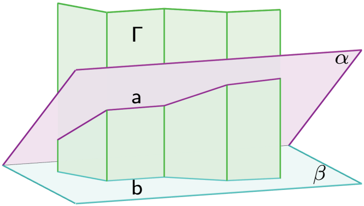
1. planes parallel to α (resp. β) also carry curves of Λ and Γ, respectively, which remain planar during ι.

   This holds true, as the intersection curve of a parallel plane to α (resp. β) results from a (resp. b) by a

   - central scaling with center in the vertex V of Λ,
   - translation along the ruling direction of Γ.





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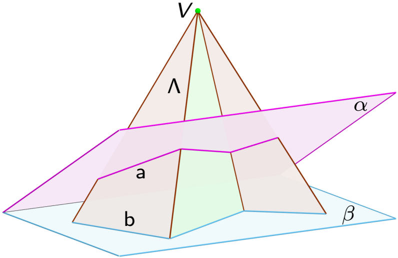
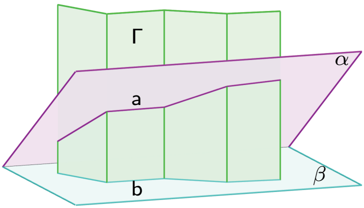
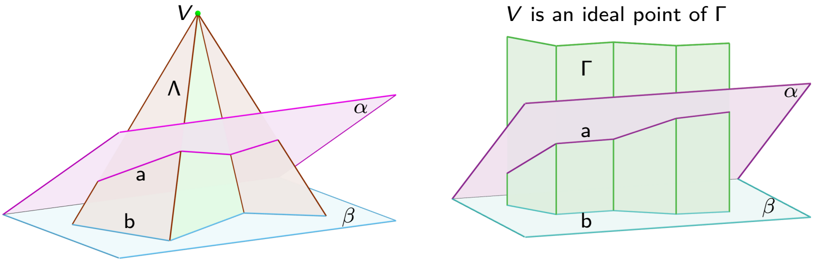




If a solution to this problem exists then

2. all planes of the pencil $\mathcal{P}$ of planes spanned by α and β intersect Λ and Γ in curves c, which remain planar during ι.

   This holds true, as we can assign an arbitrary value to the cross-ratio $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{V})$, which is evaluated along every ruling.



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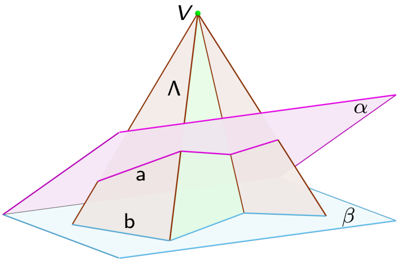
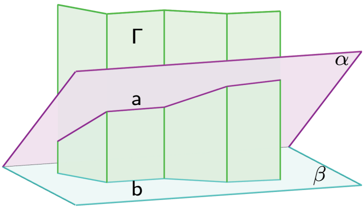
2. all planes of the pencil $\mathcal{P}$ of planes spanned by α and β intersect Λ and Γ in curves c, which remain planar during ι.

   This holds true, as we can assign an arbitrary value to the cross-ratio $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{V})$, which is evaluated along every ruling. Due to the linearity of this construction the obtained curve c is again planar and its carrier plane $\gamma \in \mathcal{P}$.





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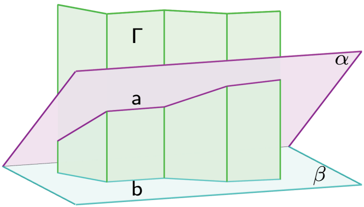
Corollary for cylindrical case:

V is an ideal point of Γ



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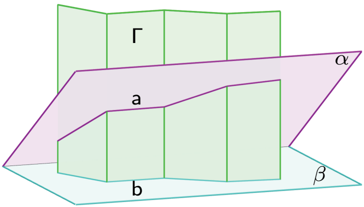

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β can be assumed to be orthogonal to the ruling direction.
$\Rightarrow$ b remains planar under all ι keeping Γ a cylinder through V.

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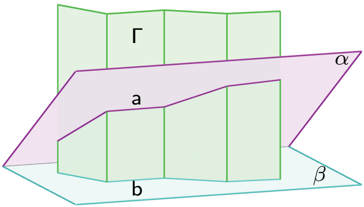

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Corollary for cylindrical case:
β can be assumed to be orthogonal to the ruling direction.
⇒ b remains planar under all ι keeping Γ a cylinder through V.
⇒ Relaxed problem: ι only has to keep the curve a planar.

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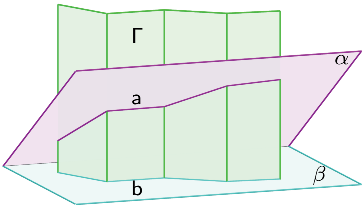

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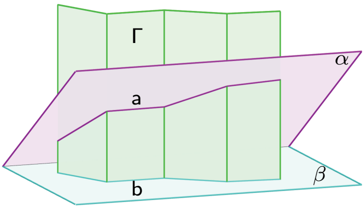
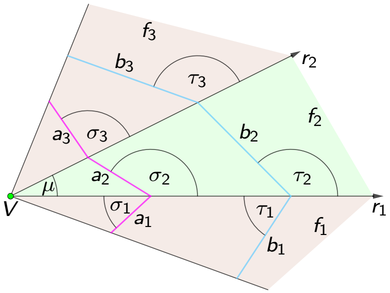
Solution for cylindrical case:

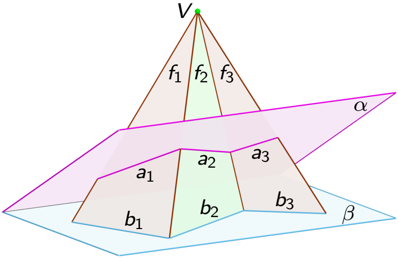
There is no condition to the shape of Γ. Only the intersection line of α and β has to be orthogonal to the rulings of Γ. This holds for the smooth and discrete case (cf. [9]).

V is an ideal point of Γ



Discrete Conical Case





It is sufficient to consider three adjacent faces f_1, f_2, f_3.

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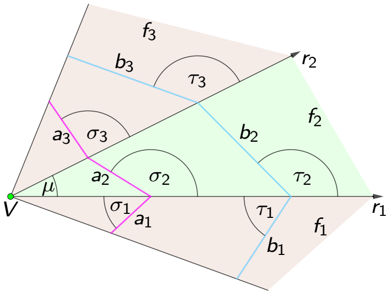

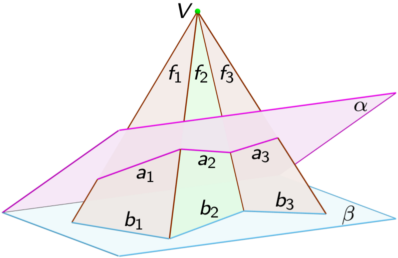

It is sufficient to consider three adjacent faces f_1, f_2, f_3.

There exists an 1-parametric rigid-folding ι along the edges r_1 and r_2 (with fold angles δ_1 and δ_2) in a way that a_1, a_2, a_3 are coplanar.

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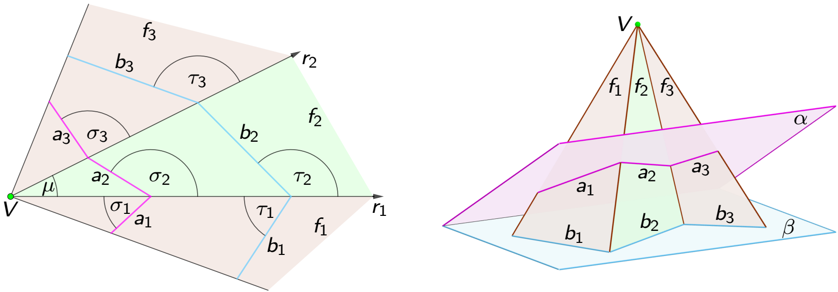

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Aim: Figure out the necessary and sufficient geometric conditions such that b_1, b_2, b_3 also remain coplanar under ι.

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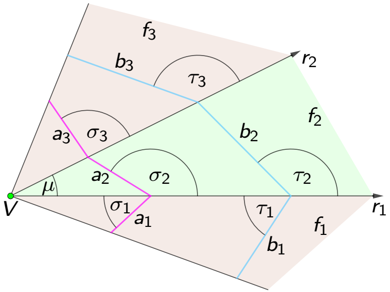

We consider f_2 as fixed and locate it in the xy-plane of the reference frame such that V equals the origin and r_1 coincides with the x-axis.

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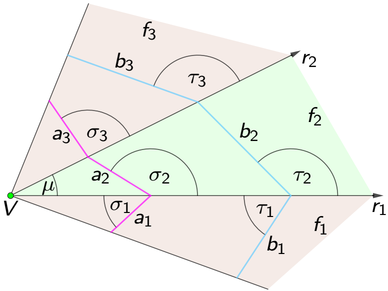

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The direction A_i of the edges a_i can be parametrized by:

$$A_3 = (\cos(\mu + \sigma_3), \sin(\mu + \sigma_3), 0)^T, \quad A_j := (\cos\sigma_j, \sin\sigma_j, 0)^T, \quad j = 1, 2.$$

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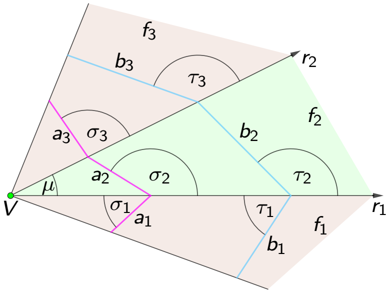

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We rotate A_1 by the angle δ_1 about the r_1-axis $\Rightarrow A_1^* = \mathbf{R}_1 A_1$ with

$$\mathbf{R}_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_1 & -\sin\delta_1 \\ 0 & \sin\delta_1 & \cos\delta_1 \end{pmatrix}.$$

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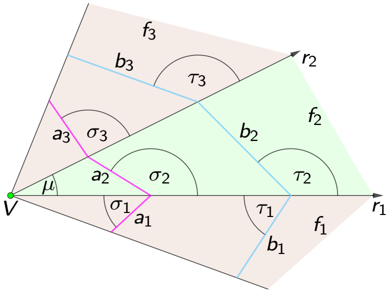

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We rotate A_3 by the angle δ_2 about the r_2-axis $\Rightarrow A_3^* = \mathbf{R}_2 A_3$ with

$$\mathbf{R}_2 := \begin{pmatrix} \sin\mu^2\cos\delta_2 + \cos\mu^2 & \cos\mu\sin\mu(1-\cos\delta_2) & \sin\mu\sin\delta_2 \\ \cos\mu\sin\mu(1-\cos\delta_2) & \cos\mu^2\cos\delta_2 + \sin\mu^2 & -\cos\mu\sin\delta_2 \\ -\sin\mu\sin\delta_2 & \cos\mu\sin\delta_2 & \cos\delta_2 \end{pmatrix}.$$

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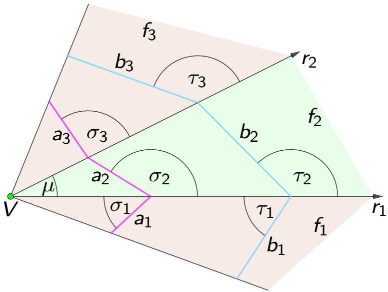
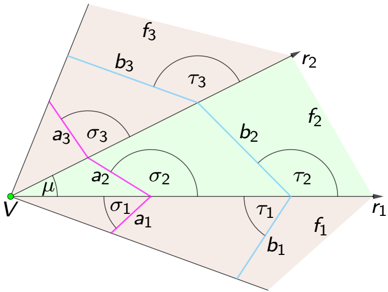


The coplanarity of a_1, a_2, a_3 can be expressed by $D_1 = 0$ with

$$D_1 := \det(A_1^*, A_2, A_3^*).$$

Discrete Conical Case



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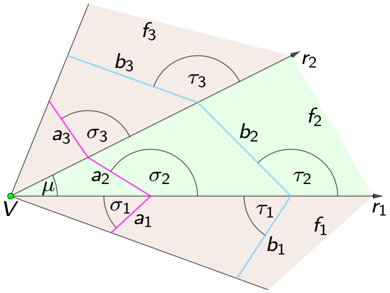
Exactly the same procedure can be done with respect to the directions B_i of the edges b_i, which are given by:

$$B_3 := (\cos(\mu + \tau_3), \sin(\mu + \tau_3), 0)^T, \quad B_j := (\cos \tau_j, \sin \tau_j, 0)^T, \quad j = 1, 2.$$

Discrete Conical Case



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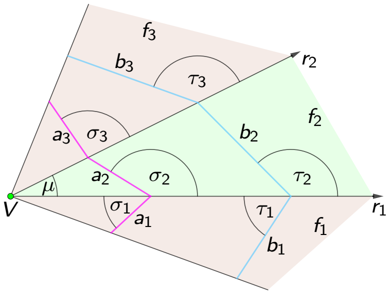
The coplanarity of b_1, b_2, b_3 equals the condition $D_2 = 0$ with

$$D_2 = \det(B_1^*, B_2, B_3^*) \quad \text{and} \quad B_1^* := \mathbf{R}_1 B_1, \quad B_3^* := \mathbf{R}_2 B_3.$$

Discrete Conical Case



In order to convert the conditions $D_1 = 0$ and $D_2 = 0$ into algebraic ones, we use the half-angle substitution; i.e.

$$d_i := \tan \frac{\delta_i}{2}, \qquad s_i := \tan \frac{\sigma_i}{2},$$
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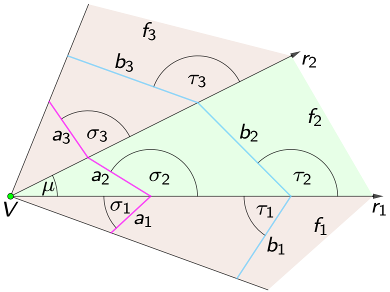
Discrete Conical Case



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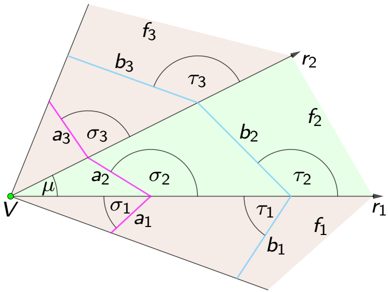
From $D_1 = 0$ and $D_2 = 0$ we eliminate d_1 by means of resultant $\Rightarrow$

$$E_4 d_2^4 + E_2 d_2^2 + E_0 = 0$$

where E_0, E_2, E_4 are functions in m, s_j, t_j with $j = 1, 2, 3$.

Discrete Conical Case



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where E_0, E_2, E_4 are functions in m, s_j, t_j with $j = 1, 2, 3$.

This condition has to hold for all $d_2 \in \mathbb{R} \Rightarrow E_0 = E_2 = E_4 = 0$.
There exists a closed form solution (for details see the paper).

Discrete Conical Case

Using the closed form solution one can easily check by direct computations (e.g. with Maple) that the following theorem holds true:

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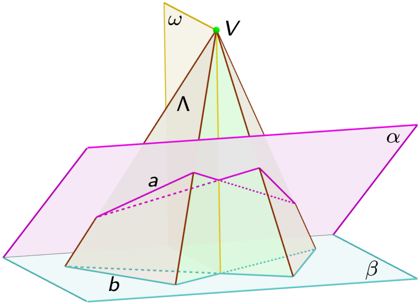
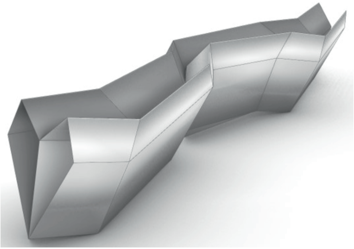


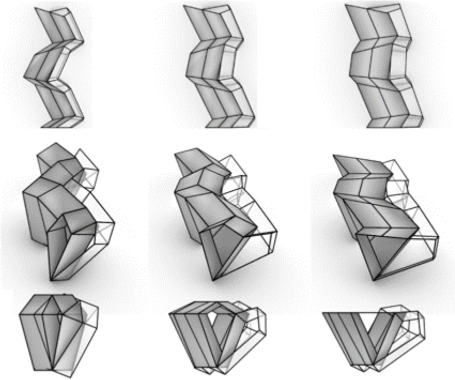
Solution for the discrete conical case:

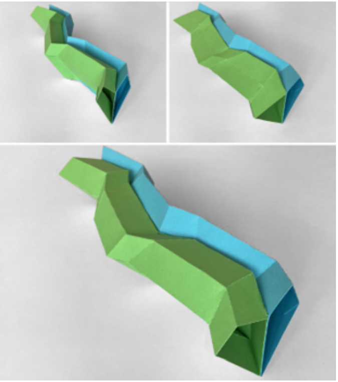
The cone Λ equals one cap of a Bricard octahedron of the plane-symmetric type. Moreover, the planes α and β have to be orthogonal to the plane of symmetry ω of this cone.

Non-translational Zipper T-Tube

Based on this result we were able to give the first example of a non-translational zipper T-tube [4].

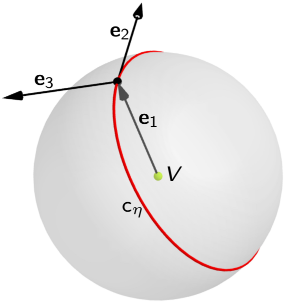






Smooth Conical Case

We presented an approach based on Euclidean differential geometry of curves and cones (cf. [10]).



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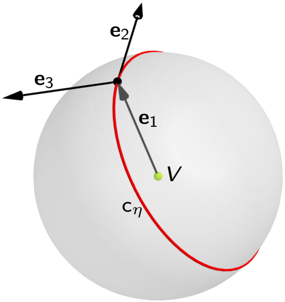
We presented an approach based on Euclidean differential geometry of curves and cones (cf. [10]).

We were able to solve the resulting system of partial differential equations symbolically up to a final ordinary differential equation.



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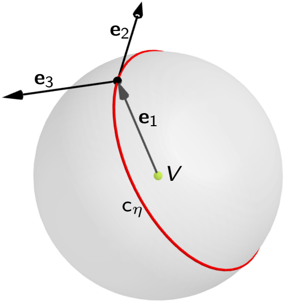
Its solution remains an open problem.



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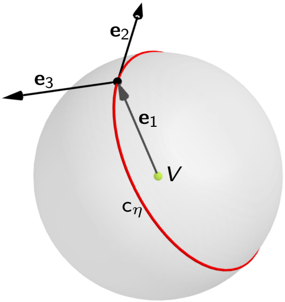

Conjecture:

There exists no non-trivial solution.

References and Acknowledgment

References

All references refer to the list of publications given in the presented paper:

Nawratil, G.: Isometrically deformable cones and cylinders carrying planar curves. Advances in Mechanism and Machine Science: Proc. of 16th IFToMM World Congress (M. Okada ed.), Springer.

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advanced computational design

Thank you for your attention!