All Planar Parallel Manipulators with Cylindrical Singularity Surface

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Abstract. In this article we give the complete list of all non-architecturally singular parallel manipulators of Stewart Gough type with planar base and platform, whose singularity set for any orientation of the platform is a cylindrical surface with rulings parallel to a given fixed direction p in the space of translations. The presented list has only two entries containing the geometric conditions of the corresponding manipulator designs possessing such a singularity surface. These manipulators have the advantage that their singularity set can easily be visualized as curve by choosing p as projection direction. Moreover the computation of singularity free zones reduces to a 5-dimensional task. We close the paper by formulating a conjecture for the non-planar case.

Key words: Stewart Gough Platform, planar parallel manipulator, cylindrical singularity surface, architecture singular manipulators

1 Introduction

1.1 Fundamentals

The geometry of parallel manipulators of Stewart Gough type is given by six base anchor points $\mathbf{M}_i := (A_i, B_i, C_i)^T$ in the fixed space and by six platform anchor points $\mathbf{m}_i := (a_i, b_i, c_i)^T$ in the moving space. By using Euler Parameters (e_0, e_1, e_2, e_3) for the parametrization of the spherical motion group the coordinates $\mathbf{m}'_i$ of the platform anchor points with respect to the fixed space can be written as $\mathbf{m}'_i = H^{-1}\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t}$ with

$$\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_1e_3+e_0e_2) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{pmatrix}, \qquad (1)$$

the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and $H := e_0^2+e_1^2+e_2^2+e_3^2$. Moreover it should be noted that H is used as homogenizing factor whenever it is suitable.

It is well known (see e.g. Merlet [6]) that the set of singular configurations is given by $Q := det(\mathbf{Q}) = 0$, where the i^{th} row of the 6×6 matrix $\mathbf{Q}$ equals the Plücker coordinates $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - H\mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$ of the carrier line of the i^{th} leg.

As we consider only manipulators with planar platform we may suppose $c_i = 0$ for $i = 1, \ldots, 6$. Moreover we can choose the Cartesian frames in the fixed space and moving space (w.l.o.g.) such that $A_1 = B_1 = a_1 = b_1 = b_2 = 0$ hold.

1.2 Preliminary Considerations

The set of Stewart Gough Platforms whose singularity set for any orientation is a cylindrical surface with rulings parallel to a given fixed direction p also contains the set of architecturally singular manipulators[1]. This is due to the fact that the singularity surface of these manipulators equals the whole space of translations for any orientation.

It can easily be seen from the following example that the above two sets are distinct: The non-planar manipulator determined by $\mathbf{m}_1 = \mathbf{m}_2$, $\mathbf{m}_3 = \mathbf{m}_4$, $\mathbf{m}_5 = \mathbf{m}_6$ and $\overline{\mathbf{M}_1\mathbf{M}_2} \parallel \overline{\mathbf{M}_3\mathbf{M}_4} \parallel \overline{\mathbf{M}_5\mathbf{M}_6} \parallel p$ has for any orientation of the platform a cylindrical surface with rulings parallel to the direction p without being architecturally singular (see Fig. 1 (a)). This manipulator is in a singular configuration if and only if the three planes $[\mathbf{M}_1, \mathbf{M}_2, \mathbf{m}_1]$, $[\mathbf{M}_3, \mathbf{M}_4, \mathbf{m}_3]$ and $[\mathbf{M}_5, \mathbf{M}_6, \mathbf{m}_5]$ have a common intersection line. The singularity surface is a quadratic cylinder (see Fig. 1 (b)) due to the (singular) affine correspondence between the base and the platform (cf. Karger [3]).

Remark: As the direct kinematics of this manipulator can be put down to that of a 3-dof RPR parallel manipulator, a rational parametrization of its singularity surface according to Husty et al. [1] can be given. Moreover it should be noted that if we assume $\mathbf{M}_1, \ldots, \mathbf{M}_6$ coplanar we get an example for a planar parallel manipulator with this property. ◇

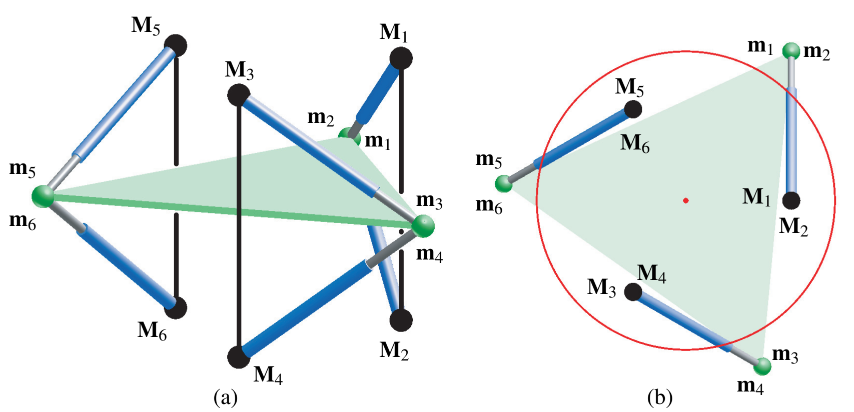

Fig. 1 Non-planar manipulator with cylindrical singularity surface: (*a*) Axonometric view. (*b*) Projection in direction p: The singularity surface (with respect to the barycenter of the platform) is displayed as conic.

Now the question arises for the whole set of non-architecturally singular manipulators possessing such a singularity surface. For the determination of this set we have to distinguish between planar and non-planar Stewart Gough Platforms because the structure (of the subset) of architecturally singular manipulators depends on the planarity of the platform and the base (cf. Karger [2, 4], Nawratil [9], Röschel and Mick [10]).

In this paper we restrict ourselves to the planar case and give only a conjecture for the non-planar one (cf. section 5).

1.3 Prior Work

The following main theorem about planar parallel manipulators with a cylindrical singularity surface was given in Nawratil [7].

[1] These manipulators which are singular in any configuration were introduced by Ma and Angeles [5].

Theorem 1. *The set of planar parallel manipulators with no four anchor points on a line which possess a cylindrical singularity surface with rulings parallel to a given fixed direction p for any orientation of the platform equals the set of planar architecture singular manipulators (with no four anchor points on a line).*

The analytically proof of this main theorem is based on the following idea:

We choose an Cartesian frame in the space of translations such that one axis t_i is parallel to the given direction p. Then $Q := det(\mathbf{Q}) = 0$ must be independent of t_i for all $e_0, \ldots, e_3, t_j, t_k$ with $j \neq k \neq i \neq j$. The proof is based on the resulting equations and Theorem 1 of Karger [2], which is used to characterize the subset of planar architecture singular manipulators (with no four anchor points on a line) algebraically. For more details we refer to Nawratil [7, 8] [2].

As byproduct of the cited proof we get another planar parallel manipulator with cylindrical singularity surface (cf. section 4 of Nawratil [7]). The geometric properties of this Stewart Gough Platform (see Fig. 2) with planar base and platform are as follows:

(i) $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ are collinear,
(ii) $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ are collinear,
(iii) $\overline{\mathbf{M}_1\mathbf{M}_2} \parallel \overline{\mathbf{M}_5\mathbf{M}_6} \parallel p$,
(iv) and $\mathbf{m}_5 = \mathbf{m}_6$.

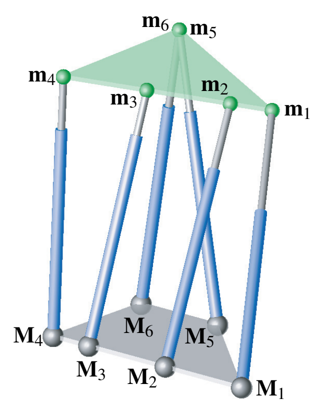

Fig. 2 A further example.

This manipulator is in a singular position if and only if $\mathbf{m}_5 = \mathbf{m}_6$ lies in the carrier plane of the base or if the carrier lines of $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ and $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ intersect each other. Therefore the quadratic singularity surface always splits into two planes (parallel to p).

2 The Complete List

In the following theorem we give the complete list of all non-architecturally singular planar parallel manipulators with cylindrical singularity surface.

Theorem 2. *The set of non-architecturally singular planar parallel manipulators which possess a cylindrical singularity surface with rulings parallel to a given fixed direction p for any orientation of the platform contains only two elements, namely:*

1. $\mathbf{m}_i = \mathbf{m}_j$, $\mathbf{m}_k = \mathbf{m}_l$, $\mathbf{m}_m = \mathbf{m}_n$ *and* $\overline{\mathbf{M}_i\mathbf{M}_j} \parallel \overline{\mathbf{M}_k\mathbf{M}_l} \parallel \overline{\mathbf{M}_m\mathbf{M}_n} \parallel p$,
2. $\mathbf{M}_i, \mathbf{M}_j, \mathbf{M}_k, \mathbf{M}_l$ *are collinear,* $\mathbf{m}_i, \mathbf{m}_j, \mathbf{m}_k, \mathbf{m}_l$ *are collinear,* $\overline{\mathbf{M}_m\mathbf{M}_n} \parallel \overline{\mathbf{M}_i\mathbf{M}_j} \parallel p$ *and* $\mathbf{m}_m = \mathbf{m}_n$,

where (i, j, k, l, m, n) *consists of all indices from* 1 *to* 6*.*

Remark: The first entry of this list is the planar case of the manipulator presented in subsection 1.2 (see Fig. 1) whereas the second entry corresponds to the design repeated in subsection 1.3 (see Fig. 2). ◇

[2] The proof of Theorem 1 was given in two parts due to its length.

2.1 Outline of the proof

The proof of Theorem 2 is organized as follows:

Due to Theorem 1 we only have to investigate planar parallel manipulators with four anchor points collinear whereas we have to distinguish if the anchor points belong to the platform (section 3) or the base (section 4). Within these sections we have to distinguish several subcases which are performed as subsections and lower hierarchical layout structures, respectively.

Basically, the analytical proof of this theorem is based on the same idea as the one of Theorem 1. The difference is that the subset of planar architecturally singular manipulators with four points collinear cannot be characterized algebraically in such a way as it was done for the case with no four points on a line (cf. Karger [2], Nawratil [7, 8]).

Therefore we use the entries 1 to 10 of the list of architecturally parallel manipulators, planar or non-planar, with four collinear anchor points given in Karger [4] Theorem 3 and the corrected two degenerated cases from the planar case (entry 11 and 12) given in Nawratil [9]. These twelve entries form the whole set of architecturally singular manipulators with four points collinear.

2.2 Notation

As already mentioned above our proof is based on the equations which result from the fact that $Q := det(\mathbf{Q}) = 0$ must be independent of t_i for all $e_0, \ldots, e_3, t_j, t_k$ with $j \neq k \neq i \neq j$ if we choose a Cartesian frame in the space of translations with $t_i \parallel p$. Therefore we denote the coefficients of $t_1^i t_2^j t_3^k$ from Q by Q^{ijk}. If possible we cross out the homogenizing factor H from Q^{ijk} and call the remaining factor again Q^{ijk}. Moreover we denote the coefficient of $e_0^a e_1^b e_2^c e_3^d$ of Q^{ijk} by P_{abcd}^{ijk}.

The case study performed in section 3 and 4 can only end up with one of the following four possibilities:

1. The i^{th} and the j^{th} leg coincide. As shortcut we use the symbol $i|j$ for $i, j \in \{1, \ldots, 6\}$.
2. We get a contradiction to any assumption. This case will be expressed by $\lightning$.
3. We end up with an architecturally singular manipulator which corresponds to the i^{th} entry of the list. As corresponding abbreviation we use E_i with $i \in \{1, \ldots, 12\}$.
4. We get one of the two possible solutions given in Theorem 2. These cases are indicated by S_i with $i \in \{1, 2\}$.

We use the introduced symbols to mark the factors of the considered equations whose vanishing cause the corresponding case. In this way it is possible to give the proof in a very compact and clear form.

Moreover it should be noted that the number n of terms of a not explicit given factor F is written into square brackets, i.e. $F[n]$.

3 Four platform anchor points are collinear

We set $b_3 = b_4 = 0$ such that the platform anchor points $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ are located on a line. In the following we must distinguish two cases depending on the parallelity of the base and p.

3.1 Base is not parallel to p

We set up the planar base as

$$C_1 = B_2 = 0, \quad C_i = [C_2(B_3A_i - A_3B_i) + A_2C_3B_i]/(A_2B_3) \quad \text{for} \quad i = 4,5,6. \tag{2}$$

In this case we eliminate the coefficients of t_3 from Q. Moreover we have to take into account that the base anchor points $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ cannot be collinear [3] due to $A_2B_3 \neq 0$. Therefore we cross out A_2 and B_3 from Q^{ijk} if possible and call the remaining factor again Q^{ijk}. If we compute Q^{003} we see that r_{33} factors out, where r_{ij} are the entries of the rotation matrix $\mathbf{R}$ given in Equ. (1). From the remaining factor we compute

$$P^{003}_{2000} - P^{003}_{0200} = b_5b_6(A_5 - A_6)[B_3(A_2a_4 - a_2A_4) + B_4(A_3a_2 - a_3A_2)], \tag{3}$$

$$P^{003}_{1001} - P^{003}_{0110} = b_5b_6(B_5 - B_6)[B_3(A_2a_4 - a_2A_4) + B_4(A_3a_2 - a_3A_2)]. \tag{4}$$

Therefore we have to distinguish the following three cases:

3.1.1 $b_5 = 0$

As $b_5 = 0$ implies $\mathbf{m}_1, \ldots, \mathbf{m}_5$ collinear we can assume $b_6 \neq 0$ (E_1) and that no four points from $\mathbf{m}_1, \ldots, \mathbf{m}_5$ coincide (E_5). In this case $A_2B_3 \neq 0$ ($\Rightarrow \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ cannot be collinear) is no restriction because if the corresponding five base anchor points $\mathbf{M}_1, \ldots, \mathbf{M}_5$ are also on a line we get E_3.

Part [A] Three points from $\mathbf{m}_1, \ldots, \mathbf{m}_5$ coincide

W.l.o.g. we set $a_2 = a_4 = 0$ ($\Rightarrow \mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m}_4$, $a_3a_5 \neq 0$) and compute

$$P^{003}_{2000} + P^{003}_{0200} = A_2B_4a_3a_5b_6(B_3 - B_5) \quad \text{and} \quad P^{003}_{1001} + P^{003}_{0110} = A_2B_4a_3a_5b_6(A_3 - A_5), \tag{5}$$

respectively. For $B_4 = 0$ we get E_6 and $\mathbf{M}_3 = \mathbf{M}_5$ yields E_4.

Part [B] At most two points from $\mathbf{m}_1, \ldots, \mathbf{m}_5$ coincide

We set $a_2 = 0$ ($\Rightarrow \mathbf{m}_1 = \mathbf{m}_2$, $a_3a_4a_5 \neq 0$) and compute again

$$P^{003}_{2000} + P^{003}_{0200} = A_2b_6[a_3a_4B_5(B_3 - B_4) + a_3a_5B_4(B_5 - B_3) + a_4a_5B_3(B_4 - B_5)], \tag{6}$$

$$P^{003}_{1001} + P^{003}_{0110} = A_2b_6[a_3a_4B_5(A_3 - A_4) + a_3a_5B_4(A_5 - A_3) + a_4a_5B_3(A_4 - A_5)]. \tag{7}$$

The resultant of these equations with respect to a_3 yields

$$A_2{}^2b_6{}^2B_3a_4a_5\,(a_4B_5 - B_4a_5)\,(B_4A_3 - B_4A_5 + A_4B_5 - B_5A_3 + A_5B_3 - B_3A_4)\,. \tag{8}$$

1. $\mathbf{M}_3, \mathbf{M}_4, \mathbf{M}_5$ collinear:
    a. Assuming $B_4 \neq B_5$ we can compute A_3 from the collinearity condition. Then Equ. (6) can only vanish for $A_2b_6 = 0$ (↯) or for a factor $T_1[6]$ indicating the special case of E_{10}.

[3] As a consequence we loss the case $\mathbf{M}_1, \ldots, \mathbf{M}_4$ collinear and $\mathbf{m}_1, \ldots, \mathbf{m}_4$ collinear. The missing cases will be discussed in subsection 4.1.2 and 4.1.3, respectively.

b. $B_3 = B_4 = B_5$: The polynomial of Equ. (7) can only vanish for $A_2b_6 = 0$ (↯), $B_5 = 0$ (E_3) or a factor $T_2[6]$ implying the special case of E_{10}.
c. $\mathbf{M}_4 = \mathbf{M}_5$: Substituting $A_4 = A_5$, $B_4 = B_5$ into Equ. (6) and (7) yields

$$\underbrace{A_2a_3b_6}_{\text{↯}}\underbrace{B_5}_{E_8}\underbrace{(a_4-a_5)}_{4|5}(B_3-B_5) \quad \text{and} \quad A_2a_3b_6B_5(a_4-a_5)(A_3-A_5), \qquad (9)$$

respectively. The remaining case $\mathbf{M}_3 = \mathbf{M}_5$ yields E_6.

2. $B_5 = B_4a_5/a_4$ and $\mathbf{M}_3, \mathbf{M}_4, \mathbf{M}_5$ not collinear: Then Equ. (6) and (7) yields

$$A_2a_5b_6B_4(a_4-a_5)(B_3a_4-a_3B_4)/a_4 \quad \text{and} \quad A_2a_5b_6(A_4-A_5)(B_3a_4-a_3B_4), \qquad (10)$$

respectively. The three remaining cases ($B_4 = 0$, $A_4 = A_5 \mid a_4 = a_5$, $A_4 = A_5 \mid B_3 = B_4a_3/a_4$) imply the collinearity of $\mathbf{M}_3, \mathbf{M}_4, \mathbf{M}_5$ (↯).

Part [C] The points $\mathbf{m}_1, \ldots, \mathbf{m}_5$ are pairwise distinct

In this case we factorize $Q^{001} = b_6F_1[10]F_2[16]F_3[132]$. As the coefficient of e_0e_2 from $F_1 = 0$ equals A_2B_3 (↯) we set $F_3 = 0$: Computing $j_0 + j_2$ yields A_2K_2 where j_i denotes the coefficient of e_i^2 from F_3. Moreover $P^{003}_{2000} + P^{003}_{0200} = b_6K_1$ holds and therefore the two conditions $K_1 = K_2 = 0$ given in Karger [4] Equ. (17) indicating E_{10} must be fulfilled.

We proceed with $F_2 = 0$: $A_2B_3A_6$ and $A_2B_3B_6$ are the coefficients of e_0e_2 and e_0e_1, respectively. As for $A_6 = B_6 = 0$ the factor F_2 vanishes we factorize $Q^{002} = b_6r_{33}F_1F_3$ which finishes this part.

3.1.2 $\mathbf{M}_5 = \mathbf{M}_6$

We set $A_5 = A_6$, $B_5 = B_6$ and compute the following four linear combinations:

$$P^{003}_{2000}+P^{003}_{0200}=N_1, \quad P^{003}_{1001}+P^{003}_{0110}=N_2, \quad P^{002}_{5010}+P^{002}_{1050}=A_2B_3N_3, \quad P^{002}_{4011}-P^{002}_{1140}=A_2B_3N_4. \qquad (11)$$

This implies $N_1 = N_2 = N_3 = N_4 = 0$ which are the four conditions given in Nawratil [9] indicating E_{12}.

Remark: The four conditions also indicate E_{11} which can be regarded as a special case of E_{12} (cf. Nawratil [9]). Moreover as we only consider planar architecturally parallel manipulators with four points collinear the cases E_2 and E_9 are also special cases of E_{12}. ◇

3.1.3 $B_3(A_2a_4 - a_2A_4) + B_4(A_3a_2 - a_3A_2) = 0$

W.l.o.g. we can assume $b_5b_6 \neq 0$ and $\mathbf{M}_5 \neq \mathbf{M}_6$. From the above condition we compute a_4. Computing $P^{003}_{2000} + P^{003}_{0200}$ yields

$$B_4D[B_3b_5(A_2a_6-a_2A_6)+B_3b_6(A_5a_2-a_5A_2)+(B_5b_6-b_5B_6)(A_2a_3-a_2A_3)] \qquad (12)$$
$$\text{with} \quad D := B_3(A_2a_3-a_2A_3)+B_4(A_3a_2-a_3A_2). \qquad (13)$$

Part [A] $B_4 = 0$

Computing Q^{002} yields

$$\underbrace{a_2}_{E_6}\underbrace{A_4}_{1|4}\underbrace{(A_2-A_4)}_{2|4}G[1736]/A_2. \tag{14}$$

Then we calculate $G_{5010}+G_{1050}-G_{0501}-G_{0105}={A_2}^2{B_3}^2b_5b_6\,(A_6-A_5)$ where G_{abcd} denotes the coefficient of $e_0^ae_1^be_2^ce_3^d$ of G. Therefore we set $A_5=A_6$ and compute $G_{4011}-G_{0411}-G_{1140}+G_{1104}={A_2}^2{B_3}^2b_5b_6\,(B_5-B_6)$ which yields $B_5=B_6$ (↯).

Part [B] $D=0$

Due to Part [A] we can assume $B_4\neq 0$. Nevertheless we have to distinguish the following two cases:

1. Assuming $B_3\neq B_4$ we can compute a_3 from this equation, which yields $\mathbf{m}_3=\mathbf{m}_4$. Factorizing $P^{003}_{1001}+P^{003}_{0110}$ yields

$$\underbrace{a_2}_{E_5}\underbrace{(B_3A_4-A_2B_3-A_3B_4+A_2B_4)}_{E_6}\underbrace{(A_3B_4-B_3A_4)}_{E_6}L[12]/((B_3-B_4)^2A_2). \tag{15}$$

   From $L=0$ we compute a_5. Of course we can also factor out all the explicit given factors of Equ. (15) from Q^{002}. Computation of the same linear combinations as in Part [A] for the remaining factor $G[5920]$ yield the same contradiction.
2. $B_3=B_4$: Then the condition of Part [B] simplifies to $a_2B_4(A_3-A_4)=0$. As $A_3=A_4$ yield $3|4$ we set $a_2=0$ ($\Rightarrow \mathbf{m}_1=\mathbf{m}_2$ and $\mathbf{m}_3=\mathbf{m}_4$). Factorizing $P^{003}_{1001}+P^{003}_{0110}$ yields

$$\underbrace{A_2}_{↯}\underbrace{a_3}_{E_5}\underbrace{(A_3-A_4)}_{3|4}[b_5(a_3B_6-a_6B_4)+b_6(a_5B_4-a_3B_5)]. \tag{16}$$

   From the last factor we express again a_5. Now the contradiction can be seen by performing the same computational steps as in the above case.

Part [C] $B_3b_5(A_2a_6-a_2A_6)+B_3b_6(A_5a_2-a_5A_2)+(B_5b_6-b_5B_6)(A_2a_3-a_2A_3)=0$

W.l.o.g. we can express a_5 from this condition. Moreover we can assume $B_4D\neq 0$. Then $P^{002}_{5100}-P^{002}_{0051}=0$ implies $A_5=A_6$. From $P^{002}_{5100}-P^{002}_{1500}=0$ we get $a_3=a_2A_3/A_2$. Then we consider

$$P^{002}_{3210}-P^{002}_{2301}=A_2B_3B_4a_2b_5b_6(A_3-A_4)(B_5-B_6). \tag{17}$$

As $a_2=0$ yields E_5, we set $A_3=A_4$. Now $P^{002}_{4011}-P^{002}_{0411}+P^{002}_{1140}-P^{002}_{1104}=0$ can only vanish for architecturally singular manipulators (or ↯).

3.2 Base is parallel to p

In this case we set $C_i=0$ for $i=1,\ldots,6$ and eliminate t_1 from Q. From Q^{100}, Q^{110}, Q^{200} we can additionally factor out r_{31} and from Q^{200}, Q^{201} the factor r_{13}. In the following we have to distinguish again three cases due to

$$P^{102}_{2101}=b_5b_6(A_5-A_6)[B_2(A_4a_3-a_4A_3)+B_3(A_2a_4-a_2A_4)+B_4(A_3a_2-a_3A_2)], \tag{18}$$
$$P^{201}_{1100}=b_5b_6(B_5-B_6)[B_2(A_4a_3-a_4A_3)+B_3(A_2a_4-a_2A_4)+B_4(A_3a_2-a_3A_2)]. \tag{19}$$

3.2.1 $b_5 = 0$

As $b_5 = 0$ implies $\mathbf{m}_1, \ldots, \mathbf{m}_5$ collinear we can assume $b_6 \neq 0$ (E_1) and that no four points from $\mathbf{m}_1, \ldots, \mathbf{m}_5$ coincide (E_5).

Part [A] Three points from $\mathbf{m}_1, \ldots, \mathbf{m}_5$ coincide

W.l.o.g. we set $a_2 = a_3 = 0$ ($\Rightarrow \mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m}_3$, $a_4a_5 \neq 0$) and compute

$$P^{201}_{1010} = a_4a_5b_6(B_5 - B_4)(A_2B_3 - B_2A_3), \quad P^{111}_{2020} = a_4a_5b_6(A_5 - A_4)(A_2B_3 - B_2A_3). \quad (20)$$

If the last factor vanishes we get E_6 and $\mathbf{M}_4 = \mathbf{M}_5$ yields E_4.

Part [B] At most two points from $\mathbf{m}_1, \ldots, \mathbf{m}_5$ coincide

We set $a_2 = 0$ ($\Rightarrow a_3a_4a_5 \neq 0$) and compute $P^{101}_{4020} - P^{101}_{2040} - P^{101}_{0402} + P^{101}_{0204}$ which yields

$$a_3a_4a_5b_6B_2(B_5A_3 - A_3B_4 + A_4B_3 + B_4A_5 - B_5A_4 - B_3A_5). \quad (21)$$

1. $\mathbf{M}_3, \mathbf{M}_4, \mathbf{M}_5$ collinear.
   a. $B_4 \neq B_5$: Under this assumption we can compute A_3. Substituting this into P^{201}_{1010} yields $b_6T_1[14]$ where $T_1[14] = 0$ implies the special case of E_{10}.
   b. $B_3 = B_4 = B_5$: Here P^{111}_{2020} yields $b_6T_2[12]$ where $T_2[12] = 0$ implies the special case of E_{10}.
   c. $\mathbf{M}_4 = \mathbf{M}_5$: Computing P^{201}_{1010} and P^{111}_{2020} yields

   $$\underbrace{a_3b_6}_{\lightning}\underbrace{(A_2B_5 - A_5B_2)}_{E_8}\underbrace{(a_4 - a_5)}_{4|5}(B_3 - B_5), \quad b_6a_3(A_2B_5 - A_5B_2)(a_4 - a_5)(A_3 - A_5). \quad (22)$$

   The remaining case $\mathbf{M}_3 = \mathbf{M}_5$ implies E_6.

2. $B_2 = 0$ and $\mathbf{M}_3, \mathbf{M}_4, \mathbf{M}_5$ not collinear: The resultant of $P^{201}_{1010} = 0$ and $P^{111}_{2020} = 0$ with respect to a_3 yields

   $$\underbrace{A_2{}^2}_{1|2}\underbrace{a_4a_5b_6^2}_{\lightning}\underbrace{(B_4A_5 - A_3B_4 + A_4B_3 - B_3A_5 - A_4B_5 + B_5A_3)}_{\lightning\ (\mathbf{M}_3,\mathbf{M}_4,\mathbf{M}_5\ \text{collinear})}B_3(a_4B_5 - a_5B_4). \quad (23)$$

   a. $B_3 = 0$: Substituting this into P^{201}_{1010} yields $A_2b_6a_3B_4B_5(a_4 - a_5)$. As $P^{111}_{2020} = 0$ implies only contradictions for $a_4 = a_5$ we set $B_4 = 0$. Now P^{111}_{2020} can only vanish for $B_5 = 0$ which yields E_3 (or $\lightning$).
   b. $B_5 = B_4a_5/a_4$: Computing P^{111}_{2020} and P^{201}_{1010} yield

   $$A_2a_5b_6(B_3a_4 - B_4a_3)(A_4 - A_5), \quad A_2a_5b_6B_4(B_3a_4 - B_4a_3)(a_4 - a_5)/a_4. \quad (24)$$

   The three remaining cases ($B_4 = 0$, $A_4 = A_5 \mid a_4 = a_5$, $A_4 = A_5 \mid B_3 = B_4a_3/a_4$) imply the collinearity of $\mathbf{M}_3, \mathbf{M}_4, \mathbf{M}_5$ ($\lightning$).

Part [C] The points $\mathbf{m}_1, \ldots, \mathbf{m}_5$ are pairwise distinct

We compute $P^{201}_{1010} = b_6K^{\star}_1[24]$ and $P^{200}_{1010} = b_6K^{\star}_2[24]$. The set of equations $K^{\star}_1 = K^{\star}_2 = 0$ is the generalized version ($B_2 \neq 0$) of the one given in Karger [4] Equ. (17) indicating E_{10}.

3.2.2 $\mathbf{M}_5 = \mathbf{M}_6$

We set $A_5 = A_6, B_5 = B_6$ and consider the following coefficients:

$$P^{201}_{1010} = N^\star_1[48], \quad P^{111}_{2020} = N^\star_2[48], \quad P^{200}_{1010} = N^\star_3[48], \quad P^{110}_{2020} = N^\star_4[48]. \tag{25}$$

The four conditions $N^\star_1 = N^\star_2 = N^\star_3 = N^\star_4 = 0$ are the generalized version ($B_2 \neq 0$) of those given in Nawratil [9] indicating E_{12}.

3.2.3 $B_2(A_4a_3 - a_4A_3) + B_3(A_2a_4 - a_2A_4) + B_4(A_3a_2 - a_3A_2) = 0$

First of all we can assume $\mathbf{M}_5 \neq \mathbf{M}_6$ as well as $b_5b_6 \neq 0$. Moreover we can assume $a_4 \neq 0$ because $a_2 = a_3 = a_4 = 0$ yields E_5.

Part [A] $A_3a_2 - a_3A_2 \neq 0$

Under this assumption we can compute B_4. Computing $P^{100}_{2110} - P^{100}_{0112} + P^{100}_{1201} - P^{100}_{1021} = 0$ and $P^{100}_{2020} + P^{100}_{0202} = 0$ shows that the following three cases must be distinguished:

1. $a_2a_3 = 0$: W.l.o.g. we set $a_2 = 0$ ($\Rightarrow \mathbf{m}_1 = \mathbf{m}_2$). We compute

$$P^{101}_{4020} + P^{101}_{2040} + P^{101}_{0402} + P^{101}_{0204} = a_4b_5b_6(A_5 - A_6)(a_3 - a_4)(B_3A_2 - B_2A_3), \tag{26}$$
$$P^{101}_{4110} + P^{101}_{0114} + P^{101}_{1401} + P^{101}_{1041} = a_4b_5b_6(B_5 - B_6)(A_3 - A_4)(B_3A_2 - B_2A_3). \tag{27}$$

   As $a_3 = a_4, A_3 = A_4$ yields 3|4 only three cases remain:

   a. $a_3 = a_4, B_5 = B_6, A_3 \neq A_4$: Then $P^{200}_{1100} = 0$ implies $B_2 = 0$. From $P^{110}_{2020} = 0$ we get the following two cases:
      i. $B_3 = B_6$: $P^{111}_{2020} = \underbrace{B_6}_{E_1} \underbrace{A_2}_{1|2} \underbrace{a_4(A_3 - A_4)}_{\lightning} (\underbrace{a_4b_5 - a_4b_6 + a_5b_6 - b_5a_6}_{S_2(+\mathbf{m}_3=\mathbf{m}_4)})$.
      ii. $a_5 = a_6b_5/b_6$: $P^{111}_{2020} = \underbrace{B_6}_{E_1} \underbrace{A_2}_{1|2} \underbrace{a_4^2(A_3 - A_4)}_{\lightning} (\underbrace{b_5 - b_6}_{S_1})$.

   b. $A_3 = A_4, A_5 = A_6, a_3 \neq a_4$: The equation $P^{101}_{4110} - P^{101}_{0114} + P^{101}_{1401} - P^{101}_{1041} = 0$ can only vanish without contradiction for $B_3 = B_2A_4/A_2$ which implies $\mathbf{M}_3 = \mathbf{M}_4$ (E_8).

   c. $B_3 = B_2A_3/A_2$ ($\Rightarrow \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ collinear): Now we consider

$$P^{100}_{2110} + P^{100}_{0112} + P^{100}_{1201} + P^{100}_{1021} = \underbrace{a_3}_{E_6} \underbrace{a_4b_5b_6}_{\lightning} (\underbrace{A_3 - A_4}_{E_8}) B_2(A_5B_6 - A_6B_5). \tag{28}$$

      i. $B_2 = 0$: As $A_2 = 0$ yields 1|2 the condition $P^{110}_{2110} + P^{110}_{1201} = a_3a_4b_5b_6A_2(B_5 - B_6)(A_3 - A_4)$ implies $B_5 = B_6$. Computing P^{111}_{2020} and P^{110}_{2020} yields

$$A_2a_3a_4B_6\,(A_3 - A_4)\,(b_5 - b_6)\,, \quad A_2a_3a_4B_6\,(A_3 - A_4)\,(a_6b_5 - a_5b_6)\,. \tag{29}$$

      $B_6 = 0$ implies E_1. The remaining case $\mathbf{m}_5 = \mathbf{m}_6$ yields $S_2(+\mathbf{m}_1 = \mathbf{m}_2)$.
      ii. $A_5B_6 - A_6B_5 = 0$, $B_2 \neq 0$: The condition $P^{200}_{1100} = 0$ can only vanish for architecturally singular manipulators (or $\lightning$).

2. $A_2(a_4 - a_3) + A_3(a_2 - a_4) + A_4(a_3 - a_2) = 0$, $a_2a_3 \neq 0$:

   a. Assuming $A_2 \neq A_3$ we can express a_4. Computing $P^{101}_{4020} + P^{101}_{2040} + P^{101}_{0402} + P^{101}_{0204}$ and $P^{101}_{4110} + P^{101}_{0114} + P^{101}_{1401} + P^{101}_{1041}$ yield

$$b_5b_6(a_2-a_3)^2(A_4-A_2)(A_3-A_4)(B_2A_3-B_3A_2)(A_5-A_6)/(A_3-A_2)^2, \quad (30)$$
$$\underbrace{b_5b_6}_{\lightning}\underbrace{(a_2-a_3)}_{E_6}\underbrace{(A_4-A_2)}_{2|4}\underbrace{(A_3-A_4)}_{3|4}(B_2A_3-B_3A_2)(B_5-B_6)/(A_3-A_2). \quad (31)$$

As $\mathbf{M}_5=\mathbf{M}_6$ yields a contradiction we set $B_3=B_2A_3/A_2$. It should be noted that we can assume $A_2\neq 0$ (w.o.l.g.) due to $A_2\neq A_3$. Computing $P^{100}_{2110}+P^{100}_{0112}+P^{100}_{1201}+P^{100}_{1021}$ yields

$$\underbrace{b_5b_6(a_2A_3-a_3A_2)}_{\lightning}\underbrace{(a_2-a_3)}_{E_6}\underbrace{(A_4-A_2)}_{2|4}\underbrace{(A_3-A_4)}_{3|4}(B_6A_5-B_5A_6)B_2/[A_2(A_3-A_2)]. \quad (32)$$

i. $B_2=0$: Now $P^{110}_{2110}+P^{110}_{1201}=0$ implies $B_5=B_6$. Then P^{111}_{2020} and P^{110}_{2020} can only vanish for $B_6=0$ (E_1) or $\mathbf{m}_5=\mathbf{m}_6$ (S_2 with an additional condition on a_4).
ii. $B_6A_5-B_5A_6=0$, $B_2\neq 0$: The condition $P^{200}_{1100}=0$ can only be fulfilled if the manipulator is architecturally singular (or $\lightning$).

b. $A_2=A_3$: Then the condition already yields $A_3=A_4$ due to $a_2\neq a_3$. Then $P^{110}_{2110}+P^{110}_{1201}=0$ implies $A_5=A_6$. Now $P^{101}_{4110}-P^{101}_{0114}+P^{101}_{1401}-P^{101}_{1041}$ can only vanish for architecturally singular designs (or $\lightning$).

3. $B_2A_3-A_2B_3=0$, $A_2(a_4-a_3)+A_3(a_2-a_4)+A_4(a_3-a_2)\neq 0$, $a_2a_3\neq 0$: W.l.o.g. we can assume $A_2\neq 0$ due to the condition of Part [A]. Therefore we set $B_3=B_2A_3/A_2$ ($\Rightarrow \mathbf{M}_1,\ldots,\mathbf{M}_4$ collinear). Now P^{201}_{1010} vanishes for $b_5b_6=0$ ($\lightning$), $CR_1=0$ with

$$CR_1:=a_4A_2A_3(a_3-a_2)+a_3A_2A_4(a_4-a_2)+a_2A_3A_4(a_4-a_3) \quad (33)$$

implying E_8 or the following two cases:

a. $B_2=0$: Now $P^{111}_{2020}=0$ implies $B_5=B_6b_5/b_6$. Then P^{110}_{2020} and $P^{110}_{2110}+P^{110}_{1201}$ can only vanish for $B_6=0$ (E_1), $b_5=0$ ($\lightning$), $CR_1=0$ (E_8) or $\mathbf{m}_5=\mathbf{m}_6$ (S_2).
b. $B_5=(A_2B_6+B_2A_5-B_2A_6)/A_2$, $B_2\neq 0$: Now $P^{111}_{2020}=0$ implies $b_5=b_6$. From $P^{110}_{2020}=0$ we get $\mathbf{m}_5=\mathbf{m}_6$. Then $P^{100}_{2200}=0$ can only be fulfilled if the manipulator is architecturally singular (or $\lightning$).

Part [B] $A_4a_3-a_4A_3=A_2a_4-a_2A_4=A_3a_2-a_3A_2=0$

From the above conditions we can express A_2 and A_3. Computing $P^{100}_{2110}-P^{100}_{0112}+P^{100}_{1201}-P^{100}_{1021}=0$ and $P^{100}_{2020}+P^{100}_{0202}=0$ shows that the following three cases must be distinguished:

1. $A_4=0$: Now P^{200}_{1100} can only vanish for $b_5b_6=0$ ($\lightning$), $CR_2=0$ with

$$CR_2:=a_4B_2B_3(a_3-a_2)+a_3B_2B_4(a_4-a_2)+a_2B_3B_4(a_4-a_3) \quad (34)$$

implying E_8 or $A_5=A_6$. From $P^{201}_{1010}=0$ follows $b_5=b_6$ and from $P^{200}_{1010}=0$ we get $a_5=a_6$. Now Q^{101} can only vanish for architecturally singular manipulators (or $\lightning$).

2. $a_2a_3=0$, $A_4\neq 0$: W.l.o.g. we set $a_2=0$ ($\Rightarrow \mathbf{m}_1=\mathbf{m}_2$). We compute

$$P^{101}_{4020}+P^{101}_{2040}+P^{101}_{0402}+P^{101}_{0204}=b_5b_6A_4B_2a_3(a_3-a_4)(A_5-A_6), \quad (35)$$
$$P^{101}_{4110}+P^{101}_{0114}+P^{101}_{1401}+P^{101}_{1041}=b_5b_6A_4B_2a_3(a_3-a_4)(B_5-B_6). \quad (36)$$

$a_3=0$ implies E_6, $B_2=0$ yields $1|2$. Therefore we set $a_3=a_4$ ($\Rightarrow \mathbf{m}_3=\mathbf{m}_4$). Now P^{200}_{1100} implies $A_5=A_6$ and from P^{100}_{2200} we get $A_4=A_6$. Then P^{201}_{1010} yields $a_5=(a_4b_6-a_4b_5+b_5a_6)/b_6$. $P^{101}_{4110}+P^{101}_{0114}-P^{101}_{1401}-P^{101}_{1041}$ shows that $a_4=a_6$ must hold. Now Q^{101} can only vanish if the manipulator is architecturally singular (or $\lightning$).

3. $B_2(a_3-a_4)+B_3(a_4-a_2)+B_4(a_2-a_3)=0$, $a_2a_3A_4\neq 0$: W.l.o.g. we can assume $a_2\neq a_3$ because for $a_2=a_3=a_4$ we get E_6. Therefore we can express B_4 from the above condition. Now $P^{101}_{4020}+P^{101}_{2040}+P^{101}_{0402}+P^{101}_{0204}$ and $P^{101}_{4110}+P^{101}_{0114}+P^{101}_{1401}+P^{101}_{1041}$ can only vanish for architecturally singular designs (or ↯).

4 Four base anchor points are collinear

In this section we assume that $\mathbf{M}_1,\dots,\mathbf{M}_4$ are collinear. Moreover we can always stop the case study if four platform anchor points are collinear; with exception of the cases discussed in subsection 4.1.2 and 4.1.3 (cf. footnote 3).

4.1 Base is not parallel to p

We set up the planar base as

$$C_1=B_2=B_3=B_4=0,\quad C_i=[C_4(B_6A_i-A_6B_i)+A_4C_6B_i]/(A_4B_6)\quad \text{for}\quad i=2,3,5. \tag{37}$$

In this case we eliminate the coefficients of t_3 from Q. We factor out A_4 and B_6 from Q^{ijk} if possible and call the remaining factor again Q^{ijk}. After crossing out r_{33} from Q^{003} we compute

$$P^{003}_{2000}+P^{003}_{0200}=B_5B_6(a_5-a_6)[b_3(a_2A_4-A_2a_4)+b_4(a_3A_2-A_3a_2)], \tag{38}$$

$$P^{003}_{1001}-P^{003}_{0110}=B_5B_6(b_5-b_6)[b_3(a_2A_4-A_2a_4)+b_4(a_3A_2-A_3a_2)]. \tag{39}$$

Therefore we have to distinguish the following three cases:

4.1.1 $B_5=0$

As $B_5=0$ implies $\mathbf{M}_1,\dots,\mathbf{M}_5$ collinear we can assume that no four points from $\mathbf{M}_1,\dots,\mathbf{M}_5$ coincide (E_5).

Part [A] Three points from $\mathbf{M}_1,\dots,\mathbf{M}_5$ coincide

W.l.o.g. we set $A_2=A_3=0$ ($\Rightarrow \mathbf{M}_1=\mathbf{M}_2=\mathbf{M}_3$, $A_4A_5\neq 0$) and compute

$$P^{003}_{2000}-P^{003}_{0200}=a_2b_3A_4A_5B_6(b_5-b_6),\quad P^{003}_{1001}+P^{003}_{0110}=a_2b_3A_4A_5B_6(a_5-a_6). \tag{40}$$

For $a_2=0$ we get $1|2$, $b_3=0$ yields E_6 and $\mathbf{m}_4=\mathbf{m}_5$ yields E_4.

Part [B] At most two points from $\mathbf{M}_1,\dots,\mathbf{M}_5$ coincide

We set $A_2=0$ ($\Rightarrow \mathbf{M}_1=\mathbf{M}_2$, $A_3A_4A_5\neq 0$) and compute again

$$P^{003}_{2000}-P^{003}_{0200}=a_2B_6(A_3A_4b_5(b_4-b_3)+A_3A_5b_4(b_3-b_5)+A_4A_5b_3(b_5-b_4)), \tag{41}$$

$$P^{003}_{1001}+P^{003}_{0110}=a_2B_6(A_3A_4b_5(a_4-a_3)+A_3A_5b_4(a_3-a_5)+A_4A_5b_3(a_5-a_4)). \tag{42}$$

The resultant of these equations with respect to A_3 yields

$$a_2{}^2 B_6^2 A_4 A_5 b_3 (b_5 A_4 - A_5 b_4)(b_4 a_5 - a_3 b_4 + b_3 a_4 - b_3 a_5 - a_4 b_5 + a_3 b_5). \quad (43)$$

The case $b_5A_4 - A_5b_4 = 0$ as well as the case of collinear points $\mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5$ can be done analogously to 3.1.1 Part [B,1] and 3.1.1 Part [B,2], respectively. Therefore we only have to discuss $b_3 = 0$ under the assumption $b_5A_4 - A_5b_4 \neq 0$ and $\mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5$ not collinear. Now Equ. (41) simplifies to $a_2A_3B_6b_4b_5(A_4 - A_5)$. As for $A_4 = A_5$ the Equ. of (42) yields a contradiction we set $b_4 = 0$. Now Equ. (42) can only vanish for $a_3 = a_4$ (special case of E_8).

Part [C] The points $\mathbf{M}_1, \ldots, \mathbf{M}_5$ are pairwise distinct

In this case we factorize $Q^{001} = B_6F_1[6]F_2[22]F_3[132]$. As the coefficient of e_0e_2 form $F_1 = 0$ equals A_4 (↯) we set $F_3 = 0$: Computing $j_0 - j_2$ yields a_2K_2 where j_i denotes the coefficient of e_i^2 from F_3. Moreover $P^{003}_{2000} - P^{003}_{0200} = B_6K_1$ holds.

1. $a_2 \neq 0$: In this case the two conditions $K_1 = K_2 = 0$ given in Karger [4] Equ. (17) indicating E_{10} must be fulfilled.
2. $a_2 = 0$: We compute

$$P^{003}_{2000} - P^{003}_{0200} = A_2B_6(b_3b_4a_5(A_4 - A_3) + b_3b_5a_4(A_3 - A_5) + b_4b_5a_3(A_5 - A_4)), \quad (44)$$
$$P^{003}_{1001} + P^{003}_{0110} = A_2B_6(a_3a_4b_5(A_4 - A_3) + a_3a_5b_4(A_3 - A_5) + a_4a_5b_3(A_5 - A_4)). \quad (45)$$

As $A_2 = 0$ yields $1|2$, we set the remaining factors equal to zero. If we solve the resulting linear equations for A_3 and A_4, we get $A_3 = A_4 = A_5$ (↯). This system cannot be solved if $\mathbf{m}_1 = \mathbf{m}_2, \mathbf{m}_i, \mathbf{m}_j$ collinear for $i, j \in \{3,4,5\}$ and $i \neq j$ (↯).

We proceed with $F_2 = 0$: $A_4B_6a_6$ and $A_4B_6b_6$ are the coefficients of e_0e_2 and e_0e_1, respectively. As for $a_6 = b_6 = 0$ the factor F_2 vanishes we factorize $Q^{002} = B_6F_1F_3H$ which finishes this part.

4.1.2 $\mathbf{m}_5 = \mathbf{m}_6$

We set $a_5 = a_6$, $b_5 = b_6$ and compute the following four linear combinations:

$$P^{003}_{2000} - P^{003}_{0200} = N_1, \quad P^{003}_{1001} + P^{003}_{0110} = N_2, \quad P^{002}_{5010} + P^{002}_{1050} = A_4B_6N_3, \quad P^{002}_{4011} + P^{002}_{1140} = A_4B_6N_4. \quad (46)$$

This implies $N_1 = N_2 = N_3 = N_4 = 0$ which are the four conditions given in Nawratil [9] indicating E_{12}.

4.1.3 $b_3(a_2A_4 - A_2a_4) + b_4(a_3A_2 - A_3a_2) = 0$

Assuming $a_2b_3 \neq 0$ we can express A_4. The rest of this case can be done 1 by 1 to section 3.1.3. Therefore we only have to discuss the special case $a_2b_3 = 0$. If $a_2 = 0$ holds, we can set $b_3 = 0$ without loss of generality. Hence, we start with $b_3 = 0$. Now the condition splits into $b_4(a_2A_3 - A_2a_3) = 0$:

1. $b_4 = 0$: Now $P^{003}_{1001} + P^{003}_{0110} = 0$ can only vanish for $CR_1 = 0$ (E_8) with CR_1 of Equ. (33) or for $b_5 = b_6B_5/B_6$. Then $P^{002}_{5001} - P^{002}_{1005} = 0$ implies $a_5 = a_6$. The condition $P^{002}_{4101} + P^{002}_{1410} = 0$ can only vanish for architecturally singular manipulators (or ↯).
2. Due to the last case we can assume $b_4 \neq 0$. Moreover we can assume $A_3 \neq 0$ as $A_2 = A_3 = 0$ yields E_6. Therefore we set $a_2 = a_3A_2/A_3$. Now $P^{003}_{1001} + P^{003}_{0110}$ implies

$$A_3[B_5(a_4b_6-b_4a_6)-B_6(a_4b_5-b_4a_5)]-a_3[B_5(A_4b_6-b_4A_6)+B_6(A_4b_5-b_4A_5)]=0. \quad (47)$$

From this factor we can express a_5. Then $P^{002}_{4011}-P^{002}_{1140}-P^{002}_{0411}+P^{002}_{1104}=0$ implies $b_5=b_6$. Computing $P^{002}_{5010}+P^{002}_{1050}-P^{002}_{0501}-P^{002}_{0105}$ yields

$$\underbrace{A_2}_{1|2}\underbrace{(A_2-A_3)}_{2|3}\underbrace{a_3}_{E_6}\underbrace{A_4B_6b_4}_{\lightning}b_6(A_4B_6+B_5A_6-B_6A_5-A_4B_5). \quad (48)$$

In both remaining cases the condition $P^{002}_{4101}-P^{002}_{0141}=0$ can only vanish for architecturally singular manipulators (or $\lightning$).

4.2 Base is parallel to p and $\mathbf{M}_1,\dots,\mathbf{M}_4$ is parallel to p

In this case we set $B_2=B_3=B_4=C_i=0$ for $i=2,\dots,6$ and eliminate t_1 from Q. From Q^{100}, Q^{200}, Q^{201} we can additionally factor out r_{31}. In the following we have to distinguish again three cases due to

$$P^{201}_{1010}=B_5B_6(a_5-a_6)[b_3(a_2A_4-A_2a_4)+b_4(a_3A_2-A_3a_2)], \quad (49)$$

$$P^{201}_{1100}=B_5B_6(b_5-b_6)[b_3(a_2A_4-A_2a_4)+b_4(a_3A_2-A_3a_2)]. \quad (50)$$

4.2.1 $B_5=0$

As $B_5=0$ implies $\mathbf{M}_1,\dots,\mathbf{M}_5$ collinear we can assume that no four points from $\mathbf{M}_1,\dots,\mathbf{M}_5$ coincide (E_5).

Part [A] Three points from $\mathbf{M}_1,\dots,\mathbf{M}_5$ coincide

W.l.o.g. we set $A_2=A_3=0$ ($\Rightarrow \mathbf{M}_1=\mathbf{M}_2=\mathbf{M}_3$, $A_4A_5\neq 0$) and compute

$$P^{111}_{2110}=a_2b_3A_4A_5B_6(b_5-b_6) \quad \text{and} \quad P^{111}_{2020}=a_2b_3A_4A_5B_6(a_5-a_6), \quad (51)$$

respectively. For $a_2=0$ we get $1|2$, $b_3=0$ yields E_6 and $\mathbf{m}_4=\mathbf{m}_5$ yields E_4.

Part [B] At most two points from $\mathbf{M}_1,\dots,\mathbf{M}_5$ coincide

We set $A_2=0$ ($\Rightarrow \mathbf{M}_1=\mathbf{M}_2$, $A_3A_4A_5\neq 0$) and compute the equations $P^{111}_{2110}=0$ and $P^{111}_{2020}=0$ which are identically with those given in Equ. (41) and (42), respectively. Therefore this case can be done analogously to 4.1.1 Part [B].

Part [C] The points $\mathbf{M}_1,\dots,\mathbf{M}_5$ are pairwise distinct

Now we compute $Q^{100}=B_6a_2r_{31}(a_6r_{13}+b_6r_{23})K_2$ and $P^{111}_{2110}=B_6K_1$. For $a_6=b_6=0$ we compute $Q^{101}=a_2B_6Hr^2_{31}K_2$.

1. $a_2\neq 0$: In this case the two conditions $K_1=K_2=0$ given in Karger [4] Equ. (17) indicating E_{10} must be fulfilled.
2. $a_2=0$: This case can be done analogously to 4.1.1 Part [C,2] if we replace the equations (44) and (45) by $P^{111}_{2110}=0$ and $P^{111}_{2020}=0$, respectively.

4.2.2 $\mathbf{m}_5 = \mathbf{m}_6$

We set $a_5 = a_6$, $b_5 = b_6$ and compute $Q^{110} = r_{31}(a_6 r_{13} + b_6 r_{23})G[120]$. As $a_6 = b_6 = 0$ yields a contradiction we consider $G_{1100} = N_1$ and $G_{1010} = N_2$. As it was shown in Nawratil [9] the conditions $N_1 = N_2 = 0$ are not sufficient for case E_{12}. There are three cases where $N_1 = N_2 = 0$ holds and the manipulator is not architecturally singular. From these three designs only one (S_2) possesses a cylindrical singularity surface (cf. Nawratil [9]).

4.2.3 $b_3(a_2 A_4 - A_2 a_4) + b_4(a_3 A_2 - A_3 a_2) = 0$

W.l.o.g. we can assume $b_3 \neq 0$, because $b_3 = b_4 = 0$ yields a contradiction (4 platform anchor points are collinear).

1. Assuming $A_2 \neq 0$ we can express a_4 from the above condition. Then we compute $P^{200}_{2200} = a_2 b_3 b_4 B_5 B_6 (A_3 - A_4)(b_5 - b_6)$. As $a_2 = 0$ yields $m_1 = m_2, m_3 m_4$ collinear (↯) there are three cases left:
   a. For $b_4 = 0$ we can compute A_5 from the only non-contradicting factor of Q^{111}. Finally Q^{200} can only vanish for architecturally singular manipulators (or ↯).
   b. $A_3 = A_4$, $b_4 \neq 0$: We can again compute A_5 from the only non-contradicting factor of Q^{111}. Now Q^{110} can only vanish for architecturally singular designs (or ↯).
   c. $b_5 = b_6$, $b_4 \neq 0$, $A_3 \neq A_4$: This can be done analogously to case b.
2. For $A_2 = 0$ the condition simplifies to $a_2(A_4 b_3 - A_3 b_4) = 0$. As $a_2 = 0$ yields $1|2$ we set $A_4 = A_3 b_4 / b_3$. Finally Q^{200} can only vanish for architecturally singular manipulators (or ↯).

4.3 *Base is parallel to* p *and* $\mathbf{M}_1, \ldots, \mathbf{M}_4$ *is not parallel to* p

In this case we set $B_2 = C_i = 0$ for $i = 2, \ldots, 6$. Moreover we rotate the Cartesian frame of the space of translation around the z-axis of the fixed frame about the angle φ. This results in a more general translation vector $\mathbf{t} := (\cos\varphi t_1 - \sin\varphi t_2, \sin\varphi t_1 + \cos\varphi t_2, t_3)^T$. Then we eliminate again t_1 from Q where we can assume $\sin\varphi \neq 0$ due to section 4.2. We start the case study by considering

$$P^{100}_{3300} = \sin\varphi b_3 b_4 B_5 B_6 (a_5 b_6 - a_6 b_5)(A_3 - A_4)(A_2 - a_2), \quad (52)$$

$$P^{100}_{0033} = \sin\varphi b_3 b_4 B_5 B_6 (a_5 b_6 - a_6 b_5)(A_3 - A_4)(A_2 + a_2). \quad (53)$$

For the discussion of the possible four cases we introduce the abbreviations:

$$P_1 := P^{100}_{3210} - P^{100}_{2301} + P^{100}_{1032} - P^{100}_{0123}, \qquad P_2 := P^{100}_{3210} - P^{100}_{2301} - P^{100}_{1032} + P^{100}_{0123}, \quad (54)$$

$$P_3 := P^{100}_{3210} + P^{100}_{2301} + P^{100}_{1032} + P^{100}_{0123}, \qquad P_4 := P^{100}_{3210} + P^{100}_{2301} - P^{100}_{1032} - P^{100}_{0123}, \quad (55)$$

$$P_5 := P^{101}_{4110} - P^{101}_{1401} - P^{101}_{1041} + P^{101}_{0114}, \qquad P_6 := P^{101}_{4110} + P^{101}_{1401} - P^{101}_{1041} - P^{101}_{0114}, \quad (56)$$

and $P_7 := P^{101}_{4200} - P^{101}_{0024}$.

4.3.1 $B_5 = 0$

As $B_5 = 0$ implies $\mathbf{M}_1, \ldots, \mathbf{M}_5$ collinear we can assume that no four points from $\mathbf{M}_1, \ldots, \mathbf{M}_5$ coincide (E_5).

Part [A] Three points from $\mathbf{M}_1, \ldots, \mathbf{M}_5$ coincide

Without loss of generality we set $A_2 = A_3 = 0$ ($\Rightarrow \mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3$, $A_4A_5 \neq 0$) and compute $P_4 = \sin\varphi a_2 b_3 A_4 A_5 B_6 b_6 (a_5 b_4 - b_5 a_4)$. For $a_2 = 0$ we get 1|2 and $b_3 = 0$ yields E_6. For both remaining cases Q^{100} and Q^{101} can only vanish for architecturally singular designs (or ↯).

Part [B] At most two points from $\mathbf{M}_1, \ldots, \mathbf{M}_5$ coincide

We set $A_2 = 0$ ($\Rightarrow \mathbf{M}_1 = \mathbf{M}_2$, $A_3A_4A_5 \neq 0$) and compute

$$P^{100}_{3030} + P^{100}_{0303} = \sin\varphi a_2 A_3 A_4 A_5 B_6 a_6 (a_3 b_4 - a_3 b_5 - b_3 a_4 + a_4 b_5 + b_3 a_5 - b_4 a_5). \quad (57)$$

1. For $\mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5$ collinear the condition $Q^{100} = 0$ implies $a_6 = b_6 = 0$. Then $Q^{111} = 0$ yields an architecturally singular design (or ↯).
2. $a_6 = 0$, $\mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5$ not collinear: We get

$$P_4 = \sin\varphi a_2 b_6 B_6 [b_3 A_4 A_5 (a_5 b_4 - a_4 b_5) + b_4 A_3 A_5 (a_3 b_5 - a_5 b_3) + b_5 A_3 A_4 (a_4 b_3 - a_3 b_4)]. \quad (58)$$

   a. $b_6 = 0$ ($\Rightarrow b_3 b_4 b_5 \neq 0$): Then Q^{201} factors into $\sin\varphi a_2 B_6 (\sin\varphi r_{32} + \cos\varphi r_{31}) G[24]$. The resultant of $G_{1010} = 0$ and $G_{1100} = 0$ with respect to A_3 can only vanish for $A_4 = A_5 b_4 / b_5$ (or ↯). Then $G_{1010} = 0$ and $G_{1100} = 0$ imply $A_3 = A_5 b_3 / b_5$. Finally Q^{101} can only vanish for architecturally singular designs (or ↯).
   b. Now we set the last factor of Equ. (58) equal to zero and assume $b_6 \neq 0$. Moreover we can assume (w.l.o.g.) that two b_i's with $i = 3,4,5$ are different from zero because otherwise 4 platform anchor points would be collinear (↯). We choose $b_3 b_4 \neq 0$.
      i. Assuming $A_3 \neq A_4$ we can express a_5. Now Q^{100} can only vanish for architecturally singular designs (or ↯).
      ii. For $A_3 = A_4$ we get $P_4 = \underbrace{\sin\varphi B_6 b_6 A_4}_{\lightning}\underbrace{a_2}_{1|2} b_5 (a_3 b_4 - a_4 b_3)(A_4 - A_5)$. For all three remaining cases $Q^{100} = 0$ can only be fulfilled if the manipulator is architecturally singular (or ↯).

Part [C] The points $\mathbf{M}_1, \ldots, \mathbf{M}_5$ are pairwise distinct

1. $a_2 \neq 0$: The following relations hold

$$P_4 = \sin\varphi a_2 B_6 b_6 K_2, \quad a_2 P_3 + A_2 P_4 = \sin\varphi a_2 A_2 B_6 b_6 K_1. \quad (59)$$

   For $b_6 = 0$ we compute $U_1 = P^{100}_{3120} - P^{100}_{2031}$ and $U_2 = P^{100}_{3120} + P^{100}_{2031}$ yielding

$$U_1 = \sin\varphi a_2 B_6 a_6 K_2, \quad a_2 U_2 + A_2 U_1 = \sin\varphi a_2 A_2 B_6 a_6 K_1. \quad (60)$$

   If also $a_6 = 0$ holds we compute $V_1 = P^{101}_{4110} - P^{101}_{0114}$ and $V_2 = P^{101}_{4110} + P^{101}_{0114}$ with

$$V_1 = \sin\varphi a_2 B_6 K_2, \quad a_2 V_2 + A_2 V_1 = \sin\varphi a_2 A_2 B_6 K_1. \quad (61)$$

   Therefore the two conditions $K_1 = K_2 = 0$ indicating E_{10} must hold.
2. $a_2 = 0$: Now Q^{111} splits up into $A_2 B_6 [2\cos\varphi(\sin\varphi r_{32} + \cos\varphi r_{31}) - r_{31}] G[24]$. This case can be done analogously to 4.1.1 Part [C,2] if we replace the equations (44) and (45) by $G_{1100} = 0$ and $G_{1010} = 0$, respectively.

4.3.2 $b_3 = 0$

As $b_3 = 0$ implies the collinearity of $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ we can assume $b_4 b_5 b_6 \neq 0$. Moreover we can assume $B_5 B_6 \neq 0$. Due to

$$P_7 = \sin\varphi b_4 B_5 B_6 A_4 (b_5 - b_6)(a_2 A_3 - a_3 A_2) \tag{62}$$

we have to distinguish the following three cases:

1. For $A_4 = 0$ ($\Rightarrow \mathbf{M}_1 = \mathbf{M}_4$) the polynomials P_2 and P_3 simplify to

$$\sin\varphi A_2 A_3 b_4 B_5 B_6 (a_5 b_6 - a_5 b_6)(a_2 - a_3), \quad \sin\varphi A_2 A_3 b_4 b_5 b_6 (A_5 B_6 - A_5 B_6)(a_2 - a_3). \tag{63}$$

   a. $\mathbf{M}_1, \mathbf{M}_5, \mathbf{M}_6$ collinear, $\mathbf{m}_1, \mathbf{m}_5, \mathbf{m}_6$ collinear: For $a_6 = a_5 b_6 / b_5$ and $A_6 = A_5 B_6 / B_5$ we get

$$P_5 = \underbrace{\sin\varphi B_5 B_6}_{\lightning} \underbrace{(a_4 b_5 - a_5 b_4)}_{\mathbf{m}_1, \mathbf{m}_4, \mathbf{m}_5, \mathbf{m}_6 \text{coll.}} A_2 A_3 (b_5 - b_6)(a_2 - a_3)/b_5. \tag{64}$$

   Moreover for $A_2 A_3 = 0$ we get five collinear base anchor points, which was already discussed in subsection 4.3.1. For both remaining cases Q^{201} can only vanish for architecturally singular designs (or $\lightning$).

   b. $A_2 = 0$ and not both triples $\mathbf{M}_1, \mathbf{M}_5, \mathbf{M}_6$ and $\mathbf{m}_1, \mathbf{m}_5, \mathbf{m}_6$ are collinear: W.l.o.g. we can assume $A_3 \neq 0$ because otherwise we would get E_5. Now $P_1 = 0$ and $P_4 = 0$ imply $a_3 = 0$. From Q^{200} we get $A_6 = A_5 B_6 / B_5$. Then Q^{111} can only vanish for architecturally singular designs (or $\lightning$).

   c. $a_2 = a_3$, $A_2 \neq 0$ and not both triples $\mathbf{M}_1, \mathbf{M}_5, \mathbf{M}_6$ and $\mathbf{m}_1, \mathbf{m}_5, \mathbf{m}_6$ are collinear: Then the polynomials $P_1 = 0$ and $P_4 = 0$ can only vanish for architecturally singular designs (or $\lightning$).

2. $b_5 = b_6$, $A_4 \neq 0$: Now $P^{101}_{4200} + P^{101}_{2400}$ yields $\sin\varphi b_4 B_5 B_6 b_6 (a_5 - a_6)(a_2 A_3 - a_3 A_2)$.

   a. For $a_5 = a_6$ ($\Rightarrow \mathbf{m}_5 = \mathbf{m}_6$) the equations $P_3 = 0$ and $P_4 = 0$ imply the collinearity of $\mathbf{M}_4, \mathbf{M}_5, \mathbf{M}_6$, i.e. $A_6 = (A_4 B_5 - A_4 B_6 + A_5 B_6)/B_5$. Then Q^{100} can only vanish for architecturally singular designs (or $\lightning$).

   b. W.o.l.g. we can set $a_2 = a_3 A_2 / A_3$ because for $A_2 = A_3 = 0$ we would get E_6. Then $P_2 = 0$ can only vanish for architecturally singular designs (or $\lightning$).

3. $a_2 = a_3 A_2 / A_3$, $b_5 \neq b_6$, $A_4 \neq 0$: Now $P_2 = 0$ implies $a_5 = a_6 b_5 / b_6$. From $P_3 = 0$ follows $A_5 = (A_4 B6 - A_4 B_5 + A_6 B_5)/B_6$ and from $P_5 = 0$ we get $a_3 = A_3 (a_4 b_6 - a_6 b_4)/(A_4 b_6)$. Now Q^{201} can only vanish for architecturally singular designs (or $\lightning$).

4.3.3 $\mathbf{M}_3 = \mathbf{M}_4$

W.l.o.g. we can assume $b_3 b_4 B_5 B_6 \neq 0$. We set $A_3 = A_4$ and compute the polynomials P_2 and $P^{100}_{3120} - P^{100}_{0213} - P^{100}_{2031} + P^{100}_{1302}$ which yield

$$\sin\varphi a_2 A_4 (A_4 - A_2) B_5 B_6 (b_5 a_6 - a_5 b_6)(b_3 - b_4), \quad \sin\varphi a_2 A_4 (A_4 - A_2) B_5 B_6 (b_5 a_6 - a_5 b_6)(a_3 - a_4). \tag{65}$$

1. $a_2 = 0$: Now Q^{200} implies $A_6 = (A_4 B_5 - A_4 B_6 + A_5 B_6)/B_5$. From $P_5 = 0$ and $P_6 = 0$ follows $A_4 = 0$. Now Q^{111} can only vanish for architecturally singular designs (or $\lightning$).
2. $A_4 = 0$ or $A_2 = A_4$, $a_2 \neq 0$: W.l.o.g. we set $A_4 = 0$ ($\Rightarrow \mathbf{M}_1 = \mathbf{M}_3 = \mathbf{M}_4$). Now $P_1 = 0$ implies the collinearity of $\mathbf{m}_1, \mathbf{m}_5, \mathbf{m}_6$, i.e. $a_6 = a_5 b_6 / b_5$ (w.l.o.g. we can assume $b_5 \neq 0$ because $b_5 = b_6 = 0$ yields the collinearity of four platform anchor points). Now Q^{100} can only vanish for $b_6 = 0$ or $A_6 = A_5 B_6 / B_5$. In both cases $Q^{110} = 0$ yields an architecturally singular design (or $\lightning$).

3. $a_6 = a_5b_6/b_5$, $a_2A_4(A_2 - A_4) \neq 0$: We have to distinguish three cases due to

$$P_4 = \sin\varphi a_2b_5b_6(A_4 - A_2)(A_5B_6 - A_6B_5 + A_4B_5 - A_4B_6)(a_4b_3 - b_4a_3). \tag{66}$$

a. For $b_6 = 0$ the condition $Q^{110} = 0$ cannot vanish without contradiction.
b. $a_4 = a_3b_4/b_3$, $b_6 \neq 0$: Now $P_6 = 0$ implies $b_5 = b_6$. We compute

$$P_3 = \sin\varphi a_2A_2A_4{b_6}^2(b_3 - b_4)(B_6A_4 - A_4B_5 + B_5A_6 - A_5B_6). \tag{67}$$

i. $\mathbf{M}_4, \mathbf{M}_5, \mathbf{M}_6$ collinear, i.e. $A_6 = (A_4B_5 - A_4B_6 + A_5B_6)/B_5$: $Q^{100} = 0$ already yields an architecturally singular design (or ↯).
ii. $A_2 = 0$, $\mathbf{M}_4, \mathbf{M}_5, \mathbf{M}_6$ not collinear: Now $Q^{100} = 0$ implies $A_6 = A_5B_6/B_5$. Then Q^{201} can only vanish for architecturally singular designs (or ↯).

c. $A_6 = (A_4B_5 - A_4B_6 + A_5B_6)/B_5$, $b_6(a_4b_3 - b_4a_3) \neq 0$: We compute

$$P_6 = \sin\varphi A_4B_5B_6(b_5 - b_6)[a_2A_4(b_4 - b_3) + A_2(a_4b_3 - a_3b_4)]. \tag{68}$$

i. $b_5 = b_6$: Q^{201} can only vanish for architecturally singular designs (or ↯).
ii. W.l.o.g. we can express A_2 from the last factor of Equ. (68). Moreover we can assume $b_5 \neq b_6$. Now $P_5 = 0$ implies the collinearity of $\mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5$, i.e. $a_3 = (a_4b_5 - b_4a_5 + a_5b_3)/b_5$. From Q^{201} we get $B_5 = B_6$. Then Q^{110} can only vanish for architecturally singular designs (or ↯).

4.3.4 $\mathbf{m}_1, \mathbf{m}_5, \mathbf{m}_6$ collinear

W.l.o.g. we can assume $b_3b_4B_5B_6(A_3 - A_4) \neq 0$. Moreover we can set $a_5 = a_6b_5/b_6$ (w.lo.g.) because for $b_5 = b_6 = 0$ we get a contradiction. We start with

$$P^{101}_{4200} + P^{101}_{0042} = \sin\varphi B_5B_6a_2b_3b_4a_6(b_5 - b_6)(A_3 - A_4)/b_6. \tag{69}$$

1. $b_5 = b_6$ ($\Rightarrow \mathbf{m}_5 = \mathbf{m}_6$): Now Q^{200} factors into

$$(a_6r_{13} + b_6r_{23})[\cos\varphi(\sin\varphi r_{31} - \cos\varphi r_{32}) + r_{32}]G[120]. \tag{70}$$

As $A_6 = b_6 = 0$ yields a contradiction we consider $G_{1100} = N_1$ and $G_{0101} = N_2$. The remaining conditions $N_3 = N_4 = 0$ indicating E_{12} can be computed as

$$P^{101}_{4110} + P^{101}_{1401} + P^{101}_{1041} + P^{101}_{0114} = \sin\varphi N_3, \quad P^{101}_{4020} + P^{101}_{2040} - P^{101}_{0402} - P^{101}_{0204} = \sin\varphi N_4. \tag{71}$$

2. $a_6 = 0$, $b_5 \neq b_6$: Moreover we can assume $a_2a_3a_4 \neq 0$ because otherwise 4 platform anchor points are collinear. The resultant of $P_5 = 0$ and $P_7 = 0$ with respect to A_2 yields

$$\sin\varphi^2{B_5}^2{B_6}^2(b_5 - b_6)^2a_2A_3A_4(b_3a_4 - b_3a_2 + b_4a_2 - b_4a_3)(A_3a_4 - A_4a_3). \tag{72}$$

Therefore we have to distinguish the following three cases:

a. $A_3 = 0$ or $A_4 = 0$: W.l.o.g. we set $A_3 = 0$. Now P_7 can only vanish for $A_2 = 0$ or $A_4 = 0$. W.l.o.g. we set $A_2 = 0$. Then Q^{100} factors into

$$\underbrace{a_2a_4b_3b_6}_{↯}\underbrace{A_4}_{E_5}b_5(B_5A_6 - A_5B_6)(\cos\varphi r_{31} + \sin\varphi r_{32})r_{23}r_{31}. \tag{73}$$

i. $b_5 = 0$: Now $Q^{200} = 0$ implies $A_5 = \cot\varphi B_5$. Then Q^{110} can only vanish for architecturally singular designs (or ↯).

ii. $A_5 = A_6 B_5/B_6$, $b_5 \neq 0$: Now $Q^{201} = 0$ implies $A_5 = \cot\varphi B_5$. Again Q^{110} can only vanish for architecturally singular designs (or $\lightning$).

b. $a_4 = a_3 A_4/A_3$: Now $P_5 = 0$ and $P_7 = 0$ imply $a_2 = a_3 A_2/A_3$. From $Q^{201} = 0$ we get $A_6 = A_5 B_6/B_5$. Finally Q^{110} can only vanish for architecturally singular designs (or $\lightning$).

c. $\mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ collinear, $A_3 A_4 (A_3 a_4 - A_4 a_3) \neq 0$: We set $a_3 = (a_4 b_3 - a_2 b_3 + a_2 b_4)/b_4$.

i. Assuming $A_2 \neq 0$ we can express a_4 from the only non-contradicting factor of $P_7 = 0$. Now $P_4 = 0$ can only vanish without contradiction for $A_6 = A_5 B_6/B_5$ or $b_5 = 0$. For both cases $Q^{100} = 0$ implies $A_5 = \cot\varphi B_5$. Then $Q^{111} = 0$ resp. $Q^{110} = 0$ can only vanish for architecturally singular designs (or $\lightning$).

ii. $A_2 = 0$: From $P_7 = 0$ we get $b_3 = b_4$ ($\Rightarrow \mathbf{m}_3 = \mathbf{m}_4$). Then Q^{100} splits up into

$$\underbrace{a_2 a_4 b_4 b_6}_{\lightning}\underbrace{(A_3 - A_4)}_{3|4} b_5 (B_5 A_6 - A_5 B_6)(\cos\varphi r_{31} + \sin\varphi r_{32}) r_{23} r_{31}. \tag{74}$$

For both remaining cases Q^{110} can only vanish for architecturally singular designs (or $\lightning$). End of all cases. □

5 Conclusion

In this article we presented the complete list of all non-architecturally singular planar parallel manipulators of Stewart Gough type which possess a cylindrical singularity surface with rulings parallel to a given fixed direction p for any orientation of the platform. The list given in Theorem 2 only has two entries containing the geometric conditions of the corresponding manipulator designs.

The determination of the whole set of non-planar parallel manipulators of Stewart Gough type which are not architecturally singular and possess a cylindrical singularity surface remains open. We conjecture that this set consists of only one element, namely the manipulator presented in subsection 1.2 (see Fig. 1).

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