THE CONTROL NUMBER AS INDEX FOR STEWART GOUGH PLATFORMS

G. Nawratil
Vienna University of Technology
Institute of Discrete Mathematics and Geometry
upuaut@controverse.net

Abstract Singular postures of Stewart Gough Platforms must be avoided because close to singularities they lose controllable degrees of freedom. Hence there is an interest in a distance measure between the instantaneous configuration and the nearest singularity. This article presents such a measure, which is invariant under Euclidean motions and similarities, which has a geometric meaning and can be computed in real-time. This measure ranging between 0 and 1 can serve as a performance index.

Keywords: Stewart Gough Platform, distance measure, perfomance index

1. Introduction

In this article we define a new measure, which allows to compare different postures of different nonredundant Stewart Gough Platforms (SGPs). Such a measure should assign to each configuration $\mathcal{K}$ a scalar $D(\mathcal{K})$ obeying the following six properties:

1. $D(\mathcal{K}) \geq 0$ for all $\mathcal{K}$ of the configuration space,
2. $D(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is singular,
3. $D(\mathcal{K})$ is invariant under Euclidean motions,
4. $D(\mathcal{K})$ is invariant under similarities,
5. $D(\mathcal{K})$ has a geometric meaning,
6. $D(\mathcal{K})$ is computable in real-time.

$\mathcal{K}$ is singular if and only if the six legs belong to a linear line complex (see Merlet, 1992) or, analytically seen, the determinant of the Jacobian

$$\mathcal{J}^T = \begin{pmatrix} \widehat{\mathbf{l}}_1 \, \|\mathbf{l}_1\|^{-1} & \vdots & \widehat{\mathbf{l}}_6 \, \|\mathbf{l}_6\|^{-1} \\ \mathbf{l}_1 \, \|\mathbf{l}_1\|^{-1} & \vdots & \mathbf{l}_6 \, \|\mathbf{l}_6\|^{-1} \end{pmatrix} \quad \text{with} \quad \begin{array}{l} \mathbf{l}_i = \mathbf{P}_i - \mathbf{B}_i \quad \text{and} \\ \\ \widehat{\mathbf{l}}_i = \mathbf{B}_i \times \mathbf{l}_i = \mathbf{P}_i \times \mathbf{l}_i \end{array} \tag{1}$$

vanishes, where $\mathbf{B}_i$ resp. $\mathbf{P}_i$ are the coordinates of the base resp. platform anchor points with respect to any fixed reference frame Σ_0 with origin O. Therefore the i^{th} row of $\mathcal{J}$ equals the normalized Plücker coordinates of the carrier line $\mathcal{L}_i$ of the i^{th} leg oriented in the direction $\mathbf{B}_i\mathbf{P}_i$. We'll assume for the rest of this article that $\mathbf{B}_i \neq \mathbf{P}_i$ for $i = 1, .., 6$.

Kinematic meaning of the Jacobian. The velocity vector $\mathbf{v}(\mathbf{P}_i)$ of $\mathbf{P}_i$ with respect to the instantaneous screw $\underline{\mathbf{q}} := (\mathbf{q}, \widehat{\mathbf{q}})$ of the platform Σ against Σ_0 can be decomposed in a component $\mathbf{v}_{\mathcal{L}}(\mathbf{P}_i)$ along the i^{th} leg $\mathcal{L}_i$ and in a component $\mathbf{v}_{\perp}(\mathbf{P}_i)$ orthogonal to it (see *Fig. 1*), thus

$$\mathbf{v}(\mathbf{P}_i) = \widehat{\mathbf{q}} + (\mathbf{q} \times \mathbf{P}_i) = \mathbf{v}_{\mathcal{L}}(\mathbf{P}_i) + \mathbf{v}_{\perp}(\mathbf{P}_i) \tag{2}$$

$$\text{with} \qquad \|\mathbf{v}_{\mathcal{L}}(\mathbf{P}_i)\| = \frac{\mathbf{l}_i}{\|\mathbf{l}_i\|}\mathbf{v}(\mathbf{P}_i) = \frac{\widehat{\mathbf{l}}_i}{\|\mathbf{l}_i\|}\mathbf{q} + \frac{\mathbf{l}_i}{\|\mathbf{l}_i\|}\widehat{\mathbf{q}} =: d_i. \tag{3}$$

Therefore the Jacobian $\mathcal{J}$ is the matrix of the linear mapping

$$\iota: \ \underline{\mathbf{q}} \mapsto \mathbf{d} = \mathcal{J}\,\underline{\mathbf{q}} \qquad \text{with} \qquad \mathbf{d} = (d_1, .., d_6)^T. \tag{4}$$

ι has at least a one-dimensional kernel ker_ι, if $\mathcal{K}$ is singular. Let $\underline{\mathbf{k}} \in ker_\iota$ and $\underline{\mathbf{k}} \neq \mathbf{o}$. Then also $\mu\underline{\mathbf{k}}$ with $\mu \in \mathbb{R}$ lies in ker_ι. Therefore we can say, that $\mathbf{v}(\mathbf{P}_i)$ can be arbitrarily large for constant translatory velocities in the six prismatic legs. The sole exeption is the case where $\mathbf{P}_i$ lies on the instantaneous screw axis (*isa*) and $\underline{\mathbf{k}}$ is an instantaneous rotation.

Review. In the following we analyze some of the in our opinion most important indices in view of the initially stated six properties.

The *manipulabilitiy* introduced by Yoshikawa, 1985 is not invariant under similarities, because for *SGP*s it equals $|det(\mathcal{J})|$. So Lee et al., 1998 used $|det(\mathcal{J})| \cdot |det(\mathcal{J})|_m^{-1}$ as index, where $|det(\mathcal{J})|_m$ denotes the maximum of $|det(\mathcal{J})|$ over the *SGP*'s configuration space. But the computation of $|det(\mathcal{J})|_m$ is a nonlinear task and was only done for planar *SGP*s with very special geometries. Only for these *SGP*s $|det(\mathcal{J})|_m$ can be interpreted geometrically as the volume of the framework.

Pottmann et al., 1998 introduced the concept of the *best fitting linear line complex* $\underline{\mathbf{c}}$. The suggested index equals the square root of the minimum of $\sum d_i^2$ with respect to $\underline{\mathbf{c}}$ under the side condition $\mathbf{cc} = 1$. The index is not invariant under similarities on the one hand and it is not defined for instantaneous translations $\underline{\mathbf{c}}$. In order to close this gap, the authors proposed to minimize a further function, which yields a second value. But how should these two values be combined to a single number?

The *rigidity rate* introduced by Lang et al., 2001 is based on the idea, that a *SGP* at any position $\mathcal{K}$ permits a one-parametric self-motion within the group of Euclidean similarities $\mathcal{G}_7$. The angle $\varphi \in [0, \pi/2]$ between the tangent of the self-motion in $\mathcal{K}$ and the subgroup of Euclidean displacements serves as an index. But the choice of the invariant symmetric bilinear form in the tangent space of $\mathcal{G}_7$, which is necassary in order to define a measure in the sense of non-Euclidean geometry, is arbitrary. Although φ fulfills all six stated properties, its applicability is limited. This becomes manifest in the remark at the end of *Section* 5.

2. Preliminary considerations

Now we take a closer look at the reciprocal of the *condition number* (cdn^{-1}) introduced by Salisbury and Craig, 1982, because it will be the starting point of our considerations. cdn^{-1} equals the ratio of the minimum $\widehat{\lambda}_{-}$ and the maximum $\widehat{\lambda}_{+}$ of the quadratic objective function

$$\widehat{\zeta}(\underline{\mathbf{q}}): \quad \underline{\mathbf{q}}^T \mathcal{I}_6 \, \underline{\mathbf{q}} = \omega^2 + \left[\widehat{\omega}^2 + \omega^2 \overline{Op}^2\right] \tag{5}$$

with p denoting the *isa*, ω the angular velocity and $\widehat{\omega}$ the translatory velocity of the screw $\underline{\mathbf{q}}$, under the quadratic side condition

$$\nu(\underline{\mathbf{q}}): \quad \mathbf{d}^T \mathbf{d} = \underline{\mathbf{q}}^T \mathcal{N} \, \underline{\mathbf{q}} = 1 \qquad \text{with} \qquad \mathcal{N} = \mathcal{J}^T \mathcal{J}. \tag{6}$$

Due to the linearity of ι in (4) the screw $\mu\underline{\mathbf{q}}$ corresponds to the μ-fold translatory velocity d_i in the six prismatic legs, and therefore the side condition $\nu(\underline{\mathbf{q}})$ is well defined. The weak point of this index is the objective function for the following reasons. First, it is not invariant under translations, because $\widehat{\zeta}(\underline{\mathbf{q}})$ depends on the choice of O. In practice O is not selected arbitrarily, but placed in the tool center point. But the real problem, which causes the variance of cdn^{-1} under similarities, occurs from the dimensional inhomogenity of $\widehat{\zeta}(\underline{\mathbf{q}})$. To overcome this deficiency, different concepts (*e.g. characteristic length*) were introduced, but they still weight the ratio of length and angle in a more or less arbitrary way. The inhomogenity and the lacking invariance of $\widehat{\zeta}(\underline{\mathbf{q}})$ do not allow a geometric interpretetion of the cdn^{-1} and they question its adequacy as a performance index for *SGP*s.

The conslusion of this considerations is, that we have to look for a new objective function $\zeta(\underline{\mathbf{q}})$ which meets our initially stated demands. But we want to add a further argument, which has the following motivation: The cdn^{-1} as well as the *manipulability* are also used to optimize the design of *SGP*s. But these two indices do not depend on the choice of $\mathbf{B}_i$ and $\mathbf{P}_i$ on $\mathcal{L}_i$ as long as $\mathbf{B}_i \neq \mathbf{P}_i$. Thus we require:

7. $D(\mathcal{K})$ depends on the geometry of the *SGP*, not only on the carrier lines $\mathcal{L}_1$,...,$\mathcal{L}_6$ of the six legs.

Pottmann et al., 1998 also presented a modified version of his method, namely the *line segment method*, which statisfies the 7^{th} demand but does not eliminate the other weak points. The *rigidity rate* is independent of the choice of the base anchor points and so it only takes the geometry of the platform into consideration. This raises the following problem: If we change the viewpoint and consider Σ as the unmoved base and Σ_0 as platform, we get another index for the same *SGP* configuration. So the instantaneous *rigidity* of the *SGP* depends on the viewpoint which is dissatisfying.

2.1 Uncontrollable postures of *SGP*s

In practice configurations must be avoided, where minor variations of the leg lengths have uncontrollable large effects on the instantaneous displacement of the platform Σ. But how should the quantity of effects be measured in relation to the variation of the leg lengths? The boarder case of this uncontrollability is, if there exists an infinitesimal motion of Σ while all actuators are locked. In such a singular position the velocities of the platform points can be arbitrarily large, and therefore the posture is uncontrollable. The question is, which measurable parameter of the *SGP* indicates the circumstance of uncontrollability in a natural way and has a geometric meaning for the manipulator.

3. Idea and definition of the control number *ctn*

Let's assume there is instantaneously a minor variation of the six leg lenghts and the *SGP* is not singular. So there exists a unique screw $\underline{\mathbf{q}}$ which describes the motion of Σ against Σ_0 according to (4). To meet our 7^{th} property, we consider the velocity $\mathbf{v}(\mathbf{P}_i)$ of $\mathbf{P}_i$ with respect to $\underline{\mathbf{q}}$. We are not interested in the instantaneous displacements of $\mathbf{P}_i$ in direction of the leg, because the leg length is an active joint which can be controlled totally. Therefore only the component $\mathbf{v}_{\perp}(\mathbf{P}_i)$ can be an indicator of uncontrollability. But $\mathbf{v}_{\perp}(\mathbf{P}_i)$ is no mechanical parameter of a *SGP* and therefore we look at the angular velociety $\omega_{\mathcal{B}_i}$ of the i^{th} passive base joint. $\omega_{\mathcal{B}_i}$ is defined as (see *Fig. 1*)

$$\omega_{\mathcal{B}_i} := \frac{\|\mathbf{v}_{\perp}(\mathbf{P}_i)\|}{\|\mathbf{l}_i\|} \quad \Rightarrow \quad {\omega_{\mathcal{B}_i}}^2 = \frac{\|\mathbf{v}_{\perp}(\mathbf{P}_i)\|^2}{\|\mathbf{l}_i\|^2} = \frac{\|\mathbf{v}(\mathbf{P}_i)\|^2 - d_i^2}{\|\mathbf{l}_i\|^2} \tag{7}$$

according to (2) and (3) and so it is proportional to $\|\mathbf{v}_{\perp}(\mathbf{P}_i)\|$. But there also exists angular velocities $\omega_{\mathcal{P}_i}$ in the passive platform joints, which are defined analogously. The sole exeption is that we regard the inverse motion of $\underline{\mathbf{q}}$. So we have to substitute $\mathbf{B}_i$ for $\mathbf{P}_i$ and $-\underline{\mathbf{q}}$ for $\underline{\mathbf{q}}$ in (2), (3) and (7). Obviously ${\omega_{\mathcal{B}_i}}^2$ and ${\omega_{\mathcal{P}_i}}^2$ are quadratic forms with the coordinates of $\underline{\mathbf{q}}$ as unkowns. Therefore we can rewrite them as

$${\omega_{\mathcal{B}_i}}^2 = \underline{\mathbf{q}}^T \mathcal{W}_{\mathcal{B}_i} \underline{\mathbf{q}} \quad \text{and} \quad {\omega_{\mathcal{P}_i}}^2 = \underline{\mathbf{q}}^T \mathcal{W}_{\mathcal{P}_i} \underline{\mathbf{q}}, \tag{8}$$

where $\mathcal{W}_{\mathcal{B}_i}$ and $\mathcal{W}_{\mathcal{P}_i}$ are symmetric 6×6 matrices.

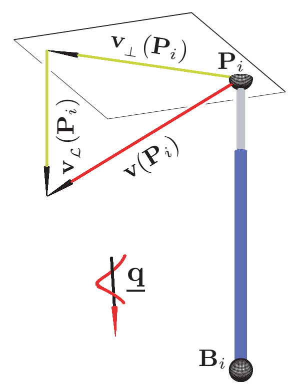

Figure 1. Defining $\omega_{\mathcal{B}_i}$

Now we define the new objective function $\zeta(\underline{\mathbf{q}})$ as

$$\zeta(\underline{\mathbf{q}}) = \sum_{i=1}^{6} {\omega_{\mathcal{B}_i}}^2 + {\omega_{\mathcal{P}_i}}^2 = \underline{\mathbf{q}}^T \mathcal{Z}\, \underline{\mathbf{q}} \quad \text{with} \quad \mathcal{Z} = \sum_{i=1}^{6} \mathcal{W}_{\mathcal{B}_i} + \mathcal{W}_{\mathcal{P}_i}. \tag{9}$$

Definition 1. *The control number of a SGP configuration $\mathcal{K}$ is defined as*

$$ctn(\mathcal{K}) := +\sqrt{\lambda_{-}/\lambda_{+}} \quad \text{with} \quad ctn(\mathcal{K}) \in [0,1]\,, \tag{10}$$

where λ_{-} resp. λ_{+} is the minimum resp. maximum of the objective function $\zeta(\underline{\mathbf{q}})$ in (9) *under the side condition $\nu(\underline{\mathbf{q}})$ in* (6)*. $ctn(\mathcal{K}) = 0$ characterizes a singular configuration and a value of* 1 *an optimal one.*

4. Computation and well-definedness of *ctn*

We solve the optimization problem in order to compute λ_{-} resp. λ_{+} by introducing a Lagrange multiplier λ. Then the approach simplifies in consideration of $\nabla\zeta = 2\,\mathcal{Z}\,\underline{\mathbf{q}}$ and $\nabla\nu = 2\mathcal{N}\,\underline{\mathbf{q}}$, to the general eigenvalue problem $(\mathcal{Z} - \lambda\mathcal{N})\,\underline{\mathbf{q}} = \mathbf{o}$. This system of linear equations has a nontrivial solution, if and only if $|\mathcal{Z} - \lambda\mathcal{N}| = 0$. The degree of the characteristic polynomial in λ corresponds with $rank(\mathcal{J})$ because of $\mathcal{N} = \mathcal{J}^T\mathcal{J}$. Every general eigenvalue λ_i is linked with an general eigenvector $\underline{\mathbf{e}}_i$. The smallest λ_{-} and the largest λ_{+} are the wanted extreme magnitudes because of

$$\mathcal{Z}\,\underline{\mathbf{e}}_i = \lambda_i \mathcal{N}\,\underline{\mathbf{e}}_i \quad \text{and} \quad \underline{\mathbf{e}}_i^T \mathcal{N}\,\underline{\mathbf{e}}_i = 1 \quad \Rightarrow \quad \zeta(\underline{\mathbf{e}}_i) = \lambda_i. \tag{11}$$

Theorem 1. *λ_{-} and λ_{+} of* Def. *1 are the extreme general eingenvalues of $\mathcal{Z}$ with respect to $\mathcal{N}$. All roots λ_i of the characteristic polynomial $|\mathcal{Z} - \lambda\mathcal{N}| = 0$ are positive if and only if $rank(\mathcal{J}) = 6$.*

Proof: According to Hestenes, 1975 all λ_i's are real. Due to (11) all λ_i's are nonnegative. If $\underline{\mathbf{q}}$ is no translation, then all angular velocities in the passive joints would vanish if and only if the 12 anchor point lie on the *isa*. But such a configuration yields $rank(\mathcal{J}) = 1$. In the case of a pure translation, there would be no angular velocities in the passive joints if and only if the legs are parallel to the direction of the translation. But such a configuration yields $rank(\mathcal{J}) \leq 3$. □

Theorem 2. *The number of roots λ_i of the characteristic polynomial $|\mathcal{Z} - \lambda\mathcal{N}| = 0$ dropping to infinity equals the defect$(\mathcal{J})$.*

Proof: All screws $\pm\mu\underline{\mathbf{q}} \in ker_\iota$ with $\mu \in \mathbb{R}$ cause arbitrarily large velocities $\mathbf{v}(\mathbf{P}_i) = \mathbf{v}_{\perp}(\mathbf{P}_i)$ resp. $\mathbf{v}(\mathbf{B}_i) = \mathbf{v}_{\perp}(\mathbf{B}_i)$ and therefore arbitrarily large $\omega_{\mathcal{B}_i}$ resp. $\omega_{\mathcal{P}_i}$. The proof follows by carring out $lim_{\mu\to\infty}$ and (11). □

Due to *Theorem 1* and *2* the *control number* is well defined. Therefore all initially stated seven properties are obviously fulfilled.

Remark. It does not make sense to define $\zeta(\underline{\mathbf{q}})$ only as $\sum \omega_{\mathcal{B}_i}{}^2$ (resp. $\sum \omega_{\mathcal{P}_i}{}^2$) for following reasons: First, the index would not fulfill our 7^{th} demand for the same reason as the *rigidity rate*. Second, the index would not fulfill our 2^{nd} demand, because there exist nonsingular *SGP* configurations, where the $\mathcal{L}_i$'s are the path tangents of $\mathbf{P}_i$ (resp. $\mathbf{B}_i$) with regard to $\underline{\mathbf{q}}$. Consequently we get $\zeta(\underline{\mathbf{q}}) = 0$ and the index would equal 0.

4.1 Instantaneous motion near singularities

According to Wolf and Shoham, 2003 the closest path normal complex of a helical motion (rotations and translations included) to $\mathcal{L}_1, .., \mathcal{L}_6$, described by its axis and pitch, provides additional information on the *SGP*'s instantaneous motion and understanding of the type of singularity when the *SGP* is at, or in the neighborhood of, a singular configuration. Since the *ctn* is a performance index as well as a distance measure, a small *ctn* indicates the closeness to a singularity. Due to *Theorem 2* and the continuity of the polynomial functions $|\mathcal{Z} - \lambda\mathcal{N}| = 0$, which arise if we move towards a singular position, we can say that the closest linear complex to $\mathcal{L}_1, .., \mathcal{L}_6$ equals the path normal complex of $\underline{\mathbf{e}}_+$ according to (11). Therefore this method additionally brings about a kind of best approximating linear line complex in the neighbourhood of singularities, and the calculation needs no case analysis like Pottmann's method.

5. Final example

We consider a two parametric set $\mathcal{S}_\mathcal{K}$ of configurations $\mathcal{K}$, given by

$$\mathbf{B}_i = (\cos\alpha_i, \sin\alpha_i, -h)^T \qquad \text{and} \qquad \mathbf{P}_i = (\cos\beta_i, \sin\beta_i, h)^T \qquad \text{with}$$

$$\alpha_1 = \beta_2 - \frac{\pi}{3} = -\alpha \qquad \alpha_3 = \beta_4 - \frac{\pi}{3} = \frac{2\pi}{3} - \alpha \qquad \alpha_5 = \beta_4 - \frac{\pi}{3} = \frac{4\pi}{3} - \alpha$$

$$\alpha_2 = \beta_1 + \frac{\pi}{3} = \alpha \qquad \alpha_4 = \beta_3 + \frac{\pi}{3} = \frac{2\pi}{3} + \alpha \qquad \alpha_6 = \beta_5 + \frac{\pi}{3} = \frac{4\pi}{3} + \alpha$$

where $\alpha \in [0, \frac{\pi}{6}]$ denotes the *design parameter* and $h \in \mathbb{R}^+$ the *posture parameter* of the *SGP*. All $\mathcal{K} \in \mathcal{S}_\mathcal{K}$ with $\alpha \neq \frac{\pi}{6}$ and $h \notin \{0, \infty\}$ are nonsingular. We study this example, because such manipulators are very relevant in practice as flight simulators. The matrix $\mathcal{Z} - \lambda\mathcal{N}$ can be manipulated by elementary row and column operations to the diagonal matrix $diag(\Delta_1, .., \Delta_6)$. Therefore the eigenvalues λ_i can be computed explicitly using $\Delta_i = 0$, whereas $\lambda_1 = \lambda_2$ and $\lambda_4 = \lambda_5$. $\mathcal{K}_+$ given by

$$h_+ = \frac{\gamma}{4} \approx 0.4, \quad \alpha_+ = -\arctan\left(\frac{\sqrt{5}\gamma - \sqrt{15}}{5}\right) \approx 4^\circ, \quad \gamma = \sqrt{2\sqrt{5} - 2} \qquad (12)$$

has the maximal *ctn* of all $\mathcal{K} \in \mathcal{S}_\mathcal{K}$ (see *Fig. 2* and *3*). For $\mathcal{K}_+$ determined by $\lambda_{1,2} = \lambda_{4,5}$ and $\lambda_3 = \lambda_6$ we get $ctn(\mathcal{K}_+) = \sqrt{2\sqrt{5} - 4} \approx 0.687$.

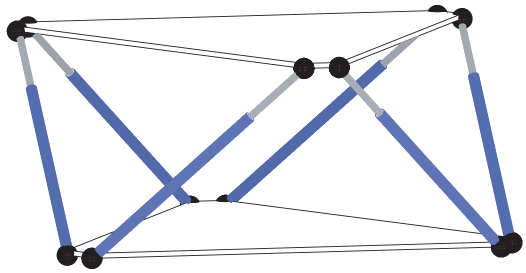

Figure 2. Axonometry of $\mathcal{K}_+$

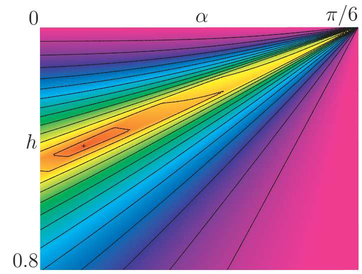

Figure 3. Contours of $ctn(\mathcal{S}_\mathcal{K})$

The *SGP* with α_+ also makes sense from the practical point of view, because contrary to the often propagandized 3-3 octahedral manipulator ($\alpha = 0$) no anchor points coincide. But coinciding anchor points are hard to manufacture. Therefore we take a closer look to this *SGP*. *Fig. 5* illustrates the graph of *ctn* depending on h. *Fig. 4* shows the contour lines of *ctn* when the platform is translated away from the central location parallel to the base plane. The difference between two neighbouring contour lines is 0.05, where the highest has the value of 0.65. *Fig. 6, 7* and *8* illustrates the graphs of *ctn* dependig on the angle of the rotation of Σ about an axis parallel to x, z or y, respectively, through $(0, 0, h_+)$.

Remark. The *rigidity rate* of all nonsingular configurations of this set $\mathcal{S}_\mathcal{K}$ is constant at the maximal value of $\pi/2$. Only in singular positions it drops to zero. So if we approach a singularity of $\mathcal{S}_\mathcal{K}$ the value of the *rigidity rate* is constant $\pi/2$. Therefore this index is not recommendable for comparing different postures of different *SGP*s.

6. Conclusion

The presented index, called *control number* (*ctn*), allows to compare different postures of different *SGP*s, because it obeys the initially stated seven conditions. Therefore *ctn* can serve as a performance index as well as a distance measure to the closest singularity. This concept can also be modified for redundant *SGP*s and 3 dof RPR manipulators.

An article about optimal configurations $\mathcal{K}$ with $ctn(\mathcal{K}) = 1$ is in preparation. It can be proved, that such configurations do exist. New performance indices for 6R robots have been presented in Nawratil, 2006.

Acknowledgment

This is part of the author's PhD-thesis. The author expresses his sincere thanks to his supervisor Prof. H. Stachel for continuous support.

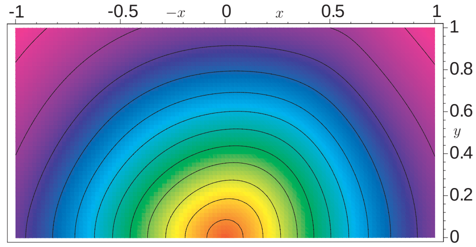

Figure 4. Translation of Σ in $z = h_+$

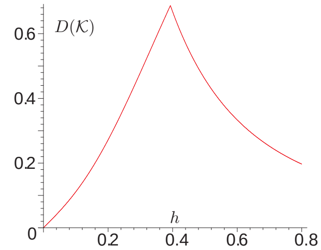

Figure 5. Variation of h

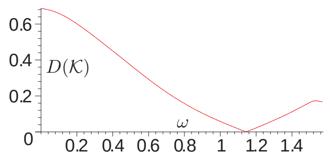

Figure 6. Rotation about a x-parallel

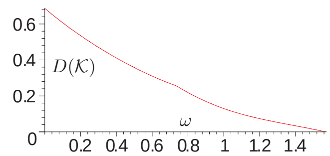

Figure 7. Rotation about a z-parallel

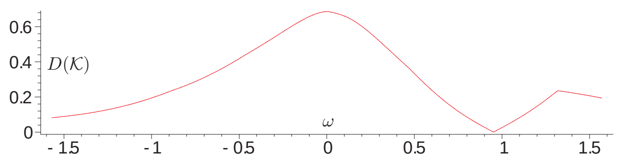

Figure 8. Rotation about a y-parallel

References

Hestenes, M.R. (1975), *Optimization theory*, New York, A Wiley publication.

Lang, J., Mick, S., and Röschel, O. (2001), The Rigidity Rate of Positions of Stewart-Gough Platforms, *Journal for Geometry and Graphics*, vol. 5, no. 2, pp. 121–132.

Lee, J., Duffy, J., and Hunt, H. (1998), A Practical Quality Index Based on the Octahedral Manipulator, *Int. Journal of Robotics Research*, vol. 17, no. 10, pp. 1081–1090.

Merlet, J.-P. (1992), Singular Configurations of Parallel Manipulators and Grassmann Geometry, *Int. Journal of Robotics Research*, vol. 8, no. 5, pp. 45–56.

Nawratil, G. (2006), New Performance Indices for 6R Robot Postures, *Proceedings of the 1st EuCoMeS*, Obergurgl February 21-26, Austria, accepted for publication.

Pottmann, H., Peternell, M., and Ravani, B. (1998), Approximation in line space: applications in robot kinematics and surface reconstruction, *Advances in Robot Kinematics: Analysis & Control* (J. Lenarcic, M. Husty, eds.), pp. 403–412, Kluwer.

Salisbury, J.K. and Craig, J.J. (1982), Articulated Hands: Force Control and Kinematic Issues, *Int. Journal of Robotics Research*, vol. 1, no. 1, pp. 4–17.

Wolf, A., and Shoham, M. (2003), Investigations of Parallel Manipulators Using Linear Complex Approximation, *Journal of Mechanical Design*, vol. 125, pp. 564–572.

Yoshikawa, T. (1985), Manipulability of Robotic Mechanisms, *Int. Journal of Robotics Research*, vol. 4, no. 2, pp. 3–9.