Reducible compositions of spherical four-bar linkages with a spherical coupler component

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Abstract

We use the output angle of a spherical four-bar linkage $C$ as the input angle of a second four-bar linkage $D$ where the two frame links are assumed in aligned position as well as the follower of $C$ and the input link of $D$. We determine all cases where the relation between the input angle of the input link of $C$ and the output angle of the follower of $D$ is reducible and where additionally at least one of these components produces a transmission which equals that of a single spherical coupler.

Key words: spherical four-bar linkage, flexible octahedra, Kokotsakis meshes

1 Introduction

Let a spherical four-bar linkage $C$ be given by the quadrangle $I_{10}A_1B_1I_{20}$ (see Fig. 1) with the frame link $I_{10}I_{20}$, the coupler $A_1B_1$ and the driving arm $I_{10}A_1$. We use the output angle $\varphi_2$ of this linkage as the input angle of a second coupler motion $D$ with vertices $I_{20}A_2B_2I_{30}$. The two frame links are assumed in aligned position as well as the driven arm $I_{20}B_1$ of $C$ and the driving arm $I_{20}A_2$ of $D$.

We want to determine all cases where the relation between the input angle $\varphi_1$ of the arm $I_{10}A_1$ and the output angle $\varphi_3$ of $I_{30}B_2$ is reducible and where additionally at least one of these components produces a transmission which equals that of a single spherical coupler. Therefore we are looking for all reducible compositions with a so-called spherical coupler component.
Fig. 1. Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, 2$ (Courtesy of H. Stachel).

The problem under consideration is of importance for the classification of flexible Kokotsakis meshes [1–3], i.e., the compounds of $3 \times 3$ planar quadrangular plates with hinges between neighboring plates. This results from the fact that the spherical image of a flexible mesh consists of two compositions of spherical four-bars sharing the transmission $\varphi_1 \mapsto \varphi_3$ (see Fig. 1). All the examples known up to recent [3] are based on reducible compositions.

The reducible compositions with a spherical coupler component are of special interest because based on their knowledge one can additionally determine all flexible octahedra in the projective extension of the Euclidean 3-space. Based on this article a proof for this open problem is in preparation (cf. [4]).

1.1 Transmission by a spherical four-bar linkage

We start with the analysis of the first spherical four-bar linkage $C$ with the frame link $I_{10}I_{20}$ and the coupler $A_1B_1$ (Fig. 1). We set $\alpha_1 := I_{10}A_1$ for the spherical length of the driving arm, $\beta_1 := I_{20}B_1$ for the output arm, $\gamma_1 := A_1B_1$, and $\delta_1 := I_{10}I_{20}$. We may suppose

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi .$$

The movement of the coupler remains unchanged when $A_1$ is replaced by its antipode $A_1$ and at the same time $\alpha_1$ and $\gamma_1$ are substituted by $\pi - \alpha_1$ and $\pi - \gamma_1$, respectively. The same holds for the other vertices. When $I_{10}$ is replaced by its antipode $I_{10}$, then also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of $S^2$ either at $I_{10}$ or at $I_{10}$. 


We use a cartesian coordinate frame with \( I_{10} \) on the positive \( x \)-axis and \( I_{10}I_{20} \) in the \( xy \)-plane such that \( I_{20} \) has a positive \( y \)-coordinate (see Fig. 1). The input angle \( \varphi_1 \) is measured between \( I_{10}I_{20} \) and the driving arm \( I_{10}A_1 \) in mathematically positive sense. The output angle \( \varphi_2 = \angle I_{10}I_{20}B_1 \) is the oriented exterior angle at vertex \( I_{20} \). This results in the following coordinates:

\[
A_1 = \begin{pmatrix} c\alpha_1 \\ s\alpha_1 c\varphi_1 \\ s\alpha_1 s\varphi_1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} c\beta_1 c\delta_1 - s\beta_1 s\delta_1 c\varphi_2 \\ c\beta_1 s\delta_1 + s\beta_1 c\delta_1 c\varphi_2 \\ s\beta_1 s\varphi_2 \end{pmatrix}.
\]

Herein \( s \) and \( c \) are abbreviations for the sine and cosine function, respectively. In these equations the spherical lengths \( \alpha_1, \beta_1 \) and \( \delta_1 \) are signed. The coordinates would also be valid for negative spherical lengths. The constant spherical length \( \gamma_1 \) of the coupler implies

\[
\begin{align*}
&c\alpha_1 c\beta_1 c\delta_1 - c\alpha_1 s\beta_1 s\delta_1 c\varphi_2 + s\alpha_1 c\beta_1 s\delta_1 c\varphi_1 \\
&+ s\alpha_1 s\beta_1 c\varphi_1 c\varphi_2 + s\alpha_1 s\beta_1 s\varphi_1 s\varphi_2 = c\gamma_1.
\end{align*}
\]

In comparison to [5] we emphasize algebraic aspects of this transmission. Hence we express \( s\varphi_i \) and \( c\varphi_i \) in terms of \( t_i := \tan(\varphi_i/2) \) since \( t_1 \) is a projective coordinate of point \( A_1 \) on the circle \( a_1 \). The same is true for \( t_2 \) and \( B_1 \in b_1 \). From (1) we obtain

\[
-K_1(1 + t_1^2)(1 - t_2^2) + L_1(1 - t_1^2)(1 + t_2^2) + M_1(1 - t_1^2)(1 - t_2^2) \\
+ 4 s\alpha_1 s\beta_1 t_1 t_2 + N_1(1 + t_1^2)(1 + t_2^2) = 0 \quad \text{with}
\]

\[
K_1 = c\alpha_1 s\beta_1 s\delta_1, \quad M_1 = s\alpha_1 s\beta_1 c\delta_1, \\
L_1 = s\alpha_1 c\beta_1 s\delta_1, \quad N_1 = c\alpha_1 c\beta_1 c\delta_1 - c\gamma_1.
\]

This biquadratic equation describes a 2-2-correspondence between points \( A_1 \) on circle \( a_1 = (I_{10}; \alpha_1) \) and \( B_1 \) on \( b_1 = (I_{20}; \beta_1) \). It can be abbreviated by

\[
c_{22} t_1^2 t_2^2 + c_{20} t_1^2 + c_{02} t_2^2 + c_{11} t_1 t_2 + c_{00} = 0
\]

setting

\[
c_{22} = K_1 - L_1 + M_1 + N_1, \quad c_{11} = 4 s\alpha_1 s\beta_1, \quad c_{02} = K_1 + L_1 - M_1 + N_1, \\
c_{20} = -K_1 - L_1 - M_1 + N_1, \quad c_{00} = -K_1 + L_1 + M_1 + N_1.
\]

under \( c_{11} \neq 0 \). Alternative expressions can be found in [3].

Remark: Note that the 2-2-correspondence (2) depends only on the ratio of the coefficients \( c_{22} : \cdots : c_{00} \) (cf. Lemma 1 of [6]).
1.2 Composition of two spherical four-bar linkages

Now we use the output angle $\phi_2$ of the first four-bar linkage $C$ as input angle of a second four-bar linkage $D$ with vertices $I_{20}A_2B_2I_{30}$ and consecutive spherical side lengths $\alpha_2, \gamma_2, \beta_2,$ and $\delta_2$ (Fig. 1). The two frame links are assumed in aligned position. In the case $\angle I_{10}I_{20}I_{30} = \pi$ the spherical length $\delta_2$ is positive, otherwise negative. Analogously, a negative $\alpha_2$ expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of $\beta_2$ means replacing the output angle $\phi_3$ by $\phi_3 - \pi$. The sign of $\gamma_2$ has no influence on the transmission.

Due to (2) the transmission between the angles $\varphi_1, \varphi_2$ and the output angle $\varphi_3$ of the second four-bar with $t_3 := \tan(\varphi_3/2)$ can be expressed by the two biquadratic equations

$$C := c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0,$$

$$D := d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0. \quad (4)$$

The $d_{ik}$ are defined by equations analogue to Eqs. (3) and (1.1). We eliminate $t_2$ by computing the resultant (cf. [7]) of the two polynomials with respect to $t_2$ and obtain

$$X := \det \begin{pmatrix}
  c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} & 0 \\
  0 & c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} \\
  d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00} & 0 \\
  0 & d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00}
\end{pmatrix} = 0. \quad (5)$$

This biquartic equation expresses a 4:4-correspondence between points $A_1$ and $B_2$ on the circles $a_1$ and $b_2,$ respectively (Fig. 1).

1.2.1 Known examples of reducible compositions with a spherical coupler component

Up to recent, to the author’s best knowledge the following examples are known. Under appropriate notation and orientation these are:

1. **Isogonal** type [1,2]: At each four-bar opposite sides are congruent; the transmission $\varphi_1 \to \varphi_3$ is the product of two projectivities and therefore again a projectivity. Each of the 4 possibilities can be obtained by one single four-bar linkage. This is the spherical image of a flexible octahedron of Type 3 (see [8]):

2. **Symmetric** type [3]: We specify the second four-bar linkage as mirror of the first one after reflection in an angle bisector at $I_{20}$ (see [3, Fig. 5b]).
Thus $\varphi_3$ is congruent to the angle opposite to $\varphi_1$ in the first quadrangle. Hence the 4-4-correspondence is reducible.

3. **Focal** type [6]: Any composition of two spherical four-bar linkages obeying the angle condition $\psi_1 = \pm \angle I_1 A_1 B_1 = \pm \angle T_{30} B_2 A_2$ (see [6, Fig. 3b]) is reducible. Each component equals the transmission $\varphi_1 \rightarrow \varphi_3$ of a single, but not necessarily real spherical four-bar linkage.

### 1.2.2 Computation of reducible compositions with a spherical coupler component

Given are the two spherical couplers $C$ and $D$ and their corresponding transmission equations $C$ and $D$, respectively (see Eq. (4)). In the following we are interested in the conditions the $c_{ij}$'s and $d_{ij}$'s have to fulfill such that $X$ of Eq. (5) splits up into the product $FG$ with:

- **Symmetric reducible composition:**
  \[
  F := f_{22}t_1^2t_3^2 + f_{20}t_1^2 + f_{02}t_3^2 + f_{11}t_1t_3 + f_{00},
  \]
  \[
  G := g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{11}t_1t_3 + g_{00}.
  \]  

As at least one of the two polynomials $F$ and $G$ should correspond to a spherical coupler component $f_{11}$ cannot vanish. Moreover we can stop the later done case study (see Section 2 and 3) if $f_{11} = g_{11} = 0$ hold.

- **First asymmetric reducible composition:**
  \[
  F := f_{11}t_1t_3 + f_{00},
  \]
  \[
  G := g_{33}t_1^3t_3^3 + g_{31}t_1^3t_3^1 + g_{13}t_1^3t_3^3 + g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{11}t_1t_3 + g_{00}.
  \]

As $F$ has to correspond with a spherical coupler component $f_{11}$ cannot vanish. Moreover we can stop the later done case study (see Section 5.1,5.2 and 5.3) if $g_{33} = g_{31} = g_{13} = 0$ hold, as this yields a special case of the symmetric composition.

- **Second asymmetric reducible composition:**
  \[
  F := f_{20}t_1^2 + f_{11}t_1t_3 + f_{00},
  \]
  \[
  G := g_{13}t_1^3 + g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{11}t_1t_3 + g_{00}.
  \]

Again $F$ has to correspond with a spherical coupler component which yields $f_{11} \neq 0$. Moreover we can stop the later done case study (see Section 6.1,6.2 and 6.3) if $g_{13} = 0$ (special case of the symmetric composition) or $f_{20} = 0$ (special case of the first asymmetric composition) holds.

It is not difficult to verify that these are all possible types of compositions with a spherical coupler component.
As we compute the resultant with respect to \( t_2 \) (cf. Eq. (5)) the coefficient of \( t_2^2 \) in \( C \) and \( D \) must not vanish. Therefore the two cases \( c_{22} = c_{02} = 0 \) and \( d_{22} = d_{20} = 0 \) are excluded. For the discussion of these excluded cases we refer to the Sections 4, 5.4 and 6.4, respectively.

In the following we denote the coefficients of \( t_i t_j \) of \( Y := FG \) and \( X \) by \( Y_{ij} \) and \( X_{ij} \), respectively. By the comparison of these coefficients we get the following 13 equations \( Q_{ij} := Y_{ij} - X_{ij} \) with 

\[
(i, j) \in \{(4, 4), (4, 2), (4, 0), (3, 3), (3, 1), (2, 4), (2, 2), (2, 0), (1, 3), (1, 1), (0, 4), (0, 2), (0, 0)\}
\]

which must be fulfilled. In the following we discuss the solution of this non-linear system of equations for the above given three possible compositions. We show how the resulting three polynomial systems can be solved explicitly by means of resultants. Note that this is a non-trivial task especially in the case of the symmetric reducible composition. In the later given elimination process we often use an principle which is explained here at hand of the following simple example (cf. footnote 4 of [9]):

Given are 3 quadratic equations \( Q_i, i = 1, 2, 3 \) in 3 variables \( x, y, z \) and one has to calculate the intersection points of these 3 quadrics. First we eliminate \( z \) by computing the resultant \( R_{ij} \) of \( Q_i \) and \( Q_j \) with respect to \( z \). Now \( R_{ij} \) is a quartic equation in \( x, y \). Computing again the resultant of e.g. \( R_{12} \) and \( R_{13} \) with respect to \( y \) yields an univariate polynomial of degree 16.

But not all roots of this polynomial are solutions of the intersection problem as it can only have 8 over \( \mathbb{C} \) due to Bezout’s Theorem. To get rid of the 8 pseudo-solutions one can compute more equations as actually necessary, i.e. the resultants of \( R_{12} \) and \( R_{23} \) as well as \( R_{13} \) and \( R_{23} \) with respect to \( y \). Now the greatest common divisor (gcd) of these 3 polynomials of degree 16 yields in general the solution-polynomial of degree 8.

The 8 pseudo-solutions stem from the geometric fact that the elimination of the variable \( z \) is geometrically equivalent of projecting the intersection curve of the quadrics \( Q_i \) and \( Q_j \) into the \( xy \)-parameter plane. Now 8 intersection points of the two projected intersection curves \( R_{12} \) and \( R_{13} \) correspond with different points which lie above each other on one projection ray.

2 Symmetric reducible composition with \( f_{20}g_{02} - f_{02}g_{20} \neq 0 \)

Under this assumption we can compute \( f_{22} \) and \( g_{22} \) from the equations \( Q_{42} \) and \( Q_{24} \), which are both linear in \( f_{22} \) and \( g_{22} \). Moreover we can also express \( f_{00} \) and \( g_{00} \) from the equations \( Q_{20} \) and \( Q_{02} \) (both linear in \( f_{00} \) and \( g_{00} \)) and also \( f_{11} \) and \( g_{11} \) from the equations \( Q_{31} \) and \( Q_{13} \) (both linear in \( f_{11} \) and \( g_{11} \)).
2.1 \( g_{20}g_{02} \neq 0 \)

Under this assumption we can express \( f_{20} \) from \( Q_{40} \) (linear in \( f_{20} \)) and \( f_{02} \) from \( Q_{04} \) (linear in \( f_{02} \)). Now we are left with 5 equations: \( Q_i \) with \( i = 0, \ldots, 4 \). In the following we denote \( Q_i \) by \( Q_i \) only:

\[
Q_0[181], \quad Q_1[94], \quad Q_2[311], \quad Q_3[94], \quad Q_4[181],
\]

where the number in the square brackets gives the number of terms of the numerator. Moreover it should be noted that we can factor out \( c_{11}d_{11} \neq 0 \) from \( Q_1 \) and \( Q_3 \).

In the following we compute the resultant of \( Q_i \) and \( Q_j \) with respect to \( g_{20} \), which is denoted by \( R_{ij} \). In the next step we compute the resultant of \( R_{13} \) and \( R_{ij} \) with respect to \( d_{11} \), which is denoted by \( T_{ij} \). Now it can easily be seen that the six expressions

\[
T_{34}, \quad T_{01}, \quad T_{14}, \quad T_{03}, \quad T_{23}, \quad T_{12},
\]

have the factors \( g_{02}c_{11}W_1W_2W_3W_4W_5W_6 \) in common with

\[
W_1 := c_{02}c_{22}d_{00}d_{02} - c_{00}c_{20}d_{20}d_{22}, \quad W_2 := c_{00}c_{22}d_{00}d_{22} - c_{20}c_{02}d_{20}d_{02}, \\
W_3 := d_{00}d_{22} - d_{20}d_{02}, \quad W_4 := d_{00}c_{22} - c_{20}d_{20}, \quad W_5 := d_{02}c_{02} - c_{00}d_{22}, \\
W_6 := c_{11}^4 - 8c_{11}^2(c_{00}c_{22} + c_{20}c_{02}) + 16(c_{00}c_{22} - c_{20}c_{02})^2.
\]

It should be noted that \( W_4 = 0 \) implies \( f_{20} = 0 \) and that \( W_5 = 0 \) implies \( f_{02} = 0 \). Moreover \( W_3 = 0 \) implies that \( D \) is an orthogonal spherical four-bar mechanism.\(^1\) The condition \( W_6 = 0 \) can be rewritten in terms of \( \alpha_1, \beta_1, \gamma_1 \) and \( \delta_1 \) as:

\[
W_6 = 256 \sin(\alpha_1)^2 \sin(\beta_1)^2 \sin(\gamma_1)^2 \sin(\delta_1)^2.
\]

Therefore this factor cannot vanish, as at least the spherical length of one spherical bar equals 0 (or \( \pi \)). Clearly, also the corresponding factor which is obtained by substituting \( c_{ij} \) by \( d_{ij} \) in \( W_6 \) cannot vanish without contradiction (w.c.). Therefore we can assume for the rest of this article that these two factors are different from zero.

**Lemma 1** Under the assumption \( g_{20}g_{02} \neq 0 \) there does not exist a symmetric reducible composition with a spherical coupler component.

**Proof:** We factor \( g_{02}c_{11}W_1W_2W_3W_4W_5W_6 \) out from \( T_{ij} \) and denote the resulting polynomial by \( L_{ij} \). Now the polynomials \( L_{34}, L_{01}, L_{14} \) and \( L_{03} \) have still the

\(^1\) The diagonals of the spherical quadrangle \( I_20A_2B_2I_30 \) are orthogonal (cf. [3,6]).
factor $H := 4W_1W_3W_7 + c_{11}^2W_2$ with

$$W_7 := c_{22}c_{00} - c_{20}c_{02}$$

in common. We distinguish two cases:

I. $H = 0$: In this case the greatest common divisor of $L_{12}$ and $L_{23}$ can only vanish w.c. for $c_{00}c_{02}c_{20}c_{22}d_{00}d_{02}d_{20}d_{22}M[14]W_7$. As $W_7 = 0$ implies together with $H = 0$ that $W_2 = 0$ must hold (a contradiction) we can assume $W_7 \neq 0$. Then the resultant of $M$ and $H$ with respect to $c_{11}$ cannot vanish w.c..

Therefore we are only left with the possibility $c_{ij}d_{ij} = 0$.

II. For the case $H \neq 0$ we proceed as follows: Beside $T_{34}$ and $T_{14}$ we also compute the resultant of $R_{14}$ and $R_{34}$ with respect to $d_{11}$, which is denoted by $T_{13}$. The common factors of $T_{34}$, $T_{14}$ and $T_{13}$ are given by:

$$c_{22}c_{20}d_{22}d_{02}g_{02}c_{11}W_1W_3W_4W_5W_6H.$$ 

Alternatively the same procedure can be done by denoting the resultant of $R_{01}$ and $R_{03}$ with respect to $d_{11}$ by $T_{13}$. The common factors of $T_{01}$, $T_{03}$ and $T_{13}$ are given by:

$$c_{00}c_{02}d_{00}d_{20}g_{02}c_{11}W_1W_3W_4W_5W_6H.$$ 

Due to I and II there can only be a reducible composition if $c_{ij}d_{ij} = 0$ holds. In the following we show that these cases also yield contradictions:

1. $c_{20}c_{22}d_{02}d_{22} = 0$: In all 4 cases $R_{24}$ yields the contradiction.
2. $c_{00}c_{02}d_{20}d_{00} = 0$: In all 4 cases $R_{02}$ yields the contradiction. \qed

We can even prove a stronger statement:

**Lemma 2** For the case $g_{20}g_{02}(f_{20}g_{02} - f_{02}g_{20})W_4W_5 \neq 0$ the condition $W_1 = 0$ is a necessary condition for a symmetric reducible composition with a spherical coupler component.

*Proof:* Due to Lemma 1 we have to show that $W_2 = 0$ and $W_3 = 0$ does not yield a solution for $W_1 \neq 0$. We start by a rough discussion of the cases $W_2 = 0$ and $W_3 = 0$ and then we go into detail:

- $W_2 = 0$: Firstly we discuss the special cases. $W_2 = 0$ holds in the following 6 cases (without contradicting $W_4W_5 \neq 0$) if two variables out of $\{c_{ij}, d_{ij}\}$ are equal to zero:

$$c_{00} = c_{20} = 0, \quad d_{00} = d_{02} = 0, \quad (7)$$

$$c_{00} = d_{20} = 0, \quad d_{02} = c_{22} = 0, \quad d_{00} = c_{02} = 0, \quad d_{22} = c_{20} = 0. \quad (8)$$

It is very easy to see that the $R_{ij}$ cannot vanish for both cases of Eq. (7). For the other cases we get:
We proceed with the detailed discussion of the open cases:

i. \( c_{00} = d_{20} = 0 \) or \( d_{00} = c_{02} = 0 \): In both cases \( Q_0 \) and \( Q_1 \) cannot vanish w.c.

ii. \( d_{00} = c_{22} = 0 \) or \( d_{22} = c_{20} = 0 \): In both cases \( Q_3 \) and \( Q_4 \) cannot vanish w.c.

We proceed with the general case. Due to the done discussion of the special cases we can set \( d_{02} := Ac_{02}c_{22}d_{22} \) and \( d_{00} := Ac_{02}c_{20}d_{20} \) with \( A \in \mathbb{R} \setminus \{0\} \) without loss of generality (w.l.o.g.). Now all \( R_{ij} \) contain the factor \( W_8 \) with

\[
W_8 := Ad_{22}d_{20}c_{11}^2 - d_{11}^2 = 0.
\]

For the case \( W_8 \neq 0 \) we consider the polynomials \( R_{ij} \), \( R_{ik} \), \( R_{jk} \). Then we compute all three possible resultants of these polynomials with respect to \( d_{11} \) and calculate the greatest common divisor \( \gcd \) (for \( i, j, k \in \{0, 1, 3, 4\} \) and \( i, j, k \) pairwise distinct).

The polynomials \( \gcd_{014} \) and \( \gcd_{034} \) have the following factors in common:

\[
(AC_{02}c_{22} - 1)(AC_{02}c_{22} + 1)W_3W_6
\]

where \((AC_{02}c_{22} - 1) = 0\) yields \( W_4W_5 = 0\), a contradiction. The vanishing of the remaining factor \( AC_{02}c_{22} + 1 = 0 \) yields \( W_1 = 0 \), a contradiction.

These are all solutions because beside \((AC_{02}c_{22} - 1)(AC_{02}c_{22} + 1)W_3W_6 = 0\) the expressions \( \gcd_{014} \) and \( \gcd_{034} \) can only vanish for \( c_{02} = c_{22} = 0 \), a contradiction. Therefore in the case \( W_2 = 0 \) one of the factors \( W_3 \) or \( W_8 \) must vanish.

- \( W_3 = 0 \): Now \( d_{00}d_{22} - d_{20}d_{02} = 0 \) must hold and \( D \) is an orthogonal coupler. For the discussion we can set \( d_{00} := Ad_{02} \) and \( d_{20} := Ad_{22} \) with \( A \in \mathbb{R} \setminus \{0\} \) w.l.o.g.. After factoring out \( g_{02}d_{11}W_4W_5 \) from \( R_{14}, R_{34} \) and \( R_{13} \) we can compute \( T_{34}, T_{14} \) and \( T_{13} \) with

\[
\gcd(T_{13}, T_{14}, T_{34}) := c_{20}c_{22}W_1W_2W_6.
\]

The alternatively way of computation yields

\[
\gcd(T_{01}, T_{03}, T_{13}) := c_{00}c_{02}W_1W_2W_6.
\]

As all possibilities of \( c_{20}c_{22} = 0 \) and \( c_{00}c_{02} = 0 \) imply \( W_1 = 0 \) or \( W_2 = 0 \), the factor \( W_2 \) must vanish.

We proceed with the detailed discussion of the open cases:

- \( W_2 = W_3 = 0 \): Due to \( W_3 = 0 \) we set \( d_{00} := Ad_{02} \) and \( d_{20} := Ad_{22} \) with \( A \in \mathbb{R} \setminus \{0\} \). Then \( W_2 = 0 \) splits up into \( Ad_{02}d_{22}W_7 \) and therefore \( W_7 = 0 \) must hold (\( \Rightarrow C \) and \( D \) are orthogonal). As a consequence we set \( c_{00} := Bc_{02} \) and \( c_{20} := Bc_{22} \) with \( B \in \mathbb{R} \setminus \{0\} \). Note that we can assume \( c_{02}c_{22}d_{02}d_{22} \neq 0 \) due to Eq. (7) and (8). Now all \( R_{ij} \) with \( i, j \in \{0, 1, 3, 4\} \) and \( i \neq j \) equal

\[
g_{02}d_{02}d_{22}ABW_4W_5c_{11}d_{11}(Bd_{22} + d_{02})W_9W_{10}
\]
with

\[ W_9 := Ad_{02}d_{22}c_{11}^2 - Bc_{02}c_{22}d_{11}^2, \quad W_{10} := 4W_4W_5 + Ad_{02}d_{22}c_{11}^2 + Bc_{02}c_{22}d_{11}^2. \]

As \( Bd_{22} + d_{00} = 0 \) yields \( W_4 = 0 \) two cases remain:

a. \( W_9 = 0 \): The computation of the resultant of \( W_9 = 0 \) and \( Q_i \) with respect to \( d_{11} \) is denoted by \( U_{i} \). Now it can easily be seen that \( U_{0} \) and \( U_{4} \) can only vanish for \((Ac_{22}g_{02} - g_{20}c_{02})F[19] = 0, U_{1} \) and \( U_{3} \) for \( G[10] = 0 \) and \( U_{2} \) for \( Bc_{02}c_{22}(Ac_{22}g_{02} - g_{20}c_{02})H[34] = 0 \). As \( Ac_{22}g_{02} - g_{20}c_{02} = 0 \) yields \( f_{02}g_{20} - f_{20}g_{02} = 0 \), a contradiction, we proceed with the case \( F = G = H = 0 \).

We compute the resultant of \( G \) and \( H \) with respect to \( g_{20} \) and the resultant of \( G \) and \( F \) with respect to \( g_{20} \). It can easily be seen that these two resultants cannot vanish w.c..

b. \( W_{10} = 0, W_9 \neq 0 \): For both possible solutions of \( W_{10} = 0 \) for \( c_{11} \) the resultant \( R_{12} \) cannot vanish w.c.

\* \( W_{2} = W_{5} = 0 \): In this case we have \( d_{02} := Ac_{00}c_{22}d_{22} \) and \( d_{00} := Ac_{02}c_{20}d_{20} \) with \( A \in \mathbb{R} \setminus \{0\} \). Moreover, the resultant of \( W_8 = 0 \) and \( Q_i \) with respect to \( d_{11} \) is denoted by \( V_{i} \). We can factor \( c_{20}d_{20}g_{02} - g_{20}c_{00}d_{22} \) out from \( V_{0}, V_{2} \) and \( V_{4} \) because its vanishing yields \( f_{02}g_{20} - f_{20}g_{02} = 0 \), a contradiction. In the following we denote the resultant of \( V_{i} \) and \( V_{j} \) with respect to \( g_{20} \) by \( P_{ij} \). We compute \( P_{13}, P_{03} \) and \( P_{01} \). Then \( P_{13} \) can only vanish w.c. for \( 4(c_{22}c_{00} + c_{20}c_{02}) - c_{11}^2 \). Now the two resultants of this factor with \( P_{03} \) and \( P_{01} \), respectively, with respect to \( c_{11} \) cannot vanish w.c.. This finishes the proof of Lemma 2.

2.1.1 \( W_1 = 0, W_4W_5 \neq 0 \)

We start by discussing the special cases. \( W_1 = 0 \) holds only in the following 8 cases (without contradicting \( W_4W_5 \neq 0 \)) if two variables out of the set \( \{c_{ij}, d_{ij}\} \) are equal to zero:

i. \( c_{22} = d_{22} = 0 \) or \( c_{20} = d_{02} = 0 \): Now \( Q_3 = 0 \) and \( Q_4 = 0 \) are fulfilled identically. Then the resultant of \( R_{02} \) and \( R_{12} \) with respect to \( d_{11} \) cannot vanish w.c..

ii. \( c_{00} = d_{00} = 0 \) or \( c_{02} = d_{20} = 0 \): Now \( Q_0 = 0 \) and \( Q_1 = 0 \) are fulfilled identically. Then the resultant of \( R_{23} \) and \( R_{24} \) with respect to \( d_{11} \) cannot vanish w.c..

We remain with the following 4 cases:

\[ c_{00} = c_{22} = 0, \quad c_{02} = c_{20} = 0, \quad d_{00} = d_{22} = 0, \quad d_{02} = d_{20} = 0. \tag{9} \]

In all 4 cases the conditions are already sufficient for a reducible composition. In each case we end up with one homogeneous quadratic equation in \( g_{20}, g_{02} \):
1. $c_{00} = c_{22} = 0$: The equation equals: $(g_{20}c_{02}d_{02} + g_{02}c_{20}d_{20})^2 - 2g_{20}g_{02}d_{20}d_{02}c_{11}$.  
2. $c_{02} = c_{20} = 0$: The equation equals: $(g_{20}c_{00}d_{22} + g_{02}c_{22}d_{00})^2 - 2g_{20}g_{02}d_{00}d_{22}c_{11}$.  
3. $d_{00} = d_{22} = 0$: The equation equals: $(g_{20}c_{02}d_{02} + g_{02}c_{20}d_{20})^2 - 2g_{20}g_{02}c_{02}d_{11}$.  
4. $c_{02} = c_{20} = 0$: The equation equals: $(g_{20}c_{00}d_{22} + g_{02}c_{22}d_{00})^2 - 2g_{20}g_{02}c_{02}c_{22}d_{11}$.

Now we discuss the general case: W.l.o.g. we can set $d_{02} := Ac_{00}c_{20}d_{22}$ and $d_{20} := Ac_{02}c_{22}d_{00}$ with $A \in \mathbb{R} \setminus \{0\}$. Then the resultant $R_{13}$ is fulfilled identically. Moreover $R_{34} = 0$ implies $R_{14} = 0$. The factors of $R_{34}$ are:

$$c_{20}g_{02}AW_1W_2W_3W_4W_5W_6W_7W_8W_9W_{10}W_{11}W_{12}W_{13}$$

with:

$W_{11} := 4d_{00}d_{22}(Ac_{00}c_{22} - 1)(Ac_{20}c_{02} - 1) + Ad_{00}d_{22}c_{11}^2 - d_{11}^2$,  
$W_{12} := d_{11}^2 + Ad_{00}d_{22}c_{11}^2$,  
$W_{13} := 4AW_1^2d_{11}^2 + 4d_{00}d_{22}(A^2c_{00}c_{22}c_{02} - 1)c_{11}^2 - (Ac_{00}c_{22} + 1)(Ac_{20}c_{02} + 1)c_{11}^2d_{11}^2$.

Therefore one of the factors $W_{11}W_{12}W_{13}$ must vanish.

$W_{11} = 0$: The computation of the resultant of $W_{11} = 0$ and $Q_i$ with respect to $d_{11}$ is denoted by $U_i$. Now it can easily be seen that $U_0$ and $U_4$ can only vanish for $(d_{00}c_{22}g_{02} - g_{20}c_{00}d_{22})F[32] = 0$, $U_1$ and $U_3$ for $G[20] = 0$ and $U_2$ for $(d_{00}c_{22}g_{02} - g_{20}c_{00}d_{22})H[70] = 0$. As $d_{00}c_{22}g_{02} - g_{20}c_{00}d_{22} = 0$ yields $f_{02}g_{20} - f_{20}g_{02} = 0$, a contradiction, we proceed with the case $F = G = H = 0$.

We compute the resultant $R_{FG}$ of $F$ and $G$ with respect to $g_{20}$, and analogously the resultants $R_{FH}$ and $R_{GH}$. Then $R_{FG}$ can only vanish w.c. for $M_1M_2 = 0$ with

$$M_1 := 2(Ac_{20}c_{02} - 1)(Ac_{22}c_{00} - 1) + Ac_{11}^2,$$
$$M_2 := 4(Ac_{22}c_{00} - 1)W_7 - (Ac_{22}c_{00} + 1)c_{11}^2.$$

- $M_1 = 0$: We compute the resultant of $M_1$ and $R_{FH}$ resp. $R_{GH}$ with respect to $c_{11}$. Then the greatest common divisor of the resulting expressions can only vanish w.c. for $Ac_{20}c_{02} + 1 = 0$ (which implies $W_2 = 0$). W.l.o.g. we can solve this equation for $A$. Then for both solution of $M_1 = 0$ for $c_{11}$ the resulting expression of $F$ cannot vanish w.c..

- $M_2 = 0$: Analogous consideration as for the case $M_1 = 0$ yield the following necessary conditions: $W_7(Ac_{20}c_{02} + 1)(A^2c_{00}c_{20}c_{02}c_{22} - 1) = 0$ where $A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$ implies $W_3 = 0$. Together with $M_2 = 0$ this yields $W_7 = Ac_{20}c_{02} + 1 = 0$ or $W_7 = A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$. The latter can be seen by computing the resultant of $A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$ and $M_2$ with respect to $A$. Therefore only the discussion of two cases remain:

- $W_7 = Ac_{20}c_{02} + 1 = 0$: As $W_7 = 0$ holds we can set $c_{00} := Bc_{02}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$ w.l.o.g. Moreover we can express $A$ from $Ac_{20}c_{02} + 1 = 0$. Then $H$ can only vanish w.c. for $c_{11}^2 - 16Bc_{02}c_{22} = 0$. For both solutions with respect to $c_{11}$ the polynomial $F$ cannot vanish w.c...
W.l.o.g. we can express these 4 expressions cannot vanish w.c.. Both remaining special cases imply spherical isograms.

1. special cases of Eq. (9) imply spherical isograms.

Proof: Due to the case study of Subsection 2.1.1 we have to show that the 4 expressions cannot vanish w.c.. For both solutions of \( A^2c_{00}c_{02}c_{22} - 1 = 0 \) with respect to \( A \) non of the resultants \( R_{FG}, R_{FH} \) and \( R_{GH} \) can vanish w.c..

\[ W_{12} = 0, \; W_{11} \neq 0: \] W.l.o.g. we can express \( A \) from \( W_{12} = 0 \). The only \( R_{ij} \)'s which are not fulfilled are \( R_{2i} \) with \( i \in \{0, 1, 3, 4\} \). It can easily be seen that these 4 expressions can only vanish w.c. for \( c_{20}c_{02}d_{11}^2 - d_{00}d_{22}c_{11}^2 = 0 \). For both solutions of this equation with respect to \( d_{11} \) the equations \( Q_1 \) and \( Q_3 \) can only vanish w.c. for \( W_7 = 0 \). Therefore we set \( c_{00} := Bc_{02} \) and \( c_{20} := Bc_{22} \) with \( B \in \mathbb{R} \setminus \{0\} \). Now the resultant of \( Q_2 \) and \( Q_0 \) (or \( Q_4 \)) with respect to \( g_{20} \) cannot vanish w.c..

\[ W_{13} = 0, \; W_{11}W_{12} \neq 0: \] Assuming \((Ac_{00}c_{22} + 1)(Ac_{20}c_{02} + 1)c_{11}^2 - 4AW_7^2 \neq 0\) we can compute \( d_{11} \) from \( W_{13} \). For both possible solutions the only \( R_{ij} \)'s which are not fulfilled are again \( R_{2i} \) with \( i \in \{0, 1, 3, 4\} \). It can easily be shown that these 4 expressions cannot vanish w.c..

Now we set \((Ac_{00}c_{22} + 1)(Ac_{20}c_{02} + 1)c_{11}^2 - 4AW_7^2 = 0\): Assuming \((Ac_{00}c_{22} + 1)(Ac_{20}c_{02} + 1) \neq 0\) we can solve the condition for \( c_{11} \). For both solutions \( W_{13} \) cannot vanish w.c.. Both remaining special cases imply \( W_7 = 0 \) and therefore we set \( c_{00} := Bc_{02} \) and \( c_{20} := Bc_{22} \) with \( B \in \mathbb{R} \setminus \{0\} \). Now \( W_{13} = 0 \) is already fulfilled identically. For both special cases the resultants \( R_{02} \) and \( R_{24} \) cannot vanish w.c..

We sum up the results of this case study in the following theorem:

**Theorem 1** For the case \( g_{20}g_{02}(f_{20}g_{02} - f_{02}g_{20}) \neq 0 \) there only exists a symmetric reducible composition with a spherical coupler component if and only if the spherical coupler \( C \) or \( D \) is a spherical isogram.

**Proof:** Due to the case study of Subsection 2.1.1 we have to show that the 4 special cases of Eq. (9) imply spherical isograms.

1. \( c_{00} = c_{22} = 0 \): It was already shown in [3] that \( c_{00} = c_{22} = 0 \) is equivalent with the conditions \( \beta_1 = \alpha_1 \) and \( \delta_1 = \gamma_1 \), i.e. \( C \) is a spherical isogram.

2. \( c_{20} = c_{02} = 0 \): Substituting of the angle expressions yields:

\[
c_{02} + c_{20} = 2 \cos (\alpha_1 + \beta_1) \cos (\delta_1) - 2 \cos (\gamma_1), \quad c_{02} - c_{20} = 2 \sin (\alpha_1 + \beta_1) \sin (\delta_1).
\]

Under consideration of \( 0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi \) we get the following solution: \( \beta_1 = \pi - \alpha_1 \) and \( \delta_1 = \pi - \gamma_1 \). The couplers of item 1 and item 2 have the same movement because we get item 2 by replacing \( I_{20} \) of item 1 by its antipode \( \overline{I}_{20} \). Clearly, the same holds for the coupler \( D \). Therefore all four special cases correspond with spherical isograms.
Now it remains to show that $W_4W_5 = 0$ only yields contradictions if we assume that none of the couplers is a spherical isogram. W.l.o.g. we set $W_4 = 0$. It is an easy task to show that no reducible composition exists (the proof is left to the reader) for the four special cases:

\[ d_{00} = c_{20} = 0, \quad d_{00} = d_{20} = 0, \quad c_{22} = c_{20} = 0, \quad c_{22} = d_{20} = 0. \]

Here we only discuss the general case: W.l.o.g. we can set $d_{00} := Ac_{20}$ and $d_{20} := Ac_{22}$ with $A \in \mathbb{R} \setminus \{0\}$ and assume $d_{00}c_{20}d_{20}c_{22} \neq 0$. Now the greatest common divisor of $T_{34}, T_{14}$ and $T_{13}$ can only vanish w.c. for $d_{02}c_{22} - c_{20}d_{22} = 0$. Therefore we can set w.l.o.g. $d_{02} := Bd_{22}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. Moreover we can assume $d_{22} \neq 0$ because otherwise $Q_4$ yields the contradiction. After factoring out all factors of $Q_1, Q_3$ and $Q_4$ which cannot vanish w.c. we compute $R_{34}$ and $R_{14}$. Finally the resultant of these two expressions with respect to $c_{11}$ yields the contradiction.

Clearly, due to the symmetry of the equations also the following theorem holds:

**Theorem 2** For the case $f_{20}f_{02}(f_{20}g_{02} - f_{02}g_{20}) \neq 0$ there only exists a symmetric reducible composition with a spherical coupler component if and only if the spherical coupler $C$ or $D$ is a spherical isogram.

### 2.2 $g_{20}g_{02} = 0$

Due to Theorem 1 and 2 we only have to discuss those cases for which $g_{20}g_{02} = 0$, $f_{20}f_{02} = 0$ and $f_{20}g_{02} - f_{02}g_{20} \neq 0$ hold. There are only the following two symmetric cases: $f_{02} = g_{20} = 0$ or $f_{20} = g_{02} = 0$. W.l.o.g. we set $f_{02} = g_{20} = 0$. Then $Q_{40}$ and $Q_{04}$ can only vanish for $W_4 = W_5 = 0$. First of all we discuss the special cases, where we distinguish four groups:

1. $C$ and $D$ are spherical isograms:

\[ d_{00} = d_{22} = c_{02} = c_{20} = 0, \quad c_{00} = c_{22} = d_{02} = d_{20} = 0. \]

2. $C$ is a spherical isogram and $D$ not:

\[ c_{00} = c_{22} = d_{02} = c_{20} = 0, \quad c_{02} = c_{20} = d_{00} = c_{00} = 0. \]

3. $D$ is a spherical isogram and $C$ not:

\[ d_{00} = d_{22} = c_{20} = d_{02} = 0, \quad d_{02} = d_{20} = c_{00} = d_{00} = 0. \]

4. $C$ and $D$ are no spherical isograms:

\[ d_{00} = c_{20} = d_{02} = c_{00} = 0, \quad d_{00} = d_{20} = c_{02} = c_{00} = 0, \quad c_{22} = c_{20} = d_{02} = d_{22} = 0. \]
It can easily be verified that all 9 special cases yield a contradiction. The proof is left to the reader. In the next step we check the semispecial cases:

1. \( d_{00} = Ac_{20}, \ d_{20} = Ac_{22} \) with \( A \in \mathbb{R} \setminus \{0\} \) and \( c_{20}c_{22} \neq 0 \):
   a. \( c_{00} = c_{02} = 0 \): In this case \( Q_{22} \) splits up into two factors. In both cases we can compute \( f_{20} \) w.l.o.g.. Then \( Q_{33} \) can only vanish w.c. for \( c_{22}d_{02} - d_{22}c_{20} = 0 \) which yields together with \( Q_{44} \) the contradiction.
   b. \( d_{02} = d_{22} = 0 \): Analogous considerations as in the last case also yield the contradiction.
   c. \( c_{00} = d_{02} = 0 \): In this case \( Q_{00} = 0 \) and \( Q_{44} = 0 \) imply \( d_{22} = c_{02} = 0 \). Then \( Q_{22} = 0 \) yields the contradiction.
   d. \( d_{22} = c_{02} = 0 \): Analogous considerations as in the last case also yield the contradiction.

2. \( d_{22} = Bc_{02}, \ d_{02} = Bc_{00} \) with \( B \in \mathbb{R} \setminus \{0\} \) and \( c_{00}c_{02} \neq 0 \):
   a. \( d_{00} = c_{20} = 0 \) or \( c_{22} = d_{20} = 0 \): Analogous considerations as in the semispecial case 1c yield the contradiction.
   b. \( d_{00} = d_{20} = 0 \) or \( c_{22} = c_{20} = 0 \): Analogous considerations as in the semispecial case 1a yield the contradiction.

Finally we can discuss the general case. W.l.o.g. we can set \( d_{00} = Ac_{20}, \ d_{20} = Ac_{22}, \ d_{22} = Bc_{02}, \ d_{02} = Bc_{00} \) for \( A, B \in \mathbb{R} \setminus \{0\} \). As for \( W_7 = 0 \) the expression \( Q_{00} \) cannot vanish w.c. we can compute \( f_{20} \) from \( Q_{00} = 0 \) w.l.o.g.. Then the resultant of \( Q_{11} \) and \( Q_{22} \) with respect to \( d_{11} \) already yields the contradiction.

We sum up the results of this subsection in the following theorem:

**Theorem 3** For the case \( g_{20}g_{02} = 0, \ f_{20}f_{02} = 0 \) and \( f_{20}g_{02} - f_{02}g_{20} \neq 0 \) there does not exist a symmetric reducible composition with a spherical coupler component.

### 3 Symmetric reducible composition with \( f_{20}g_{02} - f_{02}g_{20} = 0 \)

**3.1 Very special case:** \( f_{20} = f_{02} = g_{20} = g_{02} = 0 \)

In this case the equations \( Q_{40}, Q_{31}, Q_{13} \) and \( Q_{04} \) imply:

\[
d_{00}c_{22} = 0, \quad c_{00}d_{22} = 0, \quad d_{02}c_{02} = 0, \quad d_{20}c_{20} = 0.
\]

Now we have to discuss all possible non-contradicting combinatorial cases which can again be grouped into four classes:
1. $C$ and $D$ are spherical isograms:

$$d_{00} = d_{22} = c_{02} = c_{20} = 0, \quad c_{00} = c_{22} = d_{02} = d_{20} = 0.$$  

We only discuss the first case (for the second case we refer to analogy). As at least $f_{11}$ or $g_{11}$ must be different from zero we can assume $f_{11} \neq 0$ w.l.o.g.. Then we can express $g_{22}$ from $Q_{33}$ and $g_{00}$ from $Q_{11}$.

a. $d_{02} \neq 0$: We can compute $f_{11}$ from $Q_{44}$ w.l.o.g.. Now $Q_{00}$ splits up into:

$$c_{00}d_{20}f_{22} - f_{00}c_{22}d_{02}[c_{11}d_{11}c_{00}d_{20}d_{02}c_{22} + g_{11}(d_{02}c_{22}f_{00} + c_{00}d_{20}f_{22})].$$

The first factor can always be solved for $f_{00}$. Then only one equation $Q_{22}$ remains which can be solved for $g_{11}$ w.l.o.g..

For the second factor the same procedure also holds if we assume $g_{11} \neq 0$. For $g_{11} = 0$ the second factor can only vanish w.c. for $c_{00} = 0$. Then the remaining equation $Q_{22} = 0$ can be solved for $f_{00}$ w.l.o.g..

b. $d_{02} = 0$: Now $Q_{44}$ can only vanish for $f_{22}g_{11} = 0$.

i. For $f_{22} = 0$ we can solve $Q_{22} = 0$ for $g_{11}$ and one equation remains where $c_{00} = 0$ factors out. For $c_{00} \neq 0$ the remaining factor can be solved for $f_{11}$ w.l.o.g..

ii. For $g_{11} = 0$ we are left with $Q_{22} = 0$ and $Q_{00} = 0$. Both equations are fulfilled for $c_{00} = 0$. For $c_{00} \neq 0$ we can solve $Q_{00} = 0$ for $f_{00}$ and $Q_{22} = 0$ for $f_{22}$ w.l.o.g..

2. $C$ is a spherical isogram and $D$ not:

$$c_{00} = c_{22} = d_{02} = c_{20} = 0, \quad c_{02} = c_{20} = d_{00} = c_{00} = 0.$$  

In the following we only discuss the first case in detail (for the second case we refer to analogy). Due to item 1 we can assume w.l.o.g. $c_{02}d_{20} \neq 0$. Then $Q_{20}$ implies $d_{00} = 0$. As at least $f_{11}$ or $g_{11}$ must be different from zero we can assume w.l.o.g. that $f_{11} \neq 0$ holds. Under this assumption we can express $g_{22}$ from $Q_{33}$ and $g_{00}$ from $Q_{11}$. Now $Q_{00}$ and $Q_{44}$ can only vanish w.c. for $f_{00} = f_{22} = 0$ but then $Q_{22} = 0$ yields the contradiction.

3. $D$ is a spherical isogram and $C$ not:

$$d_{00} = d_{22} = c_{20} = d_{02} = c_{00} = 0.$$  

Analogously considerations as in item 2 yield the contradiction.

4. $C$ and $D$ are no spherical isograms:

$$d_{00} = c_{20} = d_{02} = c_{00} = 0, \quad d_{00} = d_{20} = c_{02} = c_{00} = 0, \quad c_{22} = c_{20} = d_{02} = d_{22} = 0.$$  

We start by discussing the case $d_{00} = c_{20} = d_{02} = c_{00} = 0$: Due to the above discussed cases we can assume $c_{02}c_{22}d_{20}d_{22} \neq 0$. As at least $f_{11}$ or $g_{11}$ must be different from zero we can assume $f_{11} \neq 0$ w.l.o.g.. Then we can express $g_{22}$ from $Q_{33}$ and $g_{00}$ from $Q_{11}$. Now $Q_{00}$ and $Q_{44}$ can only vanish w.c. for $f_{00} = f_{22} = 0$ but then $Q_{22} = 0$ yields the contradiction.
In the remaining two cases we get the contradiction much more easier, because $Q_{20}$ and $Q_{42}$, respectively, cannot vanish w.c..

We sum up the results of this subsection in the following theorem:

**Theorem 4** For any symmetric reducible composition with a spherical coupler component and $g_{20} = g_{02} = f_{20} = f_{02} = 0$ the couplers $C$ and $D$ are spherical isograms.

In the following we formulate the main theorem for the symmetric reducible composition:

**Theorem 5** If a symmetric reducible composition with a spherical coupler component is given, then it is one of the following cases or a special case of them, respectively:

1. One spherical coupler is a spherical isogram,
2. the spherical couplers are forming a spherical focal mechanism which is analytically given by:
   \[ c_{00}c_{20} = \lambda d_{00}d_{02}, \quad c_{22}c_{02} = \lambda d_{22}d_{20}, \quad \text{with} \quad \lambda \in \mathbb{R} \setminus \{0\} \]
   and \[ c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = \lambda[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})], \]
3. both spherical couplers are orthogonal with $c_{22} = c_{02} = d_{00} = d_{02} = 0$ resp. $d_{22} = d_{20} = c_{00} = c_{20} = 0$.

Proof: The Theorems 1 and 2 yield item 1 of Theorem 5. Moreover Theorem 4 implies a special case of item 1. Now the discussion of the special cases, the general case and the excluded case is missing. It turns out that the corresponding case studies only yield solutions which are one of the three cases of Theorem 5 or special cases of them, respectively. The detailed discussion of cases is performed in Subsection 3.2 and 3.3 and Section 4. Moreover it should be noted that item 2 is the focal type of Subsection 1.2.1.

3.2 Special cases

Due to the last subsection we can discuss the following four special cases

\[ f_{20} = f_{02} = 0, \quad f_{20} = g_{20} = 0, \quad f_{02} = g_{02} = 0, \quad g_{20} = g_{02} = 0, \]

under the assumption, that not all elements of \{ $f_{02}, f_{20}, g_{02}, g_{20}$ \} are equal to zero. Due to the symmetry of the conditions only 2 of these 4 cases must be discussed (for the other cases we can refer to analogy).
3.2.1 \( f_{20} = f_{02} = 0 \)

For the discussion we can assume w.l.o.g. that \( g_{20} \neq 0 \) holds. Under this assumption we can compute \( f_{22} \) from \( Q_{42}, f_{00} \) from \( Q_{20} \) and \( f_{11} \) from \( Q_{31} \). Then \( Q_{40} \) and \( Q_{04} \) can only vanish for \( W_4 = W_5 = 0 \). We start again with the discussion of the special cases:

1. \( \mathcal{C} \) and \( \mathcal{D} \) are spherical isograms:

\[
d_{00} = d_{22} = c_{02} = c_{20} = 0, \quad c_{00} = c_{22} = d_{02} = d_{20} = 0.
\]

These two cases yield easy contradictions as all \( f_{ij} \) vanish.

2. \( \mathcal{C} \) is a spherical isogram and \( \mathcal{D} \) not: The case \( c_{02} = c_{20} = d_{00} = c_{00} = 0 \) yields an easy contradiction as all \( f_{ij} \) vanish. For the remaining case \( c_{00} = c_{22} = d_{02} = c_{20} = 0 \) there exists the following reducible composition which is a special case of item 1 of Theorem 5: Now \( f_{11} = f_{22} = 0 \) hold. W.l.o.g. we can assume \( d_{20}d_{00} \neq 0 \) because otherwise \( f_{00} = 0 \) holds. Now the remaining three equations \( Q_{22} = 0, Q_{11} = 0 \) and \( Q_{00} = 0 \) can be solved w.l.o.g. for \( g_{22}, g_{11} \) and \( g_{00} \), respectively.

3. \( \mathcal{D} \) is a spherical isogram and \( \mathcal{C} \) not: The case \( d_{00} = d_{22} = c_{02} = d_{02} = 0 \) yields an easy contradiction as all \( f_{ij} \) vanish. For the remaining case \( d_{02} = d_{20} = c_{00} = d_{00} = 0 \) there exists an analogous reducible composition as in the last case.

4. \( \mathcal{C} \) and \( \mathcal{D} \) are no spherical isograms: The case \( d_{00} = c_{20} = d_{02} = c_{00} = 0 \) yields an easy contradiction as all \( f_{ij} \) vanish. Two cases remain:

\[
d_{00} = d_{20} = c_{02} = c_{20} = 0, \quad c_{22} = c_{20} = d_{02} = d_{22} = 0.
\]

We start discussing the first case: Now \( f_{11} = f_{00} = 0 \) hold. W.l.o.g. we can assume \( c_{22}c_{20} \neq 0 \) because otherwise \( f_{22} = 0 \) holds. Then \( Q_{22} = 0 \) implies \( g_{00} = 0 \). Now the remaining three equations \( Q_{44} = 0, Q_{33} = 0 \) and \( Q_{24} = 0 \) can be solved w.l.o.g. for \( g_{22}, g_{11} \) and \( g_{02} \), respectively. This yields a special spherical focal mechanism (item 2 of Theorem 5).

For the second case we get \( f_{11} = f_{22} = 0 \). W.l.o.g. we can assume \( d_{00}d_{20} \neq 0 \) because otherwise \( f_{00} = 0 \) holds. Then \( Q_{22} = 0 \) implies \( g_{22} = 0 \). Now the remaining three equations \( Q_{11} = 0, Q_{02} = 0 \) and \( Q_{00} = 0 \) can be solved w.l.o.g. for \( g_{11}, g_{02} \) and \( g_{00} \), respectively. This yields a special spherical focal mechanism (item 2 of Theorem 5).

In the next step we check the semispecial cases:

1. \( d_{00} = Ac_{20}, d_{20} = Ac_{22} \) with \( A \in \mathbb{R} \setminus \{0\} \) and \( c_{20}c_{22} \neq 0 \): The following 4 cases are again special spherical focal mechanisms (item 2 of Theorem 5):
   a. \( c_{00} = c_{02} = 0 \): It can easily be verified that there only exists a reducible composition if and only if \( d_{02} = d_{22} = 0 \) hold.
   b. \( d_{02} = d_{22} = 0 \): It can easily be verified that there only exists a reducible
composition if and only if $c_{00} = c_{02} = 0$ hold.

c. $c_{00} = d_{02} = 0$: It can easily be verified that there only exists a reducible composition if and only if $Ad_{22}c_{11}^2 - c_{02}d_{11}^2 = 0$ holds.

d. $d_{22} = c_{02} = 0$: It can easily be verified that there only exists a reducible composition if and only if $Ad_{02}c_{11}^2 - c_{00}d_{11}^2 = 0$ holds.

2. $d_{22} = Bc_{02}, d_{02} = Bc_{00}$ with $B \in \mathbb{R} \setminus \{0\}$ and $c_{00}c_{02} \neq 0$:

a. $d_{00} = c_{20} = 0$: We get a contradiction as all $f_{ij}$ vanish.

b. $c_{22} = d_{20} = 0$: We get a contradiction as all $f_{ij}$ vanish.

c. $d_{00} = d_{20} = 0$: In this case $f_{00} = f_{11} = 0$ hold. W.l.o.g. we can assume $c_{20}c_{22} \neq 0$ because otherwise $f_{22} = 0$ holds and all $f_{ij}$ would vanish. Then $Q_{13}$ cannot vanish w.c..

d. $c_{22} = c_{02} = 0$: In this case $f_{22} = f_{11} = 0$ hold. W.l.o.g. we can assume $d_{20}d_{00} \neq 0$ because otherwise $f_{00} = 0$ holds and all $f_{ij}$ would vanish. Then $Q_{13}$ cannot vanish w.c..

Finally we can discuss the general case. W.l.o.g. we can set $d_{00} = Ac_{20}, d_{20} = Ac_{22}, d_{22} = Bc_{02}, d_{02} = Bc_{00}$ for $A, B \in \mathbb{R} \setminus \{0\}$. Now we can express $g_{02}$ from $Q_{13}, g_{22}$ from $Q_{44}, g_{00}$ from $Q_{00}, g_{11}$ from $Q_{33}$ and one equation remains: $ABc_{11}^2 - d_{11}^2 = 0$. This case yields also a special spherical focal mechanism (item 2 of Theorem 5).

3.2.2 $f_{02} = g_{02} = 0$

W.l.o.g. we can assume $f_{20}g_{20} \neq 0$ because otherwise we would get a special case of $f_{20} = f_{02} = 0$ or of its symmetric case $g_{20} = g_{02} = 0$. Therefore we can compute $f_{22}$ from $Q_{42}$, $f_{00}$ from $Q_{20}$, $f_{11}$ from $Q_{31}$ and $f_{20}$ from $Q_{10}$ w.l.o.g.. Now $f_{20}$ can only vanish for $W_4 = 0$. Moreover $Q_{13}$ and $Q_{04}$ can only vanish w.c. for $c_{00}d_{22} = 0$ and $c_{02}d_{02} = 0$. We get the following combinatorial cases:

\[
\begin{align*}
    c_{00} &= c_{02} = 0, & c_{00} &= d_{02} = 0, & d_{22} &= c_{02} = 0, & d_{22} &= d_{02} = 0.
\end{align*}
\]

$c_{00} = c_{02} = 0 \Rightarrow c_{22} \neq 0$: Now $Q_{24}$ can only vanish w.c. for $d_{22}d_{02} = 0$:

1. $d_{22} = 0 \Rightarrow d_{20} \neq 0$: Due to $Q_{00}$ we have to distinguish two cases:

   a. $g_{00} = 0$: Due to $Q_{11}$ we have to distinguish further two cases:

   i. $d_{00} = 0$: Assuming $2c_{20}d_{20}g_{11} + c_{11}d_{11}g_{20} \neq 0$ we can express $g_{22}$ from $Q_{31} = 0$. Then one equation remains:

\[
    d_{02}c_{11}g_{20}^2 + c_{20}g_{11}c_{11}d_{11}g_{20} + c_{20}g_{11}^2d_{20} = 0.
\]

This yields a spherical focal mechanism where $\mathcal{D}$ is additionally a spherical isogram (item 1 and 2 of Theorem 5).

For $2c_{20}d_{20}g_{11} + c_{11}d_{11}g_{20} = 0$ we can express $g_{11}$ from this equation w.l.o.g.. Then $Q_{33} = 0$ cannot vanish w.c..
ii. $g_{11} = 0, d_{00} \neq 0$: Now we can compute $g_{22}$ from $Q_{22}$ w.l.o.g.. Then the remaining two equations can only vanish w.c. for $c_{20} = 0$ ($C$ is a spherical isogram; item 1 of Theorem 5) or $d_{02} = 0$, which yields a special spherical focal mechanism (item 2 of Theorem 5).

b. $g_{00} \neq 0$: In this case we solve the remaining factor of $Q_{00}$ for $g_{00}$. This can be done w.l.o.g.. Moreover we can compute $g_{11}$ from $Q_{11}$ and $g_{22}$ from $Q_{22}$ w.l.o.g.. Then the remaining two equations can only vanish w.c. for $c_{20} = 0$ ($C$ is a spherical isogram; item 1 of Theorem 5) or $d_{02} = 0$, which yields a special spherical focal mechanism (item 2 of Theorem 5).

2. $d_{02} = 0, d_{22} \neq 0$: Due to $Q_{00}$ we have to distinguish two cases:
   a. $g_{00} = 0$: Due to $Q_{11}$ we have to distinguish further two cases:
      i. $d_{00} = 0$: Assuming $2c_{20}d_{20}g_{11} + c_{11}d_{11}g_{20} \neq 0$ we can express $g_{22}$ from $Q_{33} = 0$. Now it can easily be seen that the remaining two equations $Q_{44} = 0$ and $Q_{22} = 0$ cannot vanish w.c..
         For $2c_{20}d_{20}g_{11} + c_{11}d_{11}g_{20} = 0$ we can express $g_{11}$ from this equation w.l.o.g.. Then $Q_{33} = 0$ cannot vanish w.c..
      ii. $d_{20} = 0, d_{00} \neq 0$: Assuming $2c_{22}d_{00}g_{11} + c_{11}d_{11}g_{20} \neq 0$ we can express $g_{22}$ from $Q_{33} = 0$. Then one equation remains:

      \[ d_{22}c_{11}g_{20}^2 + c_{22}g_{11}c_{11}d_{11}g_{20} + c_{22}g_{11}^2d_{00} = 0. \]

      This yields a spherical focal mechanism where $D$ is additionally a spherical isogram (item 1 and 2 of Theorem 5).

   b. $g_{00} \neq 0$: In this case we solve the remaining factor of $Q_{00}$ for $g_{00}$. This can be done w.l.o.g.. Moreover we can compute $g_{11}$ from $Q_{11}$ and $g_{22}$ from $Q_{22}$ w.l.o.g.. Then $Q_{22}$ cannot vanish w.c..

\[ d_{22} = d_{02} = 0 \Rightarrow d_{20} \neq 0: \] For this case we refer to analogy. It can be done similarly to the last case if the variables are substituted as follows:

\[ c_{00} \leftrightarrow d_{02}, \quad c_{02} \leftrightarrow d_{22}, \quad c_{20} \leftrightarrow d_{00}, \quad c_{22} \leftrightarrow d_{20}, \quad c_{11} \leftrightarrow d_{11}. \]

\[ c_{00} = d_{02} = 0: \] In the following we distinguish two cases:

1. Assuming $g_{20}c_{11}d_{20}(d_{00}c_{22} + c_{20}d_{20}) + 2(d_{00}c_{22} - c_{20}d_{20})^2g_{11} \neq 0$ we can express $g_{00}$ from $Q_{11}$ and $g_{22}$ from $Q_{33}$. Then we compute the resultants $R_{ij}$ of $Q_{ii}$ and $Q_{jj}$ with respect to $g_{11}$ for $i, j \in \{0, 2, 4\}$ and $i \neq j$. The greatest common divisor of these three resultants can only vanish w.c. for:
a. $c_{22} = 0$: Now $\mathcal{C}$ is a spherical isogram (item 1 of Theorem 5). It can easily be seen that the remaining two equations $Q_{00} = 0$ and $Q_{22} = 0$ can only vanish w.c. for:

$$g_{11}^2 d_{20}^2 c_{20} + g_{20}^2 d_{11}^2 c_{02} + d_{11} c_{11} g_{20} g_{11} d_{20} = 0.$$ 

b. $d_{20} = 0$, $c_{22} \neq 0$: Now $\mathcal{D}$ is a spherical isogram (item 1 of Theorem 5). It can easily be seen that the remaining two equations $Q_{44} = 0$ and $Q_{22} = 0$ can only vanish w.c. for:

$$g_{20}^2 c_{11} d_{22} + g_{11}^2 c_{22} d_{00} + g_{11} c_{22} d_{11} c_{11} g_{20} = 0.$$ 

c. $4d_{22} c_{02} (d_{00} c_{22} - c_{20} d_{20}) + d_{22} d_{20} c_{11}^2 - c_{02} d_{11}^2 c_{22} = 0$, $c_{22} d_{20} \neq 0$: We distinguish again three cases:

i. Assuming $d_{22} c_{02} \neq 0$ we can compute $c_{20}$ from this equation. Now it can easily be seen that the remaining 3 equations can only vanish w.c. if a homogeneous quadratic equation in $g_{20}$ and $g_{11}$ (with 10 terms) is fulfilled. This equation can be solved w.l.o.g. for $g_{11}$. This yields a special spherical focal mechanism (item 2 of Theorem 5).

ii. $d_{22} = 0$: Then the equation can only vanish for $c_{02} = 0$. Now the remaining equations can only vanish w.c. if a homogeneous linear equation in $g_{20}$ and $g_{11}$ (with 5 terms) is fulfilled. This equation can be solved w.l.o.g. for $g_{11}$. This yields a special spherical focal mechanism (item 2 of Theorem 5).

iii. $c_{02} = 0$, $d_{22} \neq 0$: Then the equation cannot vanish w.c.

Now we assume that the greatest common divisor of the resultants $R_{02}$, $R_{04}$, $R_{24}$ is different from zero. Moreover we can set $c_{20} d_{00} \neq 0$ because both cases imply a contradiction. Therefore we can express $c_{22}$ from the only non-contradicting factor of $R_{24}$. Then $R_{02} = 0$ implies $d_{22} d_{00} c_{11}^2 + c_{02} c_{20} d_{11}^2 = 0$ which can be solved for $d_{22}$ w.l.o.g.. Now $Q_{00}$ and $Q_{44}$ cannot vanish w.c.

2. $g_{20} c_{11} d_{11} (d_{00} c_{22} + c_{20} d_{20}) + 2 (d_{00} c_{22} - c_{20} d_{20})^2 g_{11} = 0$: W.l.o.g. we can solve this equation for $g_{11}$. Now we proceed as follows: $Q_{00} = 0$ is a homogeneous quadratic equation in $g_{20}, g_{00}$ and $Q_{44} = 0$ is a homogeneous quadratic equation in $g_{20}, g_{22}$. Moreover, $Q_{22} = 0$ is also a homogeneous quadratic equation in $g_{20}, g_{22}, g_{00}$ where $g_{00}$ and $g_{22}$ appear only linear. From these 3 equations we eliminate $g_{00}$ and $g_{22}$ by applying the resultant method. We compute the resultant $R$ of $Q_{44}$ and $Q_{22}$ with respect to $g_{22}$. Then we compute the resultant of $R$ and $Q_{00}$ with respect to $g_{00}$. This resultant can only vanish w.c. if a homogeneous factor $F[22]$ of degree 16 in the unknowns $c_{ij}, d_{ij}$ is fulfilled. Moreover $Q_{11}$ and $Q_{33}$ can only vanish w.c. for:

$$d_{00} d_{20} (4 c_{02} c_{20}^2 d_{20} - 4 c_{02} d_{00} c_{22} c_{20} - d_{00} c_{11}^2 c_{22} - d_{20} c_{11}^2 c_{20}) = 0$$

$$c_{20} c_{22} (4 d_{22} d_{00} c_{22} - 4 d_{22} c_{20} d_{00} c_{20} - c_{22} c_{22} c_{11} d_{00} - c_{20} d_{11} c_{20}) = 0$$

It can easily be seen that the cases $d_{00} d_{20} c_{20} c_{22} = 0$ only imply contradictions. Therefore we can assume w.l.o.g. $d_{00} d_{20} c_{20} c_{22} \neq 0$. Moreover we
distinguish two cases:

a. \( d_{00}c_{22} + c_{20}d_{20} \neq 0 \): Under this assumption we can compute \( c_{11} \) and \( d_{11} \) from the remaining factors of Eq. (10) and Eq. (11), respectively. For all four branches \( F \) is fulfilled identically. In all cases we end up with a special spherical focal mechanism (item 2 of Theorem 5).

b. \( d_{00}c_{22} + c_{20}d_{20} = 0 \): W.l.o.g. we can solve this equation for \( d_{00} \). Then \( Q_{11} \) and \( Q_{33} \) can only vanish w.c. for \( c_{02} = 0 \) and \( d_{22} = 0 \). Again \( F \) is fulfilled identically. This yields a special spherical focal mechanism (item 2 of Theorem 5).

d\(_{22} = c_{02} = 0 \Rightarrow d_{20}c_{22} \neq 0 \): For this case we refer to analogy. It can be done similarly to the last case if the variables are substituted as follows:

\[
c_{00} \leftrightarrow d_{22}, \quad c_{02} \leftrightarrow d_{02}, \quad c_{20} \leftrightarrow d_{20}, \quad c_{22} \leftrightarrow d_{00}, \quad c_{11} \leftrightarrow d_{11}.
\]

3.3 General case

Due to the discussion of the last two subsection we can assume w.l.o.g. that \( f_{20}g_{02}f_{02}g_{20} \neq 0 \) holds. Therefore we can set \( f_{20} = Ag_{20} \) and \( f_{02} = Ag_{02} \) with \( A \in \mathbb{R} \setminus \{0\} \).

In the next step we compute \( g_{20} \) from \( Q_{40} = 0 \) which yields \( \pm W_4/\sqrt{A} \). Moreover we can express \( g_{02} \) from \( Q_{04} = 0 \) which yields \( \pm W_5/\sqrt{A} \). Therefore we have to distinguish the following cases:

3.3.1 \( g_{20} = -W_4/\sqrt{A}, \ g_{02} = +W_5/\sqrt{A} \) or \( g_{20} = +W_4/\sqrt{A}, \ g_{02} = -W_5/\sqrt{A} \)

W.l.o.g. we can compute \( g_{22} \) and \( g_{00} \) from \( Q_{42} = 0 \) and \( Q_{02} = 0 \). In the following we distinguish two cases:

1. \( g_{11}A - f_{11} \neq 0 \): Now we can express \( f_{00} \) and \( f_{22} \) from \( Q_{11} = 0 \) and \( Q_{33} = 0 \) w.l.o.g.. Moreover we can compute \( f_{11} \) from \( Q_{31} = 0 \) w.l.o.g.. Then \( Q_{13} \) can only vanish w.c. for \( W_1 = 0 \). First of all we discuss again the special cases: \( W_1 = 0 \) holds only in the following 8 cases (without contradicting \( W_4W_5 \neq 0 \)) if two variables out of the set \( \{c_{ij}, d_{ij}\} \) are equal to zero:

a. \( c_{22} = d_{22} = 0 \): It can easily be seen that the following expression has to vanish in order to get a reducible composition:

\[
4c_{20}d_{02}(c_{00}d_{20} - c_{02}d_{00}) + c_{11}^2d_{00}d_{02} - c_{20}d_{11}^2c_{00}.
\]

b. \( c_{20} = d_{02} = 0 \): It can easily be seen that the following expression has to
vanish in order to get a reducible composition:
\[ 4c_{22}d_{22}(c_{00}d_{20} - c_{02}d_{00}) - c_{11}^2d_{22}d_{20} + c_{02}d_{11}^2c_{22}. \]

c. \( c_{02} = d_{20} = 0 \): It can easily be seen that the following expression has to vanish in order to get a reducible composition:
\[ 4c_{00}d_{00}(d_{22}c_{20} - c_{22}d_{02}) + c_{11}^2d_{00}d_{02} - c_{20}d_{11}^2c_{00}. \]

d. \( c_{00} = d_{00} = 0 \): It can easily be seen that the following expression has to vanish in order to get a reducible composition:
\[ 4d_{20}c_{02}(d_{22}c_{20} - c_{22}d_{02}) - c_{11}^2d_{22}d_{20} + c_{02}d_{11}^2c_{22}. \]

e. \( c_{00} = c_{22} = 0 \): In this case \( Q_{24} = 0 \) and \( Q_{20} = 0 \) imply \( d_{00} = d_{22} = 0 \), which already yields a reducible composition.

f. \( d_{00} = d_{22} = 0 \): In this case \( Q_{24} = 0 \) and \( Q_{20} = 0 \) imply \( c_{00} = c_{22} = 0 \), which already yields a reducible composition.

g. \( c_{20} = c_{02} = 0 \): In this case \( Q_{24} = 0 \) and \( Q_{20} = 0 \) imply \( d_{20} = d_{02} = 0 \), which already yields a reducible composition.

h. \( d_{20} = d_{02} = 0 \): In this case \( Q_{24} = 0 \) and \( Q_{20} = 0 \) imply \( c_{20} = c_{02} = 0 \), which already yields a reducible composition.

The cases a-d imply special spherical focal mechanisms (item 2 of Theorem 5). In the cases e-h we get spherical focal mechanisms where both couplers are spherical isograms (item 1 and 2 of Theorem 5).

Now we can discuss the general case: W.l.o.g. we can set \( c_{00} := Bc_{22}d_{02}c_{02} \) and \( d_{00} := Bd_{22}c_{20}d_{20} \) with \( B \in \mathbb{R} \setminus \{0\} \). Then \( Q_{24} \) and \( Q_{20} \) can only vanish w.c. if their common factor
\[ 4d_{20}c_{02}(c_{22}d_{02} - d_{22}c_{20})(Bd_{22}c_{22} - 1) + c_{22}c_{02}d_{11}^2 - d_{22}c_{11}^2d_{20} \]
(12) vanishes. This condition is already sufficient for a reducible composition and it yields the general spherical focal mechanism case given in item 2 of Theorem 5.

2. \( g_{11} A - f_{11} = 0 \): W.l.o.g. we can set \( f_{11} = g_{11} A \). Moreover we can compute \( g_{11} \) from \( Q_{31} = 0 \) w.l.o.g.. Then \( Q_{13} \) can only vanish w.c. for \( W_1 = 0 \).

In the 8 special cases of \( W_1 = 0 \) it can easily be seen that the remaining equations cannot vanish w.c.. Therefore we only discuss the general case in more detail: W.l.o.g. we can set \( c_{00} := Bc_{22}d_{02}c_{02} \) and \( d_{00} := Ad_{22}c_{20}d_{20} \) with \( B \in \mathbb{R} \setminus \{0\} \). Now \( Q_{11} \) and \( Q_{33} \) can only vanish w.c. for:
\[ 4d_{20}(c_{22}Bd_{22} - 1)(d_{02} - c_{20}d_{22}^2B) + d_{11}^2(1 + c_{22}Bd_{22}) = 0. \]

Moreover \( Q_{24} \) and \( Q_{20} \) can only vanish w.c. for Eq. (12). If these two conditions are fulfilled then we already get a reducible composition. Clearly this yields a special spherical focal mechanism (item 2 of Theorem 5).
3.3.2 \[ g_{20} = +W_4/\sqrt{A}, \ g_{02} = +W_5/\sqrt{A} \text{ or } g_{20} = -W_4/\sqrt{A}, \ g_{02} = -W_5/\sqrt{A} \]

W.l.o.g. we can compute \( g_{22} \) and \( g_{00} \) from \( Q_{12} = 0 \) and \( Q_{02} = 0 \).

1. \( g_{11} A - f_{11} \neq 0 \): Under this assumption we can express \( f_{00} \) and \( f_{22} \) from \( Q_{11} = 0 \) and \( Q_{33} = 0 \) w.l.o.g.. Moreover we can compute \( f_{11} \) from \( Q_{31} = 0 \) w.l.o.g.. Then \( Q_{13} \) can only vanish w.c. for \( W_2 = 0 \). First of all we discuss again the special cases: \( W_2 = 0 \) holds only in the 6 cases given in Eq. (7) and (8) (without contradicting \( W_4 W_5 \neq 0 \)) if two variables out of the set \( \{c_{ij}, d_{ij}\} \) are equal to zero. It is very easy to verify that these cases do not yield a solution (the proof is left to the reader).

Here we only discuss the general case in more detail: W.l.o.g. we can set \( d_{00} := Bc_{02}c_{20}d_{20} \) and \( d_{02} := Ac_{22}c_{00}d_{22} \) with \( B \in \mathbb{R} \setminus \{0\} \). Now \( Q_{24} \) and \( Q_{20} \) can only vanish w.c. for \( d_{20}Bc_{11}^2 d_{22} - d_{11}^2 = 0 \), which can be solved for \( d_{22} \) w.l.o.g.. Then the resultant of \( Q_{00} \) and \( Q_{44} \) with respect to \( g_{11} \) can only vanish w.c. in the following two cases:

a. \( Bc_{02}c_{22} + 1 = 0 \): W.l.o.g. we can solve this equation for \( B \). Now the resultant of \( Q_{22} \) and the only non-contradicting factor of \( Q_{00} \) and \( Q_{44} \) with respect to \( g_{11} \) can only vanish w.c. for \( W_7 = 0 \). Therefore we set \( c_{00} = Lc_{02} \) and \( c_{20} = Lc_{22} \) with \( L \in \mathbb{R} \setminus \{0\} \). Then \( Q_{00} \) and \( Q_{44} \) imply the contradiction, as \( g_{11} = 0 \) yields \( f_{11} = 0 \).

b. \( W_7 = 0, Bc_{02}c_{22} + 1 \neq 0 \): W.l.o.g. we set \( c_{00} = Lc_{02} \) and \( c_{20} = Lc_{22} \) with \( L \in \mathbb{R} \setminus \{0\} \). Now the resultant of \( Q_{22} \) and the only non-contradicting factor of \( Q_{00} \) and \( Q_{44} \) with respect to \( g_{11} \) cannot vanish w.c..

2. \( g_{11} A - f_{11} = 0 \): W.l.o.g. we can set \( f_{11} = g_{11} A \). Moreover we can compute \( g_{11} \) from \( Q_{31} = 0 \) w.l.o.g.. Then \( Q_{13} \) can only vanish w.c. for \( W_2 = 0 \).

For the six special cases given in Eq. (7) and (8) it can easily be seen that the remaining equations cannot vanish w.c. Moreover we can discuss the general case in more detail: W.l.o.g. we can set \( d_{00} := Bc_{02}c_{20}d_{20} \) and \( d_{02} := Bc_{22}c_{00}d_{22} \) with \( B \in \mathbb{R} \setminus \{0\} \). Now \( Q_{24} \) and \( Q_{20} \) can only vanish w.c. for \( d_{20}Bc_{11}^2 d_{22} - d_{11}^2 = 0 \), which can be solved for \( d_{22} \) w.l.o.g.. Then we can compute \( c_{20} \) from \( Q_{33} = 0 \) w.l.o.g.. Finally \( Q_{11} \) cannot vanish w.c..

4 Excluded cases of the symmetric reducible composition

4.1 \( c_{22} = c_{02} = 0 \)

As we compute the resultant \( X \) of \( C \) and \( D \) with respect to \( t_2 \) the coefficient of \( t_2^2 \) in \( D \) must not vanish. Moreover, as still at least one of the two polynomials \( F \) and \( G \) should correspond to a spherical coupler, we can stop the discussion if \( d_{22} = d_{20} = 0 \) or \( f_{11} = g_{11} = 0 \) hold.
Due to $Q_{44}$, $Q_{04}$ and $Q_{24}$ either $f_{22} = f_{02} = 0$ or $g_{22} = g_{02} = 0$ must hold. W.l.o.g. we can assume $f_{22} = f_{02} = 0$. Then two cases have to be distinguished:

1. $f_{11} = 0$: As a consequence we can assume $g_{11} \neq 0$ w.l.o.g.. Therefore we can express $f_{20}$ and $f_{00}$ from $Q_{31}$ and $Q_{11}$, respectively. Moreover due to $c_{11}d_{11} \neq 0$ we can compute $g_{22}$ from $Q_{42}$, $g_{20}$ from $Q_{40}$, $g_{02}$ from $Q_{02}$ and $g_{00}$ from $Q_{00}$. Now the remaining two conditions $Q_{22}$ and $Q_{20}$ can only vanish w.c. for $d_{00} = d_{02} = 0$ (item 3 of Theorem 5). From the other excluded case we get the second possibility $d_{22} = d_{20} = c_{00} = c_{20} = 0$ given in item 3 of Theorem 5.

2. $f_{11} \neq 0$: From $Q_{33} = 0$ and $Q_{13} = 0$ we get $g_{22} = g_{02} = 0$. Moreover we can compute $g_{20}$ from $Q_{31}$, $g_{11}$ from $Q_{22}$ and $g_{00}$ from $Q_{11}$ w.l.o.g.. Then we distinguish again two cases:
   a. $d_{22} = 0$: Now only the three conditions $Q_{40} = 0$, $Q_{20} = 0$ and $Q_{00} = 0$ remain. It can easily be seen that there exists a reducible composition if $d_{00} = 0$ ($\Rightarrow \mathcal{D}$ is a spherical isogram; item 1 of Theorem 5) or if $d_{02}d_{00}c_{21}^2 + 4d_{02}c_{20}c_{00}d_{20} - c_{00}d_{11}^2c_{20} = 0$ with $d_{02} \neq 0$ holds ($\Rightarrow$ special spherical focal mechanism; item 2 of Theorem 5).
   b. $d_{22} \neq 0$: Then $Q_{42}$ and $Q_{02}$ can only vanish w.c. for $c_{00} = c_{20} = 0$ ($\Rightarrow \mathcal{C}$ degenerates into a special spherical isogram as $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = \pi/2$ hold). The remaining conditions $Q_{40} = 0$, $Q_{20} = 0$ and $Q_{00} = 0$ can only vanish w.c. for:
      i. $d_{00} = d_{02} = 0$: This is a special case of item 1 and 3 of Theorem 5.
      ii. $d_{00} = f_{20} = f_{00} = 0$: We get a special case of item 1 of Theorem 5.

4.2 $c_{22} = c_{02} = d_{22} = d_{20} = 0$

In this very special case the resultant yields:

$$X := c_{11}d_{00}t_1 - c_{00}d_{11}t_3 + c_{11}d_{02}t_1t_3^2 - c_{20}d_{11}t_3t_1^2.$$  

This expression cannot have a reducible compositions of the form $X = FG$ with $F$ and $G$ of Eq. (6).

It can immediately be seen that this expression has only a reducible composition with a spherical coupler component if and only if $d_{00} = 0$ or $c_{00} = 0$ holds and therefore at least one of the couplers is a spherical isogram (item 1 of Theorem 5). For the special case $d_{00} = c_{00} = 0$ we get a spherical focal mechanism where $\mathcal{C}$ and $\mathcal{D}$ are spherical isograms (item 1 and 2 of Theorem 5). This finishes the proof of Theorem 5. $\square$

Note that this case has not to be considered any more for the two asymmetric reducible compositions, which are discussed in the following two sections.
5 First asymmetric reducible composition

Computation of \(Q_{04}\) and \(Q_{40}\) shows that \(W_4 = W_5 = 0\) must hold. First of all we discuss the special cases:

5.1 Special cases

We distinguish the following 4 groups:

1. \(C\) and \(D\) are spherical isograms:
   \[
   d_{00} = d_{22} = c_{02} = c_{20} = 0, \quad c_{00} = c_{22} = d_{02} = d_{20} = 0.
   \]
   We only discuss the first case (for the other we refer to analogy): Due to \(Q_{42}\) and \(Q_{24}\) we get \(g_{31} = g_{13} = 0\). Moreover we get \(g_{20} = g_{02} = 0\) from \(Q_{31}\) and \(Q_{13}\), respectively. Then we can express \(g_{33}\) from \(Q_{44}, g_{22}\) from \(Q_{33}, g_{11}\) from \(Q_{22}\) and \(g_{00}\) from \(Q_{11}\) w.l.o.g.. Then the equation \(Q_{00} = 0\) remains, which is a homogeneous quartic equation in \(f_{00}\) and \(f_{11}\). This equation can be solved w.l.o.g. for \(f_{00}\). This yields a spherical focal mechanism where \(C\) and \(D\) are spherical isograms (item 1 and 2 of Theorem 5).

2. \(C\) is a spherical isogram and \(D\) not: For \(c_{00} = c_{22} = d_{02} = c_{20} = 0\) the equations \(Q_{44} = Q_{42} = Q_{24} = 0\) imply \(g_{31} = g_{13} = g_{33} = 0\), a contradiction.

   For the second possibility \(c_{02} = c_{20} = d_{00} = c_{00} = 0\) we get \(g_{31} = 0\) from \(Q_{44} = 0\). Moreover we can compute \(g_{33}\) from \(Q_{44}, g_{22}\) from \(Q_{33}, g_{11}\) from \(Q_{22}, g_{20}\) from \(Q_{31}\) and \(g_{13}\) from \(Q_{24}\) w.l.o.g.. Then we distinguish 2 cases:
   a. \(f_{00} = 0\): The remaining two equations \(Q_{13}\) and \(Q_{11}\) imply \(g_{02} = g_{00} = 0\) (item 1 of Theorem 5).
   b. \(f_{00} \neq 0\): \(Q_{02}\) and \(Q_{00}\) imply \(g_{02} = g_{00} = 0\). Then \(Q_{13}\) cannot vanish w.c.

3. \(D\) is a spherical isogram and \(C\) not: This can be done analogously to 2.

4. \(C\) and \(D\) are no spherical isograms: For the cases
   \[
   d_{00} = c_{20} = d_{02} = c_{00} = 0, \quad c_{22} = c_{20} = d_{02} = d_{22} = 0.
   \]
   we immediately get a contradiction as \(Q_{44} = 0, Q_{42} = 0\) and \(Q_{24} = 0\) imply \(g_{31} = g_{13} = g_{33} = 0\).

   For the third case \(d_{00} = d_{20} = c_{02} = c_{00} = 0\) we distinguish two cases:
   a. \(f_{00} = 0\): Now we get \(g_{00} = g_{02} = g_{11} = g_{20} = 0\) from \(Q_{31} = 0, Q_{22} = 0, Q_{13} = 0\) and \(Q_{11} = 0\). The remaining four equations can be solved for \(g_{33}, g_{31}, g_{22},\) and \(g_{13}\) w.l.o.g.. This yields a special spherical focal mechanism (item 2 of Theorem 5).
   b. \(f_{00} \neq 0\): Due to \(Q_{20} = 0, Q_{02} = 0\) and \(Q_{00} = 0\) we get \(g_{20} = g_{02} = g_{00} = 0\). Then \(Q_{31} = 0, Q_{13} = 0\) and \(Q_{11} = 0\) imply \(g_{31} = g_{13} = g_{11} = 0\). Moreover \(Q_{42} = 0, Q_{24} = 0\) and \(Q_{22} = 0\) can only vanish w.c. for \(g_{22} = c_{20} = d_{02} = 0\). Now \(Q_{44}\) and \(Q_{33}\) cannot vanish w.c.
5.2 Semispecial cases

1. In the first part we set \( d_{00} = Ac_{20} \) and \( d_{20} = Ac_{22} \) with \( A \in \mathbb{R} \setminus \{0\} \) and \( c_{20}c_{22} \neq 0 \). For the following 4 cases we can assume \( f_{00} \neq 0 \) w.l.o.g. because for \( f_{00} = 0 \) the equation \( Q_{20} = 0 \) cannot vanish w.c.

a. \( c_{00} = c_{02} = 0 \): We get \( g_{00} = g_{02} = 0 \) from \( Q_{02} = 0 \) and \( Q_{00} = 0 \), respectively. Moreover we can compute \( g_{33} \) from \( Q_{44} \), \( g_{31} \) from \( Q_{42} \), \( g_{22} \) from \( Q_{33} \), \( g_{20} \) from \( Q_{31} \), \( g_{13} \) from \( Q_{24} \) and \( g_{11} \) from \( Q_{22} \). Then \( Q_{20} = 0 \) and \( Q_{00} = 0 \) imply \( f_{00} = -Ac_{11}f_{11}/d_{11} \). Then the remaining two equations can only vanish w.c. for \( d_{22} = d_{02} = 0 \). This corresponds to a special spherical focal mechanism (item 2 of Theorem 5).

b. \( d_{22} = d_{02} = 0 \): This case can be done analogously. Finally in this case we end up with the equation \( Ad_{02}c_{11}^2 - c_{00}d_{11}^2 = 0 \).

c. \( c_{00} = d_{02} = 0 \): We get \( g_{13} = 0 \) from \( Q_{24} \) and \( g_{02} = 0 \) from \( Q_{02} \). Moreover we can compute \( g_{33} \) from \( Q_{44} \), \( g_{31} \) from \( Q_{42} \), \( g_{22} \) from \( Q_{33} \), \( g_{20} \) from \( Q_{31} \) and \( g_{11} \) from \( Q_{22} \). Then \( Q_{20} = 0 \) and \( Q_{00} = 0 \) imply \( f_{00} = -Ac_{11}f_{11}/d_{11} \) and \( g_{00} = -Ad_{02}c_{20}c_{22}^2/(c_{11}f_{11}) \), respectively. Finally one equation remains: \( Ad_{22}c_{11}^2 - c_{02}d_{11}^2 = 0 \). This also yields a special spherical focal mechanism (item 2 of Theorem 5).

d. \( d_{22} = c_{02} = 0 \): This case can be done analogously. Finally in this case we end up with the equation \( Ad_{02}c_{11}^2 - c_{00}d_{11}^2 = 0 \).

2. For the second semispecial case \( d_{22} = Bc_{02} \), \( d_{02} = Bc_{00} \) with \( B \in \mathbb{R} \setminus \{0\} \) and \( c_{00}c_{02} \neq 0 \) we refer to analogy.

5.3 General case

W.l.o.g. we can set \( d_{00} = Ac_{20} \), \( d_{20} = Ac_{22} \), \( d_{22} = Bc_{02} \) and \( d_{02} = Bc_{00} \) with \( A, B \in \mathbb{R} \setminus \{0\} \). We can compute \( g_{33} \) from \( Q_{44} \), \( g_{31} \) from \( Q_{42} \), \( g_{22} \) from \( Q_{33} \), \( g_{20} \) from \( Q_{31} \), \( g_{13} \) from \( Q_{24} \) and \( g_{11} \) from \( Q_{22} \). As for \( f_{00} = 0 \) the equation \( Q_{20} = 0 \) cannot vanish w.c. we can assume \( f_{00} \neq 0 \). Therefore we can express \( g_{02} \) from \( Q_{02} \) and \( g_{00} \) from \( Q_{00} \) w.l.o.g.. Then \( Q_{20} = 0 \) implies \( f_{00} = -Ac_{11}f_{11}/d_{11} \). Finally one equation remains, namely: \( ABCc_{11}^2 - d_{11}^2 = 0 \), which indicates a special spherical focal mechanism (item 2 of Theorem 5).

5.4 Excluded case

Now we discuss the case \( c_{22} = c_{02} = 0 \). The equations \( Q_{44} = 0 \) and \( Q_{24} = 0 \) imply \( g_{33} = g_{13} = 0 \). Then the equations \( Q_{33} = 0 \) and \( Q_{13} = 0 \) yield \( g_{22} = g_{02} = 0 \). Moreover we can assume \( c_{20}d_{22} \neq 0 \) because otherwise \( Q_{42} = 0 \) yields \( g_{31} = 0 \), a contradiction. Therefore the equations \( Q_{40} = 0 \) and \( Q_{02} = 0 \) imply
$d_{20} = c_{00} = 0 \implies C$ is a spherical isogram. We proceed by expressing $g_{31}$ from $Q_{42}$, $g_{20}$ from $Q_{31}$ and $g_{11}$ from $Q_{22}$. Then we distinguish two cases:

1. $f_{00} = 0$: In this case the remaining two equations can only vanish w.c. for $g_{00} = d_{00} = 0$. This yields a spherical focal mechanism where $C$ is a spherical isogram (item 1 and 2 of Theorem 5).
2. $f_{00} \neq 0$: Then $Q_{00} = 0$ implies $g_{00} = 0$ and from $Q_{11} = 0$ we get $d_{02} = 0$. Finally a homogeneous quadratic equation in $f_{00}, f_{11}$ remains which can be solved for $f_{00}$ w.l.o.g.. This also yields a spherical focal mechanism where $C$ is a spherical isogram (item 1 and 2 of Theorem 5).

6 Second asymmetric reducible composition

Computation of $Q_{44}$ and $Q_{40}$ shows that $c_{22}d_{02} - d_{22}c_{20} = 0$ and $W_5 = 0$, respectively, must hold. First of all we discuss the special cases:

6.1 Special cases

It can easily be seen that there only exists two possible special cases:

1. $d_{22} = d_{02} = 0$: In this case $Q_{24}$ cannot vanish w.c..
2. $d_{02} = c_{00} = c_{20} = 0$: Again $Q_{24} = 0$ yields the contradiction.

6.2 Semispecial cases

In this asymmetric case there is only one possible semispecial case, namely: $d_{02} := Ac_{20}, d_{22} := Ac_{22}$ and $c_{02} = c_{00} = 0$ with $A \in \mathbb{R} \setminus \{0\}$ and $c_{20}c_{22} \neq 0$. W.l.o.g. we can compute $g_{22}$ from $Q_{42} = 0$ and $g_{13}$ from $Q_{24} = 0$. In the following we distinguish two cases:

1. $f_{00} = 0$: Then $Q_{13} = 0$ and $Q_{11} = 0$ imply $g_{02} = g_{00} = 0$. Now we can compute $g_{20}$ from $Q_{40} = 0$ and $g_{11}$ from $Q_{22} = 0$ w.l.o.g.. Moreover we get $f_{20} = -d_{11}f_{11}/(Ac_{11})$ from $Q_{33}$. Finally, the remaining two equations can only vanish w.c. for $d_{00} = d_{20} = 0$. This yields a special case ($B = 0$) of the later given Theorem 6.
2. $f_{00} \neq 0$: $Q_{02} = Q_{00} = 0$ imply $g_{02} = g_{00} = 0$. Then $Q_{13}$ cannot vanish w.c.
6.3 General case

Due to the last two subsections we can set \( c_{20} := Ad_{02}, c_{22} := Ad_{22}, c_{02} := Bd_{22} \) and \( c_{00} := Bd_{02} \) with \( A, B \in \mathbb{R} \setminus \{0\} \) and \( d_{02} = 0 \). W.l.o.g., we can compute \( g_{22} \) from \( Q_{22} = 0 \), \( g_{20} \) from \( Q_{40} = 0 \) and \( g_{13} \) from \( Q_{24} = 0 \). Then \( Q_{33} \) can only vanish w.c. for \( f_{20} = -Ad_{11}/c_{11} \). Moreover we can express \( g_{02} \) from \( Q_{13} = 0 \). Then \( Q_{33} \) can only vanish w.c. for \( f_{20} = -Ad_{11}/c_{11} \). Now \( Q_{02} \) implies \( f_{00} = -Bd_{11}/c_{11} \) and from \( Q_{22} = 0 \) we can compute \( g_{11} \). W.l.o.g. then \( Q_{11} = 0 \) yields \( g_{00} = 0 \) and \( Q_{31} \) and \( Q_{00} \) can only vanish w.c. for \( W_{3} = 0 \). It can easily be seen that \( W_{3} = 0 \) and the last remaining equation \( Q_{20} = 0 \) can only vanish w.c. for \( d_{00} = d_{20} = 0 \). This yields a new reducible composition (cf. Theorem 6).

6.4 Excluded cases

The case \( c_{22} = c_{02} = 0 \) does not yield a solution as \( Q_{24} \) cannot vanish w.c. Therefore we consider the other excluded case \( d_{22} = d_{02} = 0 \). Now \( Q_{42} = 0 \) implies \( g_{22} = 0 \). Then \( Q_{33} \) cannot vanish w.c. End of all cases.

We sum up the results of Section 5 and 6 into the following theorem:

**Theorem 6** Beside special cases of the isogram type and focal type of Theorem 5 there exists one special asymmetric reducible composition with a spherical coupler component, namely the following:

\[
\begin{align*}
c_{20} &:= Ad_{02}, \quad c_{22} := Ad_{22}, \quad c_{02} := Bd_{22}, \quad c_{00} := Bd_{02}, \quad d_{00} = d_{20} = 0, \quad d_{02}d_{22} \neq 0 \\
d_{02} &:= Ac_{20}, \quad d_{22} := Ac_{22}, \quad d_{20} := Bc_{22}, \quad d_{00} := Bc_{20}, \quad c_{00} = c_{02} = 0, \quad c_{20}c_{22} \neq 0
\end{align*}
\]

with \( A \in \mathbb{R} \setminus \{0\} \) and \( B \in \mathbb{R} \).

7 Conclusion

A comparison of Theorem 5 and 6 with the known examples of reducible compositions with a spherical coupler component given in Subsection 1.2.1 shows that we have found 3 new cases; namely item 1 and 3 of Theorem 5 and the case of Theorem 6. We close the paper with the following concluding remarks:

- It can be proven by computation that the symmetric type given in Subsection 1.2.1 is a special case of the focal type (item 2 of Theorem 5).
- Clearly, the isogonal type given in Subsection 1.2.1 is a special case of the isogram type (item 1 of Theorem 5) but also of the focal type (item 2 of Theorem 5).
• Beside the compositions given in Subsection 1.2.1 also the orthogonal type [3] is known which is as follows: Two orthogonal four-bars are combined such that they have one diagonal in common (see [3, Fig. 5a]), i.e., under $\alpha_2 = \beta_1$ and $\delta_2 = -\delta_1$, hence $I_{30} = I_{10}$. Then the 4-4-correspondence between $A_1$ and $B_2$ is the square of a 2-2-correspondence of the form

$$s_{21}t_1^2 + s_{12}t_1t_3^2 + s_{10}t_1 + s_{01}t_3 = 0$$

(cf. [3]) and therefore this component cannot produce a transmission which equals that of a single spherical coupler. A paper on such reducible compositions without a spherical coupler component is in preparation. At this point it should only be noted that the given orthogonal type can be generalized as follows:

$d_{00} := Ac_{20}, d_{20} := Ac_{22}, d_{02} := Bc_{02}, d_{22} := Bc_{00}$ with $A, B \in \mathbb{R}$ and $C$ being an orthogonal coupler.

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