ON ELLIPTIC SELF-MOTIONS OF PLANAR PROJECTIVE STEWART GOUGH PLATFORMS

Georg Nawratil
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Austria
Email: nawratil@geometrie.tuwien.ac.at

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ABSTRACT

It has been previously shown that non-architecturally singular parallel manipulators of Stewart Gough type, where the planar platform and the planar base are related by a projectivity, have either so-called elliptic self-motions or pure translational self-motions. As the geometry of all manipulators with translational self-motions is already known, we focus on elliptic self-motions. We show that these necessarily one-parameter self-motions have a second, instantaneously local, degree of freedom in each pose of the self-motion. Moreover, we introduce a geometrically motivated classification of elliptic self-motions and study the so-called orthogonal ones in detail.

Keywords: Self-motion, Stewart Gough platform, Borel Bricard problem.

1. INTRODUCTION

The geometry of a planar Stewart Gough (SG) platform is given by the six base anchor points M_i ($i = 1,\ldots,6$) with coordinates $\mathbf{M}_i := (A_i, B_i)^T$ with respect to the xy-plane π_{M} of the fixed system and by the six platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i)^T$ with respect to the xy-plane π_{m} of the moving system. If the geometry of the manipulator is given as well as the six leg lengths, then the SG platform is in general rigid, but under particular conditions, it can perform an n-parametric motion ($n > 0$), which is called self-motion. Note, that these motions are also solutions of the famous Borel Bricard problem (cf. [1–3]).

It is well known, that planar SG platforms, which are singular in every possible configuration, possess self-motions in each pose (over $\mathbb{C}$). These so-called architecturally singular planar SG platforms were extensively studied in [4–7]. Therefore, we are only interested in self-motions of planar SG platforms, which are not architecturally singular.

In this paper, we discuss the case where the base anchor points M_i and the platform anchor points m_i are related by a non-singular projectivity κ; i.e.

$$\kappa: \quad \pi_{\mathsf{m}} \rightarrow \pi_{\mathsf{M}} \quad \text{with} \quad \mathsf{m}_i \mapsto \mathsf{M}_i. \tag{1}$$

For the remainder of this article, we call these manipulators planar projective SG platforms.

Remark 1 *A non-singular projectivity κ is a bijective linear mapping between two planes, which are projectively extended; i.e. we also consider the ideal points of these planes. Moreover, κ preserves incidences and cross-ratios. Finally it should be mentioned, that projectivites, which map all ideal points again onto ideal points, are so-called affinities (or affine mappings).* ◇

It is well known (cf. Chasles [8]), that a planar projective SG platform is architecturally singular if and only if one set of anchor points is located on a conic section, which can also be reducible. Moreover, it was shown by the author in [9], that one can attach a two-parametric set $\mathscr{L}$ of additional legs to planar projective SG platforms without changing the forward kinematics and singularity surface. The platform anchor points n_i and the base anchor points N_i of these additional legs are also related by κ, i.e. $\mathsf{n}_i\kappa = \mathsf{N}_i$.

Under the assumption that s denotes the line of intersection of π_{M} and π_{m} in the projective extension of the Euclidean 3-space, an elliptic self-motion can be defined as follows (cf. Def. 1 of [9]):

Definition 1 *A self-motion of a non-architecturally singular planar projective SG platform is called elliptic, if in each pose of this motion s exists with $\mathsf{s} = \mathsf{s}\kappa$ and the projectivity from s onto itself is elliptic ($\Leftrightarrow$ the projectivity from s onto itself has no real fixed points).*

It should be noted, that Definition 1 implies that neither π_{M} and π_{m} nor two related points of the platform and the base coincide during an elliptic self-motion.

It was also shown by the author in [9], that non-architecturally singular planar projective SG platforms can either have elliptic self-motions or pure translational ones. The latter are the only self-motions of non-architecturally singular planar affine SG platforms.[1] In this case, the projectivity has to be an affinity $\mathbf{a} + \mathbf{A}\cdot\mathbf{x}$, where the singular values s_1 and s_2 of the 2×2 transformation matrix $\mathbf{A}$ with $0 < s_1 \leq s_2$ fulfill the condition $s_1 \leq 1 \leq s_2$. But, it should be mentioned, that all planar affine SG platforms, which do not fulfill this condition, only have "*ordinary*" singularities causing a local shakiness of the manipulator but no actual self-motion.

[1] Corresponding anchor points of the planar platform and the planar base are related by an affinity (cf. Remark 1).

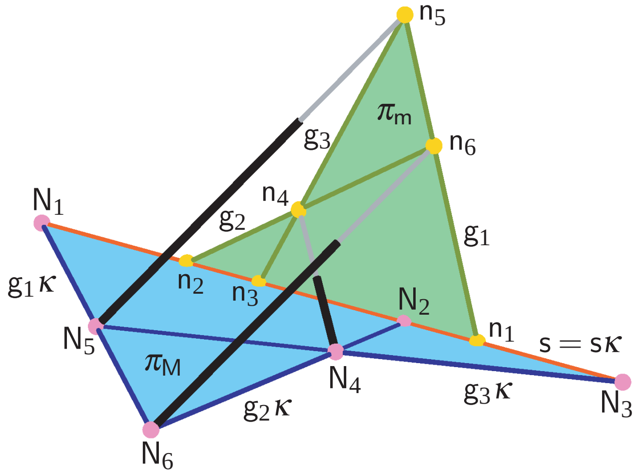

Figure 1: Sketch and notation of points and lines used for the proof of Theorem 1. Note, that the projectivity from s onto itself is determined by $\mathsf{n}_i \mapsto \mathsf{N}_i$ for $i = 1,2,3$. As it is elliptic, it does not have real fixed points.

Note, that a self-motion is dangerous because it is uncontrollable thus a hazard to man and machine. Therefore being able to avoid manipulator designs that engender self-motion is of interest to engineers. Is the large class of planar projective SG manipulators, where κ is no affinity, free of self-motions? An answer to this question, which is given at the very end of the paper (cf. Sect. 7), is based on the following study of non-architecturally singular planar projective SG platforms with elliptic self-motions. Let us start with the following basic result:

Theorem 1 *If a non-architecturally singular planar projective SG platform allowing an elliptic self-motion exists, then it possesses in each pose of the elliptic self-motion exactly two instantaneous degrees of freedom.*

Proof: Due to Lemma 1 of [9] and the results of [10], we can replace the original six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ with $i = 1,\ldots,6$ by a new set of six legs $\overline{\mathsf{n}_i\mathsf{N}_i}$ without changing the direct kinematics and singularity surface, if $\mathsf{n}_i\kappa = \mathsf{N}_i$ holds and $\mathsf{n}_1,\ldots,\mathsf{n}_6$ are not located on a conic section. Therefore, $\mathsf{n}_1,\ldots,\mathsf{n}_6$ can be selected as follows (cf. Fig. 1): We choose three lines $\mathsf{g}_1,\mathsf{g}_2,\mathsf{g}_3 \in \pi_\mathsf{m}$ so that $\mathsf{g}_1,\mathsf{g}_2,\mathsf{g}_3,\mathsf{s}$ are pairwise distinct and that no three of them belong to a pencil of lines. Then, we can define n_i as the intersection point of g_i and $\mathsf{s} = \mathsf{s}\kappa$ for $i = 1,2,3$. Moreover, the intersection point of g_i and g_j is denoted by n_{k+3} with pairwise distinct $i,j,k \in \{1,2,3\}$. By applying κ to $\mathsf{n}_1,\ldots,\mathsf{n}_6$, we get the corresponding base anchor points $\mathsf{N}_1,\ldots,\mathsf{N}_6$. The resulting planar projective SG platform $\mathsf{n}_1,\ldots,\mathsf{N}_6$ is not architecturally singular, as n_4, n_5 and n_6 are not collinear.

As the three legs $\overline{\mathsf{n}_1\mathsf{N}_1}$, $\overline{\mathsf{n}_2\mathsf{N}_2}$, $\overline{\mathsf{n}_3\mathsf{N}_3}$ coincide with $\mathsf{s} = \mathsf{s}\kappa$, we get at least two instantaneous degrees of freedom (dofs). Moreover, due to the above given construction, the remaining three legs $\overline{\mathsf{n}_4\mathsf{N}_4}$, $\overline{\mathsf{n}_5\mathsf{N}_5}$, $\overline{\mathsf{n}_6\mathsf{N}_6}$ are pairwise skew. Therefore, the carrier lines of these legs span a regulus $\mathscr{R}$ of a regular ruled quadric, based on a well known fact from line geometry (e.g. page 182 of [11]).

Due to this fact, there only exist three instantaneous dofs if $\mathsf{s} = \mathsf{s}\kappa$ also belongs to $\mathscr{R}$. In the following, we show by contradiction that this cannot be the case:

If we assume that $\mathsf{s} = \mathsf{s}\kappa$ belongs to $\mathscr{R}$, then each plane ε through $\mathsf{s} = \mathsf{s}\kappa$ has to intersect $\mathscr{R}$ in a conic, which splits into $\mathsf{s} = \mathsf{s}\kappa$ and another line t. By setting $\varepsilon = \pi_\mathsf{m}$ we see, that t has to contain the points n_4, n_5 and n_6, which already yields the contradiction, as these points are not collinear. □

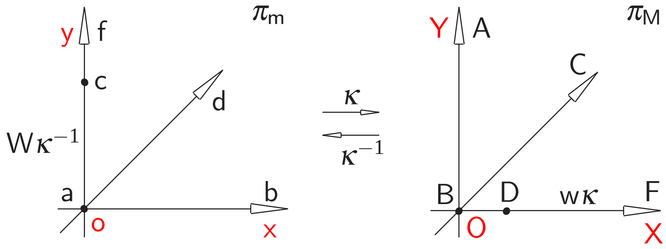

Figure 2: Sketch of basic quadrangles $(a,b,c,d) \in \pi_m$ and $(A,B,C,D) \in \pi_M$ of κ and the Cartesian coordinate systems (o,x,y) in π_m and (O,X,Y) in π_M.

Theorem 1 shows that elliptic self-motions are highly singular motions, if indeed non-architecturally singular planar projective SG platforms with these motions even exist.[2] We try to shed light on this still open question within this paper, which is structured as follows:

In Sect. 2, we present the basic quadrangles of κ and their coordinatization. In Sect. 3, we give the analytical version of the Darboux constraints and Mannheim constraints determined by the basic quadrangles and we introduce the so-called angle constraint. Based on this mathematical framework, we study the geometry of elliptic self-motions in Sect. 4, which also yields a deeper understanding of Theorem 1. In Sect. 5, we give two approaches to the determination of planar projective SG platforms with a special class of elliptic self-motions; namely orthogonal ones. Moreover, in Sect. 6 we present the main theorem on orthogonal elliptic self-motions and in Sect. 7 we give a conjecture for the non-orthogonal case and some final remarks.

2. BASIC QUADRANGLES OF κ

The projectivity κ of the planar projective SG manipulator $m_1,\ldots,M_6$ is given. It is well known, that κ is uniquely determined by any pair of quadrangles, which correspond within κ. In the following, we choose special quadrangles, on which we base the coordinatization of the manipulator in order to get the most compact equations in the computations given later (cf. Sect. 3-6). Without loss of generality (w.l.o.g.) the selection can be done as follows (cf. Fig. 2):

Assume w and W denote the ideal line of π_m and π_M, respectively. Due to Sect. 1, we can assume that κ is no affinity, i.e. $w\kappa \neq W$. Therefore, $W\kappa^{-1}$ and $w\kappa$ are finite lines. We denote the ideal point of $W\kappa^{-1}$ by f, which is mapped under κ onto the ideal point F of $w\kappa$.

Further, we denote the infinite vertex of the pencil of lines orthogonal to $w\kappa$ by A. Analogously, the infinite vertex of the pencil of lines orthogonal to $W\kappa^{-1}$ is denoted by b. Therefore, $a := A\kappa^{-1}$ and $B := b\kappa$ are finite points of π_m and π_M, respectively. Then, we choose a point c with $f \neq c \neq a$ on $W\kappa^{-1}$ and a point D with $F \neq D \neq B$ on $w\kappa$. We can still specify the points c and D by assuming that the angles $\sphericalangle(d,a,f)$ and $\sphericalangle(F,B,C)$ equal $\pi/4$ with $d := D\kappa^{-1}$ and $C := c\kappa$. This yields our basic quadrangles (a,b,c,d) and (A,B,C,D) of κ.

As the basic quadrangles are related by κ, the legs, which are spanned by corresponding vertices, can be added to the planar projective SG platform $m_1,\ldots,M_6$ without changing the direct kinematics and singularity surface. Therefore, we can attach the special legs $\overline{aA},\ldots,\overline{dD} \in \mathscr{L}$. It was shown by the author in [12], that the attachment of the special leg $\overline{aA}$ (resp. $\overline{cC}$) corresponds with the so-called Darboux constraint, that the platform anchor point a (resp. c) moves in a plane of the fixed system orthogonal to the direction of the ideal

[2] Until now, no example is known to the author.

point A (resp. C). The attachment of the special leg $\overline{\mathsf{bB}}$ (resp. $\overline{\mathsf{dD}}$) corresponds with the so-called Mannheim constraint, that a plane of the moving system orthogonal to b (resp. d) slides through the point B (resp. D).

2.1. Coordinatization of the basic quadrangles

As we are dealing with planar projective SG platforms, we have to consider the projective extension of π_{m} and π_{M}, i.e.

$$(a_i, b_i)^T \mapsto (w_i : x_i : y_i)^T \quad \text{and} \quad (A_i, B_i)^T \mapsto (W_i : X_i : Y_i)^T. \tag{2}$$

Note, that ideal points are characterized by $w_i = 0$ and $W_i = 0$, respectively.

W.l.o.g. we can choose the following system $(\mathsf{O}, \mathsf{X}, \mathsf{Y})$ of Cartesian coordinates in π_{M} (cf. Fig. 2): B is the origin O , F is the ideal point X of the x-axis and A the ideal point Y of the y-axis. This yields the following homogeneous Cartesian coordinates for the quadrangle $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$:

$$\mathbf{A} = (0:0:1)^T, \quad \mathbf{B} = (1:0:0)^T, \quad \mathbf{C} = (0:1:1)^T, \quad \mathbf{D} = (1:\alpha:0)^T, \tag{3}$$

where we can additionally assume w.l.o.g. that $\alpha > 0$ holds. Moreover, we can choose the following system $(\mathsf{o}, \mathsf{x}, \mathsf{y})$ of Cartesian coordinates in π_{m} w.l.o.g. (cf. Fig. 2): a is the origin o, b is the ideal point x of the x-axis and f the ideal point y of the y-axis. This yields the following homogeneous Cartesian coordinates for the quadrangle $(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d})$:

$$\mathbf{a} = (1:0:0)^T, \quad \mathbf{b} = (0:1:0)^T, \quad \mathbf{c} = (1:0:\beta)^T, \quad \mathbf{d} = (0:1:1)^T. \tag{4}$$

In addition, we can eliminate the factor of similarity by setting $\alpha = 1$. As a consequence κ only depends on $\beta \in \mathbb{R} \setminus \{0\}$ and the matrix $\mathbf{P}$ (resp. $\mathbf{P}^{-1}$) of κ (resp. κ^{-1}) equals:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta & 0 & 0 \end{pmatrix} \mathbb{R}, \qquad \mathbf{P}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \beta & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \mathbb{R}. \tag{5}$$

3. MATHEMATICAL FRAMEWORK

For the determination of self-motions, it is advantageous to work in the Study parameter space $P^7_{\mathbb{R}}$, which is a 7-dimensional real projective space with homogeneous coordinates $e_0, \ldots, e_3, f_0, \ldots, f_3$. By using these so-called Study parameters for the parametrization of Euclidean displacements, the coordinates $\mathbf{m}'_i$ of the platform anchor points with respect to fixed system can be written as $K\mathbf{m}'_i = \mathbf{R}\,\mathbf{m}_i + (t_1, t_2, t_3)^T$ with

$$\begin{aligned} t_1 &:= 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \\ t_2 &:= 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3), \\ t_3 &:= 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1), \end{aligned}$$

and the rotation matrix $\mathbf{R} := (r_{ij})$ (i denotes the row and j the column) given by:

$$\begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix},$$

with Euler parameters $e_0, \ldots, e_3$. Now, all points of $P^7_{\mathbb{R}}$, which are located on the so-called Study quadric $\Phi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to Euclidean displacements with exception of the subspace $e_0 = \ldots = e_3 = 0$ of Φ, as these points cannot fulfill the normalizing condition $K = 1$ with $K := e_0^2 + e_1^2 + e_2^2 + e_3^2$.

The formulas for the analytical versions of the Darboux and Mannheim constraints were given by the author in [12]. If we introduce the coordinates of our anchor points $\mathsf{a},\ldots,\mathsf{d},\mathsf{A},\ldots,\mathsf{D}$ of Eqs. (3) and (4) into these formulas, we get the following expressions for the two Darboux constraints:

$$\Omega_{\mathsf{A}}^{\mathsf{a}}:\ t_2+L_aK=0,\qquad \Omega_{\mathsf{C}}^{\mathsf{c}}:\ t_1+t_2+L_cK+\beta(r_{12}+r_{22})=0, \tag{6}$$

with variables $L_a,L_c\in\mathbb{R}$ and the following expressions for the two Mannheim constraints:

$$\Pi_{\mathsf{B}}^{\mathsf{b}}:\ \bar{t}_1+g_bK=0,\qquad \Pi_{\mathsf{D}}^{\mathsf{d}}:\ \bar{t}_1+\bar{t}_2+g_dK-(r_{11}+r_{12})=0, \tag{7}$$

with variables $g_b,g_d\in\mathbb{R}$ and

$$\bar{t}_1:=2(e_0f_1-e_1f_0-e_2f_3+e_3f_2),\quad \bar{t}_2:=2(e_0f_2+e_1f_3-e_2f_0-e_3f_1).$$

Therefore, we get four homogeneous quadratic equations in Study parameters which are linear in $f_0,\ldots,f_3$.

3.1. The angle constraint

In the following, we attach a further leg $\overline{\mathsf{gG}}$ of $\mathscr{L}$ to the manipulator with $\mathbf{g}=(1:0:-\beta)^T$ and $\mathbf{G}=(0:-1:1)^T$, which gives rise to a third Darboux constraint:

$$\Omega_{\mathsf{G}}^{\mathsf{g}}:\ -t_1+t_2+L_gK+\beta(r_{12}-r_{22})=0,$$

with the variable $L_g\in\mathbb{R}$. Now, the linear combination

$$-\Omega_{\mathsf{A}}^{\mathsf{a}}/\beta+\Omega_{\mathsf{C}}^{\mathsf{c}}/(2\beta)+\Omega_{\mathsf{G}}^{\mathsf{g}}/(2\beta) \tag{8}$$

equals $r_{12}-\gamma_\Omega K$ with $\gamma_\Omega:=(2L_a-L_c-L_g)/(2\beta)$.

Moreover, we attach a further leg $\overline{\mathsf{hH}}$ of $\mathscr{L}$ with $\mathbf{h}=(0:1:-1)^T$ and $\mathbf{H}=(1:-1:0)^T$, which causes a third Mannheim constraint:

$$\Pi_{\mathsf{H}}^{\mathsf{h}}:\ \bar{t}_1-\bar{t}_2+g_hK+(r_{11}-r_{12})=0,$$

with the variable $g_h\in\mathbb{R}$. Now, the linear combination

$$-\Pi_{\mathsf{B}}^{\mathsf{b}}+\Pi_{\mathsf{D}}^{\mathsf{d}}/2+\Pi_{\mathsf{H}}^{\mathsf{h}}/2 \tag{9}$$

equals $r_{12}-\gamma_\Pi K$ with $\gamma_\Pi:=-(2g_b-g_d-g_h)/2$.

As the anchor points $\mathsf{a},\ldots,\mathsf{d},\mathsf{g},\mathsf{h}$ and $\mathsf{A},\ldots,\mathsf{D},\mathsf{G},\mathsf{H}$ are located on two lines (degenerate conic), the manipulator, defined by these six pairs of anchor points, is architecturally singular. This is the reason for the existence of the following linear combination:

$$2\Omega_{\mathsf{A}}^{\mathsf{a}}-\Omega_{\mathsf{C}}^{\mathsf{c}}-\Omega_{\mathsf{G}}^{\mathsf{g}}-2\beta\Pi_{\mathsf{B}}^{\mathsf{b}}+\beta\Pi_{\mathsf{D}}^{\mathsf{d}}+\beta\Pi_{\mathsf{H}}^{\mathsf{h}}=K(2L_a-L_c-L_g+2\beta g_b-\beta g_d-\beta g_h),$$

which immediately implies that $\gamma:=\gamma_\Omega=\gamma_\Pi$ has to hold. Clearly, due to the projectivity, the resulting special condition $r_{12}-\gamma K=0$ can be generated by the linear combination of any three of either Darboux or Mannheim constraints, which correspond with legs of $\mathscr{L}$.

Moreover, it can easily be verified that this remarkable equation has the following geometric interpretation: The y-axis of the moving frame (with ideal point f) and the x-axis of the fixed frame (with ideal point F) subtend a constant angle $\arccos(\gamma)$. Therefore, we introduce the following notation:

$$\sphericalangle_{\mathsf{F}}^{\mathsf{f}}:\ r_{12}-\gamma K=0, \tag{10}$$

where the variable $\gamma\in]-1,1[$ has to hold.[3] This condition is a homogeneous quadratic equation in Euler parameters. Finally, it should be noted, that this equation is a special case of Eq. (3) of [2].

[3] As $\gamma=\pm1$ implies $\mathsf{f}=\mathsf{F}$ (= fixed point) this contradicts the definition of an elliptic self-motion. For this case, please see [9].

4. ON THE GEOMETRY OF ELLIPTIC SELF-MOTIONS

Based on the same arguments as in the proof of Theorem 1, we can replace the original six legs $\overline{\mathsf{m}_i\mathsf{M}_i}$ with $i = 1,\dots,6$ by a new set of six legs $\overline{\mathsf{aA}},\dots,\overline{\mathsf{dD}},\overline{\mathsf{fF}}$ and $\overline{\mathsf{mM}}$, where m and M has to be chosen in a way that the manipulator is not architecturally singular. E.g. we can set:

$$\mathbf{m} = (1 : -\beta : 0)^T \quad \text{and} \quad \mathbf{M} = (1 : 0 : -1)^T. \tag{11}$$

Note, that for this selection the following triples of anchor points are collinear:

$$(\mathsf{m},\mathsf{c},\mathsf{d}) \Leftrightarrow (\mathsf{M},\mathsf{C},\mathsf{D}), \quad (\mathsf{m},\mathsf{a},\mathsf{b}) \Leftrightarrow (\mathsf{M},\mathsf{A},\mathsf{B}), \quad (\mathsf{b},\mathsf{d},\mathsf{f}) \Leftrightarrow (\mathsf{B},\mathsf{D},\mathsf{F}), \quad (\mathsf{a},\mathsf{c},\mathsf{f}) \Leftrightarrow (\mathsf{A},\mathsf{C},\mathsf{F}).$$

These collinearity properties imply, under consideration of Lemma 2 of Karger [4], that the resulting 6-legged manipulator cannot be architecturally singular. Based on this special 6-legged manipulator, we can prove the following theorem.

Theorem 2 *If a non-architecturally singular planar projective SG platform allowing an elliptic self-motion exists, then the elliptic self-motion has to be a one-parametric motion.*

Proof: By removing the leg $\overline{\mathsf{mM}}$, there remains a 5-legged manipulator $\overline{\mathsf{aA}},\dots,\overline{\mathsf{dD}},\overline{\mathsf{fF}}$. In the following we show, that this 5-legged manipulator cannot have a two-parametric motion. This is equivalent with the statement, that the six hyperquadrics $\Omega_{\mathsf{A}}^{\mathsf{a}}$, $\Omega_{\mathsf{C}}^{\mathsf{d}}$, $\Pi_{\mathsf{B}}^{\mathsf{b}}$, $\Pi_{\mathsf{D}}^{\mathsf{d}}$, $\triangleleft_{\mathsf{F}}^{\mathsf{f}}$, Φ in the Study parameter space, cannot have a surface in common. This can be proven as follows:

We solve Φ together with three of the following four conditions $\Omega_{\mathsf{A}}^{\mathsf{a}}$, $\Omega_{\mathsf{C}}^{\mathsf{d}}$, $\Pi_{\mathsf{B}}^{\mathsf{b}}$, $\Pi_{\mathsf{D}}^{\mathsf{d}}$ for $f_0,\dots,f_3$. Then, we introduce the resulting values into the remaining equation, which yields for all four possible cases the same polynomial $\Lambda[72]$, where the number in the brackets gives the number of terms.

We proceed by computing the resultant Γ of $\triangleleft_{\mathsf{F}}^{\mathsf{f}}$ and Λ with respect to e_3; i.e. $\Gamma[684] := Res_{e_3}(\triangleleft_{\mathsf{F}}^{\mathsf{f}},\Lambda)$. Now, all coefficients of Γ, with respect to the remaining Study parameters e_0, e_1 and e_2, have to vanish identically. The coefficient of e_1^8 of Γ equals γ and therefore we get a contradiction, as γ is the coefficient of the highest exponent of e_3 in $\triangleleft_{\mathsf{F}}^{\mathsf{f}}$ (cf. Eq. (1.1) of Chapter 3 of [13]).

Therefore, we have to set $\gamma = 0$ and recompute Γ, which yields an expression with only 36 terms. Now, the coefficients of $e_0^6e_1$ and $e_0^6e_2$ of Γ imply $L_c = L_a + g_b - \beta$ and $g_d = L_a + g_b + 1$, respectively. Finally, the coefficient of $e_0^4e_1^3$ equals β, which contradicts the assumption $\beta \neq 0$.

It can easily be seen, that $f_0,\dots,f_3$ can not be computed with the above approach, if either $e_0 = e_3 = 0$ or $e_1 = e_2 = 0$ hold. In both cases π_{M} and π_{m} have to be parallel, as they cannot coincide due to Definition 1. Therefore, κ maps the line at infinity of π_{m} (= the line s) onto the ideal line of π_{M}. This implies a planar affine SG platform, which contradicts our assumptions.

Now, we sum up our results: From the computation above we see, that a two-parametric motion of the 5-legged manipulator is not possible and therefore it can only have one-parametric ones. As a consequence, the elliptic self-motion of the planar projective SG platform can only be one-parametric. □

Based on this result, we can proceed with the following geometric analysis of elliptic self-motions:

If s does not change during the self-motion, then the self-motion can only be a pure rotation about the line $\mathsf{s} = \mathsf{s}\kappa$ (one-parametric motion), as no instantaneous motion along $\mathsf{s} = \mathsf{s}\kappa$ is possible.[4] Now it can easily be seen, that any leg of $\mathscr{L}$ with anchor points not on $\mathsf{s} = \mathsf{s}\kappa$ prevents this rotational mobility.

[4]This follows immediately from the fact, that we can attach legs of $\mathscr{L}$ with anchor points on $\mathsf{s} = \mathsf{s}\kappa$.

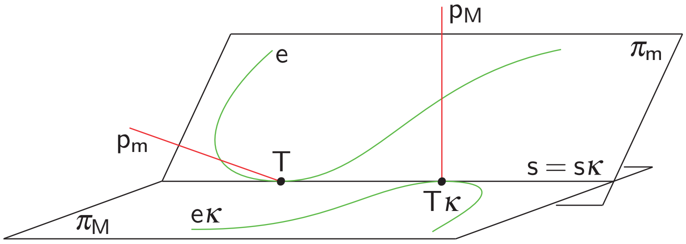

Figure 3: Sketch of the decomposition of an elliptic self-motion.

Therefore, s has to change during the one-parametric elliptic self-motion. Now, e denotes the envelope of the resulting one-parametric set of lines s generated during the self-motion (cf. Fig. 3). Due to the projective correspondence, the envelope of $s\kappa$ in π_M is given by $e\kappa$. Moreover, as no instantaneous sliding along $s = s\kappa$ is possible (cf. Footnote 4), any elliptic self-motion can be decomposed into: (1) a rolling (without slipping) of $s\kappa$ on $e\kappa$, (2) a rotation of π_m about $s = s\kappa$, and (3) a rolling[5] (without slipping) of e on s.

This has the following consequences for the instantaneous behavior of a planar projective SG platform: On the velocity level, each of the rolling motions can be replaced by an instantaneous rotation about a line p_m (resp. p_M) perpendicular to π_m (resp. π_M) through the contact point T (resp. $T\kappa$) of $s = s\kappa$ and e (resp. $e\kappa$). As the angular velocities of these two rotations are coupled by κ (cf. Footnote 5), we get only one dof. The second dof arises from the instantaneous rotation of π_m about $s = s\kappa$. This gives the geometric interpretation/reasoning for the two instantaneous dofs noted in Theorem 1.

Finally, we can also introduce a classification of elliptic self-motions based on the geometric interpretation of $\sphericalangle_F^f$. This implies the following definition:

Definition 2 *An elliptic self-motion is called orthogonal, if the angle enclosed by the unique pair of ideal points* (f, F) *with* $f\kappa = F$ *equals* $\pi/2$.

In the remainder of this article, we answer the question of existence of orthogonal elliptic self-motions. Therefore, we set $\gamma = 0$.

5. DETERMINATION OF ORTHOGONAL ELLIPTIC SELF-MOTIONS

In this section, we introduce two approaches to the determination of planar projective SG platforms with orthogonal elliptic self-motions.

5.1. The classical approach

Until now, we have a set of six constraints Ω_A^a, Ω_C^c, Π_B^b, Π_D^d, $\sphericalangle_F^f$, Φ in eight homogeneous unknowns $e_0 : \ldots : f_3$. We get a self-motion, if we can attach the sixth leg $\overline{mM}$ of $\mathscr{L}$ to the 5-legged manipulator pertaining to Theorem 2 without restricting at least one component of its one-parametric motion. The attachment of the leg $\overline{mM}$ corresponds to the so-called sphere constraint, that m has to be located on a sphere with center M and radius[6] $R \in \mathbb{R}$. Due to Husty [14], this condition can again be expressed by a homogeneous quadratic equation

[5] Clearly, the two rolling motions of item 1 and item 3 are coupled by the projectivity κ.

[6] Note, that the variables $L_a, L_c, g_b, g_d, \gamma$ of the Darboux, Mannheim and angle constraint, respectively, can be seen as the analogue to the variable R of the sphere constraint.

$\Theta^{\mathsf{m}}_{\mathsf{M}}$ in Study parameters. For our selection given in Eq. (11), we get the following compact expression:

$$\Theta^{\mathsf{m}}_{\mathsf{M}}: \; (R^2-\beta^2-1)K-4(f_0^2+f_1^2+f_2^2+f_3^2)-2t_2+2\beta(\bar{t}_1+r_{21})=0. \tag{12}$$

As a consequence, each common curve of the seven hyperquadrics in the Study parameter space, which are given by $\Omega^{\mathsf{a}}_{\mathsf{A}}$, $\Omega^{\mathsf{c}}_{\mathsf{C}}$, $\Pi^{\mathsf{b}}_{\mathsf{B}}$, $\Pi^{\mathsf{d}}_{\mathsf{D}}$, $\lhd^{\mathsf{f}}_{\mathsf{F}}$, Φ, $\Theta^{\mathsf{m}}_{\mathsf{M}}$, corresponds with an elliptic self-motion. This is a necessary and sufficient condition.

Therefore, the classical approach for the determination of these self-motions can be done with a modified version of Husty's algorithm [14] as follows: We consider the following two ideals:

$$\mathscr{I}^{\mathsf{c}}_{\mathsf{C}} := \left\{\Omega^{\mathsf{a}}_{\mathsf{A}},\Pi^{\mathsf{b}}_{\mathsf{B}},\Pi^{\mathsf{d}}_{\mathsf{D}},\lhd^{\mathsf{f}}_{\mathsf{F}},\Phi,\Theta^{\mathsf{m}}_{\mathsf{M}}\right\}, \qquad \mathscr{I}^{\mathsf{d}}_{\mathsf{D}} := \left\{\Omega^{\mathsf{a}}_{\mathsf{A}},\Omega^{\mathsf{c}}_{\mathsf{C}},\Pi^{\mathsf{b}}_{\mathsf{B}},\lhd^{\mathsf{f}}_{\mathsf{F}},\Phi,\Theta^{\mathsf{m}}_{\mathsf{M}}\right\}. \tag{13}$$

For $\mathscr{I}^{\mathsf{c}}_{\mathsf{C}}$ (resp. $\mathscr{I}^{\mathsf{d}}_{\mathsf{D}}$), we compute $f_0,\ldots,f_3$ from $\Omega^{\mathsf{a}}_{\mathsf{A}},\Pi^{\mathsf{b}}_{\mathsf{B}},\Pi^{\mathsf{d}}_{\mathsf{D}},\Phi$ (resp. $\Omega^{\mathsf{a}}_{\mathsf{A}},\Omega^{\mathsf{c}}_{\mathsf{C}},\Pi^{\mathsf{b}}_{\mathsf{B}},\Phi$), which are linear in these unknowns. Substituting the resulting values into $\Theta^{\mathsf{m}}_{\mathsf{M}}$ yields $\Lambda^{\mathsf{c}}_{\mathsf{C}}[737]$ (resp. $\Lambda^{\mathsf{d}}_{\mathsf{D}}[737]$). Now, we eliminate in each of the two ideals the variable e_3 by computing the following resultants with respect to e_3:

$$Res_{e_3}(\lhd^{\mathsf{f}}_{\mathsf{F}},\Lambda^{\mathsf{c}}_{\mathsf{C}}) = (e_0^2+e_1^2)\Gamma^{\mathsf{c}}_{\mathsf{C}}[244], \quad Res_{e_3}(\lhd^{\mathsf{f}}_{\mathsf{F}},\Lambda^{\mathsf{d}}_{\mathsf{D}}) = (e_0^2+e_2^2)\Gamma^{\mathsf{d}}_{\mathsf{D}}[244]. \tag{14}$$

Analogous considerations can be done for the ideals:

$$\mathscr{I}^{\mathsf{a}}_{\mathsf{A}} := \left\{\Omega^{\mathsf{c}}_{\mathsf{C}},\Pi^{\mathsf{b}}_{\mathsf{B}},\Pi^{\mathsf{d}}_{\mathsf{D}},\lhd^{\mathsf{f}}_{\mathsf{F}},\Phi,\Theta^{\mathsf{m}}_{\mathsf{M}}\right\}, \qquad \mathscr{I}^{\mathsf{b}}_{\mathsf{B}} := \left\{\Omega^{\mathsf{a}}_{\mathsf{A}},\Omega^{\mathsf{c}}_{\mathsf{C}},\Pi^{\mathsf{d}}_{\mathsf{D}},\lhd^{\mathsf{f}}_{\mathsf{F}},\Phi,\Theta^{\mathsf{m}}_{\mathsf{M}}\right\}, \tag{15}$$

which yield

$$Res_{e_3}(\lhd^{\mathsf{f}}_{\mathsf{F}},\Lambda^{\mathsf{a}}_{\mathsf{A}}) = (e_0^2+e_1^2)\Gamma^{\mathsf{a}}_{\mathsf{A}}[440], \quad Res_{e_3}(\lhd^{\mathsf{f}}_{\mathsf{F}},\Lambda^{\mathsf{b}}_{\mathsf{B}}) = (e_0^2+e_2^2)\Gamma^{\mathsf{b}}_{\mathsf{B}}[440]. \tag{16}$$

Now, $\Gamma^{\mathsf{a}}_{\mathsf{A}},\ldots,\Gamma^{\mathsf{d}}_{\mathsf{D}}$ are polynomials of degree 14 in e_0 and $\Gamma[36]$ (cf. proof of Theorem 2) is a polynomial of degree 6 in e_0. Moreover, as we deal with planar SG platforms, only even powers of e_0 in all five polynomials appear. Therefore, we can substitute e_0^2 by $\overline{e}_0$ and compute

$$\Psi^{\mathsf{a}}_{\mathsf{A}} := Res_{\overline{e}_0}(\Gamma^{\mathsf{a}}_{\mathsf{A}},\Gamma), \quad \ldots, \quad \Psi^{\mathsf{d}}_{\mathsf{D}} := Res_{\overline{e}_0}(\Gamma^{\mathsf{d}}_{\mathsf{D}},\Gamma). \tag{17}$$

Finally, the greatest common divisor (gcd) of $\Psi^{\mathsf{a}}_{\mathsf{A}},\ldots,\Psi^{\mathsf{d}}_{\mathsf{D}}$ yields

$$\beta e_1^9 e_2^9 (e_1^4-e_2^4)\Upsilon[24685], \tag{18}$$

which corresponds to the solution of the direct kinematics problem. Therefore, in the general case, the coefficients of Υ with respect to e_1 and e_2 have to vanish identically. As Υ is of degree 16 in e_1 and e_2 this yields 17 equations in the six unknowns L_a,L_c,g_b,g_d,β,R. As the resulting system of equations is highly non-linear, we were not able to solve it explicitly. Therefore, we developed the following alternative approach, which is based on two necessary conditions for an elliptic self-motion.

Remark 2 *For non-orthogonal elliptic self-motions, which additionally depend on* $\gamma \neq 0$*, we were not able to compute* $\Psi^{\mathsf{a}}_{\mathsf{A}},\ldots,\Psi^{\mathsf{d}}_{\mathsf{D}}$ *symbolically (with* MAPLE*) because intermediate expressions were so large as to exceed capacity of largest available computational resources (78GB RAM). This also shows that the classic approach is hopeless for solving the non-orthogonal case, which is a further motivation for the following alternative method.* ◇

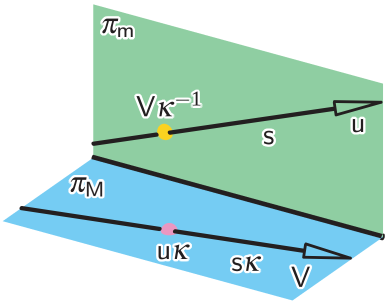

Figure 4: Left: Sketch of the points and lines used for the alternative approach. Right: Situation for $\mathsf{u} = \mathsf{V}$.

5.2. The alternative approach

Due to the properties of elliptic self-motions derived up to now, we can formulate two necessary conditions, so that at least a component of the one-parametric motion of the 5-legged manipulator of Theorem 2 corresponds with an elliptic self-motion. These two conditions arise from the property that $\mathsf{s} = \mathsf{s}\kappa$ must remain valid throughout the entire motion.

As s (resp. $\mathsf{s}\kappa$) cannot be parallel to the y-axis of the moving frame[7] (resp. x-axis of the fixed frame), we can assume that the ideal point u of s (resp. V of $\mathsf{s}\kappa$) can be given by (cf. Fig. 4, left):

$$\mathbf{u} = (0:1:u)^T, \qquad \mathbf{V} = (0:v:1)^T, \tag{19}$$

with $u, v \in \mathbb{R}$. In order that u and V coincide during the motion the condition

$$(\mathbf{R}\cdot(1,u,0)^T) \times (v,1,0)^T = (0,0,0)^T \tag{20}$$

has to hold. It can easily be checked, that for $r_{32}r_{13} = 0$ this system of equations can only be fulfilled for $e_0 = e_3 = 0$ or $e_1 = e_2 = 0$. Due to reasoning in the penultimate paragraph of the proof of Theorem 2, this already yields a contradiction. Under consideration of $r_{32}r_{13} \neq 0$, we can solve Eq. (20) w.l.o.g., which implies $u = -r_{31}/r_{32}$ and $v = -r_{23}/r_{13}$. Now, we have the situation displayed on the right side of Fig. 4.

As $\mathsf{s} = \mathsf{s}\kappa$ has to hold, $\mathsf{V}\kappa^{-1} \in \pi_\mathsf{m}$ has to be located also in π_M (cf. Fig. 4, right). Due to Eq. (5), $\mathsf{V}\kappa^{-1}$ has homogeneous coordinates $(1:0:v\beta)^T$ with respect to the moving frame. Therefore, this condition can be expressed by

$$\left(\mathbf{R}\cdot(0,v\beta,0)^T + (t_1,t_2,t_3)^T\right)(0,0,1)^T = 0, \tag{21}$$

which yields

$$\Xi_1: \; r_{12}t_3 - \beta r_{23}r_{32} = 0. \tag{22}$$

Moreover, as $\mathsf{s} = \mathsf{s}\kappa$ has to hold, $\mathsf{u}\kappa \in \pi_\mathsf{M}$ has to be located also in π_m (cf. Fig. 4, right). Due to Eq. (5), $\mathsf{u}\kappa$ has homogeneous coordinates $(1:u:0)^T$ with respect to the fixed frame. Therefore, this condition can be expressed by

$$\left(\mathbf{R}\cdot(0,0,1)^T\right)(t_1,t_2,t_3)^T - K\left(\mathbf{R}\cdot(0,0,1)^T\right)(u,0,0)^T = 0. \tag{23}$$

K factors out from this equations and we remain with:

$$\Xi_2: \; r_{32}\bar{t}_3 + r_{13}r_{31} = 0 \quad \text{with} \quad \bar{t}_3 := 2(e_0f_3 - e_1f_2 + e_2f_1 - e_3f_0). \tag{24}$$

[7] As $\mathsf{f} \in \mathsf{s}$ implies that $\mathsf{f} = \mathsf{F}$ is a fixed point of the projectivity of s onto itself.

The obtained equations Ξ_1 and Ξ_2 are quartic equations in Study parameters, but they are only linear in $f_0,\ldots,f_3$ (in contrast to $\Theta_{\mathsf{M}}^{\mathsf{m}}$). A further advantage of these two equations is, that no additional unknown is introduced, as was in the case for $\Theta_{\mathsf{M}}^{\mathsf{m}}$ (variable R) of Sect. 5.1.

Note, that for an elliptic self-motion it is necessary, that the six hyperquadrics $\Omega_{\mathsf{A}}^{\mathsf{a}}$, $\Omega_{\mathsf{C}}^{\mathsf{c}}$, $\Pi_{\mathsf{B}}^{\mathsf{b}}$, $\Pi_{\mathsf{D}}^{\mathsf{d}}$, $\lhd_{\mathsf{F}}^{\mathsf{f}}$, Φ and the two hyperquartics Ξ_1 and Ξ_2 have a common curve in the Study parameter space.[8] Therefore, this set of equations can be used to prove the main theorem on orthogonal elliptic self-motions, which is given next.

Remark 3 *Note, that* $\Xi_1 = 0$ *and* $\Xi_2 = 0$ *already imply that there exist two instantaneous dofs. Therefore, we cannot generate any additional conditions from the property that the rank of the manipulator's Jacobian has to be equal to four during the self-motion (cf. Theorem 1).* ◇

6. MAIN THEOREM ON ORTHOGONAL ELLIPTIC SELF-MOTIONS

Theorem 3 *There do not exist non-architecturally singular planar projective SG platforms with an orthogonal elliptic self-motion.*

Proof: We compute $f_0,\ldots,f_3$ from the system of equations $\Omega_{\mathsf{A}}^{\mathsf{a}},\Pi_{\mathsf{B}}^{\mathsf{b}},\Pi_{\mathsf{D}}^{\mathsf{d}},\Phi$ (resp. $\Omega_{\mathsf{A}}^{\mathsf{a}},\Omega_{\mathsf{C}}^{\mathsf{c}},\Pi_{\mathsf{B}}^{\mathsf{b}},\Phi$), which are linear in these unknowns. Then, we substitute the resulting values into Ξ_1, which yields $\Lambda_{\mathsf{C}}^{\mathsf{c}}[84]$ (resp. $\Lambda_{\mathsf{D}}^{\mathsf{d}}[86]$). This can only be done for $e_0e_1 - e_2e_3 \neq 0$ (resp. $e_0e_2 - e_1e_3 \neq 0$).

Now, we eliminate e_3 by computing the following resultants with respect to e_3:

$$Res_{e_3}(\lhd_{\mathsf{F}}^{\mathsf{f}},\Lambda_{\mathsf{C}}^{\mathsf{c}}) = (e_0^2 - e_2^2)\Gamma_{\mathsf{C}}^{\mathsf{c}}[34], \qquad Res_{e_3}(\lhd_{\mathsf{F}}^{\mathsf{f}},\Lambda_{\mathsf{D}}^{\mathsf{d}}) = e_2(e_0^2 + e_2^2)\Gamma_{\mathsf{D}}^{\mathsf{d}}[35].$$

This elimination is only valid if the coefficients of the greatest power of e_3 do not vanish, which are e_0, e_1N_2 and e_1N_3 with

$$N_2 := \beta + L_a - L_c - g_b, \quad N_3 := 1 + L_a - g_b + g_d. \tag{25}$$

Now, $\Gamma_{\mathsf{C}}^{\mathsf{c}}$ and $\Gamma_{\mathsf{D}}^{\mathsf{d}}$ are polynomials of degree 8 in e_0, and $\Gamma[36]$ (cf. proof of Theorem 2) is of degree 6 in e_0. As again only even powers of e_0 appear in all three polynomials, we replace e_0^2 with $\overline{e}_0$ and compute

$$\Psi_{\mathsf{C}}^{\mathsf{c}} := Res_{\overline{e}_0}(\Gamma_{\mathsf{C}}^{\mathsf{c}},\Gamma), \qquad \Psi_{\mathsf{D}}^{\mathsf{d}} := Res_{\overline{e}_0}(\Gamma_{\mathsf{D}}^{\mathsf{d}},\Gamma). \tag{26}$$

This elimination is only valid if the coefficients of the greatest power of $\overline{e}_0$ do not vanish, which are N_1, e_2N_4 and $e_1N_1 + e_2N_4$ with

$$N_1 := \beta - L_a + L_c - g_b, \quad N_4 := 1 + L_a + g_b - g_d. \tag{27}$$

Finally, the greatest common divisor of $\Psi_{\mathsf{C}}^{\mathsf{c}}$ and $\Psi_{\mathsf{D}}^{\mathsf{d}}$ yields

$$\beta e_1^9 e_2^3 (e_1^2 + e_2^2)\Upsilon_1[1960]. \tag{28}$$

Moreover, the same procedure can be done with respect to Ξ_2, where we only get one additional condition, namely $(e_0^2 - e_1^2) \neq 0$. In this case, we obtain

$$e_1^3 e_2^9 (e_1^2 + e_2^2)\Upsilon_2[1960]. \tag{29}$$

Therefore, in the general case the coefficients of Υ_1 and Υ_2 with respect to the remaining Study parameters e_1 and e_2 have to vanish identically. As both equations are homogeneous of degree 12 in e_1 and e_2, this yields 26 necessary conditions. Now, we are looking for solutions of this system under the assumption

$$e_0e_1e_2(e_0^2 - e_1^2)(e_0^2 - e_2^2)(e_0e_1 - e_2e_3)(e_0e_2 - e_1e_3) \neq 0. \tag{30}$$

[8] Note, that these conditions are not sufficient for an elliptic self-motion.

6.1. Case study of the general case

In the following, we denote the coefficient of $e_1^j e_2^k$ of Υ_i by $\Upsilon_i^{j,k}$ for $i = 1,2$. Now, $\Upsilon_i^{12,0}$ and $\Upsilon_i^{0,12}$ split up into:

$$\Upsilon_1^{12,0} = N_1 P_1^2 Q_1, \quad \Upsilon_1^{0,12} = N_2 P_2^2 Q_2, \quad \Upsilon_2^{12,0} = N_3 P_3^2 Q_3, \quad \Upsilon_2^{0,12} = N_4 P_4^2 Q_4,$$

$$P_1 := \beta + L_a - L_c + g_b, \qquad Q_1 := (L_a - g_b + g_d)(L_a - L_c + \beta) + g_b, \tag{31}$$
$$P_2 := \beta - L_a + L_c + g_b, \qquad Q_2 := (L_a + g_b - g_d)(L_a - L_c - \beta) - g_b, \tag{32}$$
$$P_3 := 1 - L_a + g_b - g_d, \qquad Q_3 := (g_b + L_a - L_c)(g_b - g_d + 1) + \beta L_a, \tag{33}$$
$$P_4 := 1 - L_a - g_b + g_d, \qquad Q_4 := (g_b - L_a + L_c)(g_b - g_d - 1) - \beta L_a. \tag{34}$$

Note, that $N_1 N_2 N_3 N_4 = 0$ is not allowed, as otherwise at least one of the greatest powers of e_3 and e_0, respectively, within the above given resultant based elimination vanishes. Therefore, these cases have to be discussed separately (see Sect. 6.1.1). As a consequence, $P_j Q_j = 0$ for $j = 1, \ldots, 4$ has to hold, where all possible combinatorial cases are discussed within the following listing. Note, that for this discussion the assumptions $N_1 N_2 N_3 N_4 \neq 0$, $L_a, L_c, g_b, g_d \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{0\}$ hold in addition to Eq. (30):

1. $P_1 = P_2 = P_3 = P_4 = 0$: In this case, we solve the four equations, which yield $L_a = L_c = 1$ and $g_b = g_d = -\beta$. Then, we get:

$$\Upsilon_1 = e_1^3 e_2^3 (\beta e_1^2 + \beta e_2^2 + 2 e_1 e_2)(e_1^2 + e_2^2 + \beta^2 e_1^2 + \beta^2 e_2^2 + 4\beta e_1 e_2)^2, \tag{35}$$
$$\Upsilon_2 = e_1^3 e_2^3 (e_1^2 + e_2^2 + \beta 2 e_1 e_2)(e_1^2 + e_2^2 + \beta^2 e_1^2 + \beta^2 e_2^2 + 4\beta e_1 e_2)^2, \tag{36}$$

   which cannot vanish without contradiction (w.c.).

2. $P_1 = P_2 = P_3 = 0$, $P_4 \neq 0$: In this case, we solve the three equations, which yield $L_a = L_c = 1 - g_d - \beta$ and $g_b = -\beta$. Now, Q_4 cannot vanish w.c.

   Analogously, it can be shown for the cases $P_4 = P_i = P_j = 0$, $P_k \neq 0$ with pairwise distinct $i, j, k \in \{1,2,3\}$, that no solution exists.

3. $P_1 P_2 \neq 0$: Therefore, $Q_1 = Q_2 = 0$ has to hold and we can express g_b and g_d from these equations. Now, $\Upsilon_2^{12,0}$ and $\Upsilon_2^{0,12}$ can only vanish w.c. for $L_a = L_c$. Then, $\Upsilon_2^{11,1}$ implies $L_c = 0$ ($\Rightarrow L_a = g_b = g_d = 0$), which yields a solution to the necessary system of equations.

   Note, that the case $P_3 P_4 \neq 0$ can be done analogously. This case also yields the same solution.

4. $P_1 P_3 \neq 0$: Therefore, $Q_1 = Q_3 = 0$ has to hold and we distinguish the following two cases:

   a) $L_a - g_b + g_d \neq 0$: Under this assumption, we can solve $Q_1 = Q_3 = 0$ for L_c and β. Now, $\Upsilon_1^{0,12}$ and $\Upsilon_2^{0,12}$ can only vanish w.c. for

      i. $g_b = g_d$: Then, $\Upsilon_2^{11,1}$ cannot vanish w.c.

      ii. $L_a - 1 - 3(g_b - g_d) = 0$, $L_a - 1 + g_b - g_d = 0$, $g_b \neq g_d$: The difference of these equations already yields the contradiction.

   b) $L_a = g_b - g_d$: Now, Q_1 implies $g_b = 0$. We distinguish two cases:

      i. $g_d \neq 0$: Under this assumption, we can express β from Q_3. Then, $\Upsilon_1^{0,12}$ cannot vanish w.c.

ii. $g_d = 0$: Then, Q_3 implies $L_c = 0$ and we get again the above solution.

Note, that the case $P_2P_4 \neq 0$ can be done analogously. This case also yields the same solution.

5. $P_1P_4 \neq 0$: Therefore, $Q_1 = Q_4 = 0$ has to hold and we distinguish the following two cases:
   a) $L_a - g_b + g_d \neq 0$: Under this assumption, we can solve $Q_1 = Q_4 = 0$ for L_c and β. Now, $\Upsilon_1^{0,12}$ and $\Upsilon_2^{0,12}$ can only vanish w.c. for $g_b = g_d$. Then, $\Upsilon_2^{11,1}$ cannot vanish w.c.
   b) $L_a = g_b - g_d$: Now, Q_1 implies $g_b = 0$. We distinguish two cases:
      i. $g_d \neq 0$: In this case, we can express β from Q_3. Then, $\Upsilon_1^{0,12}$ cannot vanish w.c.
      ii. $g_d = 0$: Then, Q_4 implies $L_c = 0$ and we get again the above solution.

   Note, that the case $P_2P_3 \neq 0$ can be done analogously. This case also yields the same solution.

Now it remains to check, if the obtained solution $L_a = L_c = g_b = g_d = 0$ yields an elliptic self-motion. Therefore, we compute

$$gcd(\Gamma, \Gamma_{\mathsf{C}}^{\mathsf{c}}) = e_0^2 - e_1^2, \quad gcd(\Gamma, \Gamma_{\mathsf{D}}^{\mathsf{d}}) = e_0^2 - e_2^2, \tag{37}$$

which implies $e_0^2 = e_1^2 = e_2^2$, but this contradicts Eq. (30). Therefore, the general case does not yield a solution to our problem. In the following, we discuss the excluded cases $N_1N_2N_3N_4 = 0$ under the assumptions $L_a, L_c, g_b, g_d \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$.

6.1.1. Excluded case $N_1 = 0$

Under consideration of $g_b = \beta - L_a + L_c$, computations similar to those at the beginning of Sect. 6 yield $\Upsilon_1[527]$ of degree 11 in e_1, e_2 and $\Upsilon_2[708]$ of degree 12 in e_1, e_2. Then, we get

$$\Upsilon_1^{0,11} = N_2(L_a - L_c - \beta)^3, \quad \Upsilon_2^{12,0} = N_3\beta(g_d - 1 + 2L_a - L_c - \beta)^2. \tag{38}$$

As $N_2N_3 = 0$ is not allowed, due to the elimination process, we have to discuss these cases again separately. Therefore, we can assume w.l.o.g. $N_2N_3 \neq 0$. As a consequence, the remaining factors of Eq. (38) can be solved for L_a and L_c. Then, $\Upsilon_1^{11,0}$ cannot vanish w.c. Hence, we remain with the discussion of $N_2N_3 = 0$:

1. $N_2 = 0$: In this case, we have $L_a = L_c$. We end up with $\Upsilon_1[53]$ of degree 10 in e_1, e_2 and $\Upsilon_2[101]$ of degree 12 in e_1, e_2. Now, already $\Upsilon_1^{10,0} = \beta^3$ yields the contradiction.

2. $N_3 = 0$: Now, we have $g_d = \beta + L_c - 1 - 2L_a$. We end up with $\Upsilon_1[105]$ and $\Upsilon_2[113]$ of degree 11 in e_1 and e_2. Then, $\Upsilon_1^{11,0} = \beta^2$ yields the contradiction.

The other excluded cases $N_i = 0$ for $i \in \{2,3,4\}$ can be discussed in a fashion similar to that above in Sect. 6.1.1. They also yield no solution. Therefore, we are only left with the discussion of the special cases where Eq. (30) does not hold for $L_a, L_c, g_b, g_d \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$.

6.2. Case study of the special cases

According to the reasoning given in the penultimate paragraph of the proof of Theorem 2, we can assume that $e_0 = e_3 = 0$ and $e_1 = e_2 = 0$ do not hold. Moreover, due to $\triangleleft_{\mathsf{F}}^{\mathsf{f}}: e_1e_2 - e_0e_3 = 0$, we only have to distinguish the following four cases, if at least one e_i equals zero:

$$e_0 = e_1 = 0, \quad e_0 = e_2 = 0, \quad e_3 = e_1 = 0, \quad e_3 = e_2 = 0. \tag{39}$$

But in all these cases $r_{32}r_{13} = 0$ holds, which contradicts the assumption of Sect. 5.2. Therefore, we can assume for the remaining discussion $e_i \neq 0$ with $i = 0, \ldots, 3$.

6.2.1. Special case $e_0^2 = e_1^2$

Under consideration of $\sphericalangle_{\mathsf{F}}^{\mathsf{f}}$, we have to distinguish the following two cases:

1. $e_0 = e_1$ and $e_2 = e_3$: W.l.o.g., we can compute $f_0, \dots, f_3$ from the system of equations $\Omega_{\mathsf{A}}^{\mathsf{a}}, \Omega_{\mathsf{C}}^{\mathsf{c}}, \Pi_{\mathsf{D}}^{\mathsf{d}}, \Phi$. Then, the coefficient of e_1^4 of Ξ_1 equals β, which cannot vanish w.c.

2. $e_0 = -e_1$ and $e_2 = -e_3$: The argumentation of item 1 also holds in this case.

6.2.2. Special case $e_0^2 = e_2^2$

Under consideration of $\sphericalangle_{\mathsf{F}}^{\mathsf{f}}$, we have to distinguish the following two cases:

1. $e_0 = e_2$ and $e_1 = e_3$: W.l.o.g., we can compute $f_0, \dots, f_3$ from the system of equations $\Omega_{\mathsf{C}}^{\mathsf{c}}, \Pi_{\mathsf{B}}^{\mathsf{b}}, \Pi_{\mathsf{D}}^{\mathsf{d}}, \Phi$. Then, the coefficient of e_1^4 of Ξ_2 equals 1, which yields the contradiction.

2. $e_0 = -e_2$ and $e_1 = -e_3$: Similarly, the contradiction applies here as well.

6.2.3. Remaining special cases

1. $e_0e_2 - e_1e_3 = 0$: This condition yields together with the angle constraint $\sphericalangle_{\mathsf{F}}^{\mathsf{f}}$ the following two cases:

$$e_0 = e_1 \quad \wedge \quad e_2 = e_3 \qquad \text{and} \qquad e_0 = -e_1 \quad \wedge \quad e_2 = -e_3. \tag{40}$$

   These are exactly the same cases already discussed in Sect. 6.2.1.

2. $e_0e_1 - e_2e_3 = 0$: This condition yields together with the angle constraint $\sphericalangle_{\mathsf{F}}^{\mathsf{f}}$ the following two cases:

$$e_0 = e_2 \quad \wedge \quad e_1 = e_3 \qquad \text{and} \qquad e_0 = -e_2 \quad \wedge \quad e_1 = -e_3. \tag{41}$$

   These are exactly the same cases already discussed in Sect. 6.2.2.

This closes the proof of Theorem 3. □

Remark 4 *In view of Remark 2, it should be noted, that with the alternative method presented, the corresponding expressions of Υ_1 and Υ_2 given in Eqs. (28) and (29) for non-orthogonal elliptic self-motions ($\gamma \neq 0$), can be computed without major difficulties. Each of these two expressions, which additionally depend on the variable γ, has 8259 terms and implies again 13 equations (cf. paragraph after Eq. (29)).* ⋄

7. CONJECTURE AND CONCLUSION

We proved that non-architecturally singular planar projective SG platforms with orthogonal elliptic self-motions do not exist (cf. Definitions 1 and 2, Theorem 3). Moreover, we conjecture the following: *"Non-architecturally singular planar projective SG platforms with non-orthogonal elliptic self-motions do not exist"*.

In this context it should be noted, that we were recently able to prove this conjecture in [15], based on Theorem 3. On one side, this result is a bit frustrating from the kinematican's point of view, as one-parametric self-motions, which have everywhere two instantaneous dofs (cf. Theorems 1 and 2), would have been of theoretical interest. But on the other side, this result is useful to designers who may be sure that a planar projective SG manipulator, where the projectivity is no affinity, can never have self-motion (cf. last paragraph before Theorem 1).

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