Special cases of Schönflies-singular planar Stewart Gough platforms

Georg Nawratil

Technische Universität Wien
Vienna University of Technology

Institute of Discrete Mathematics and Geometry

Differential Geometry and Geometric Structures

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[1] Singular configurations of SGPs

The geometry of a Stewart Gough Platform (SGP) is given by the six base anchor points

$\mathbf{M}_i := (A_i, B_i, 0)^T$ in the fixed space Σ_0

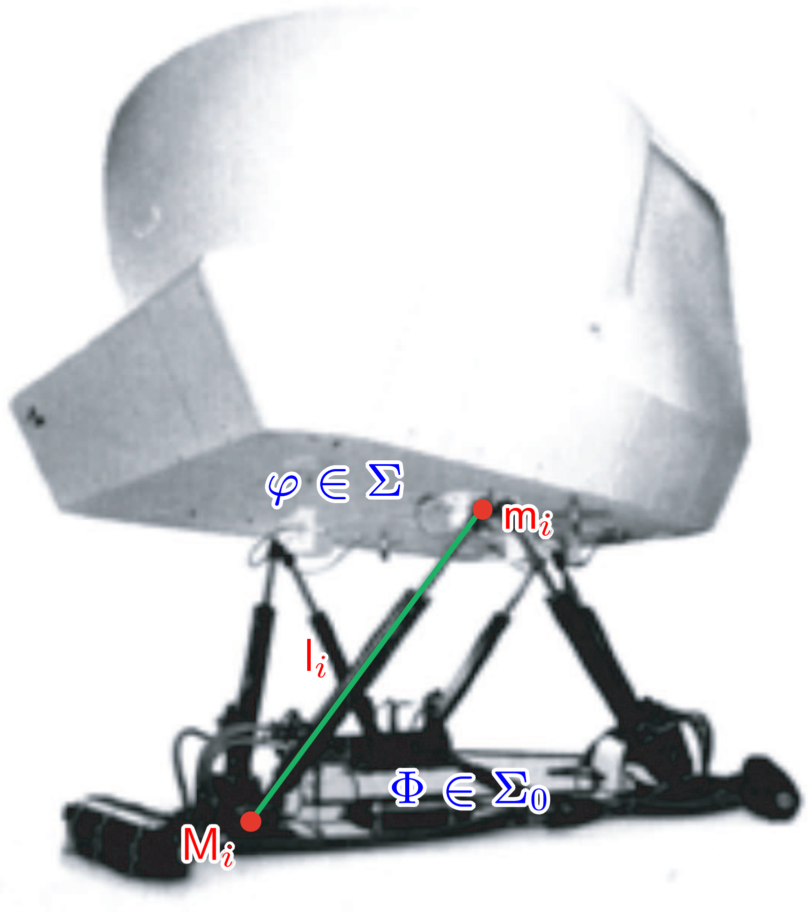
and by the six platform anchor points

$\mathbf{m}_i := (a_i, b_i, 0)^T$ in the moving space Σ.

$\Phi \in \Sigma_0$ denotes the carrier plane of the M_i's.

$\varphi \in \Sigma$ denotes the carrier plane of the m_i's.

Theorem
A SGP is singular iff the carrier lines l_i of the six legs belong to a linear line complex.



[1] Schönflies-singular SGPs

The Schönflies motion group X(a) consists of three linearly independent translations and all rotations about the infinity of axes with direction a.

Definition Schönflies-singular SGP
A SGP is called Schönflies-singular (or more precisely X(a)-singular) if there exists a Schönflies group X(a) such that the manipulator is singular for all transformations from X(a) (applied to the moving part of the SGP).

Every Schönflies-singular manipulator belongs to one of these cases:

1. $\alpha \neq \beta$: (a) $\alpha = \pi/2$, $\beta \in [0, \pi/2[$ (b) $\alpha, \beta \in [0, \pi/2[$
2. $\alpha = \beta$: (a) $\alpha = \pi/2$ (b) $\alpha \in]0, \pi/2[$ (c) $\alpha = 0$

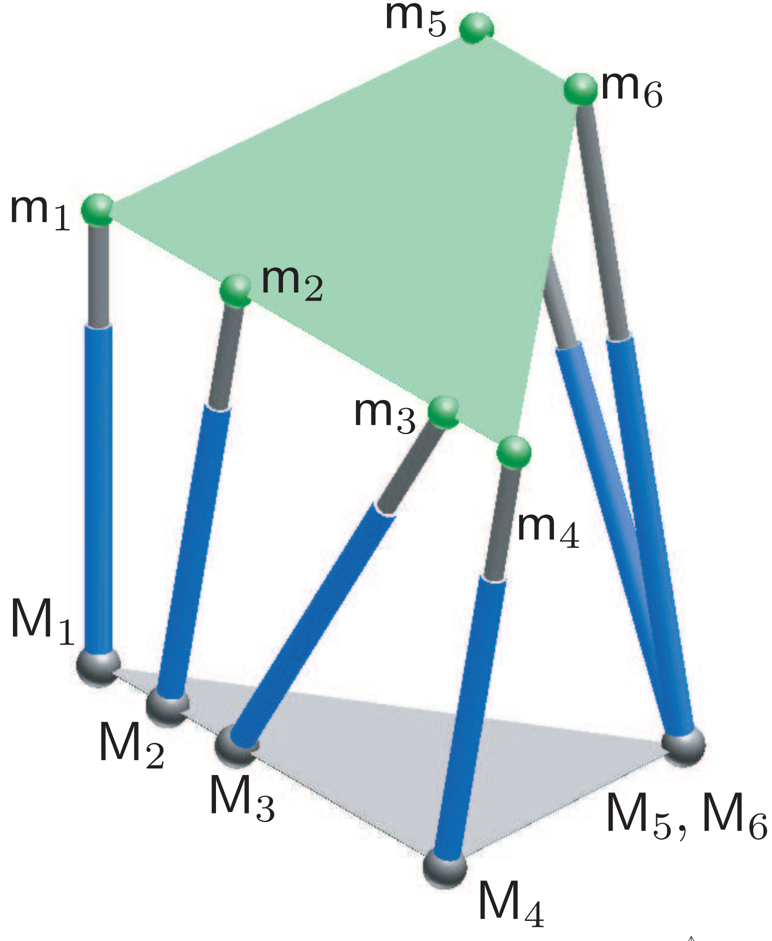
with $\alpha := \angle(\mathsf{a}, \Phi) \in [0, \pi/2]$ and $\beta := \angle(\mathsf{a}, \varphi) \in [0, \pi/2]$.

[2] Schönflies-singular SGPs of case (1a)

According to **Nawratil [2010,A]** the solution set of case (1a) can be characterized as:

Theorem Nawratil [2010,A]
A non-architecturally singular planar SGP is X(a)-singular, where a is orthogonal to Φ and orthogonal to the x-axis of the moving frame iff $rk(\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{Bb}, \mathbf{a}, \mathbf{b}, \mathbf{Ab})_1^6 = 4$ holds with

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_6 \end{bmatrix}, \quad \mathbf{Xy} = \begin{bmatrix} X_1 y_1 \\ X_2 y_2 \\ \vdots \\ X_6 y_6 \end{bmatrix}.$$



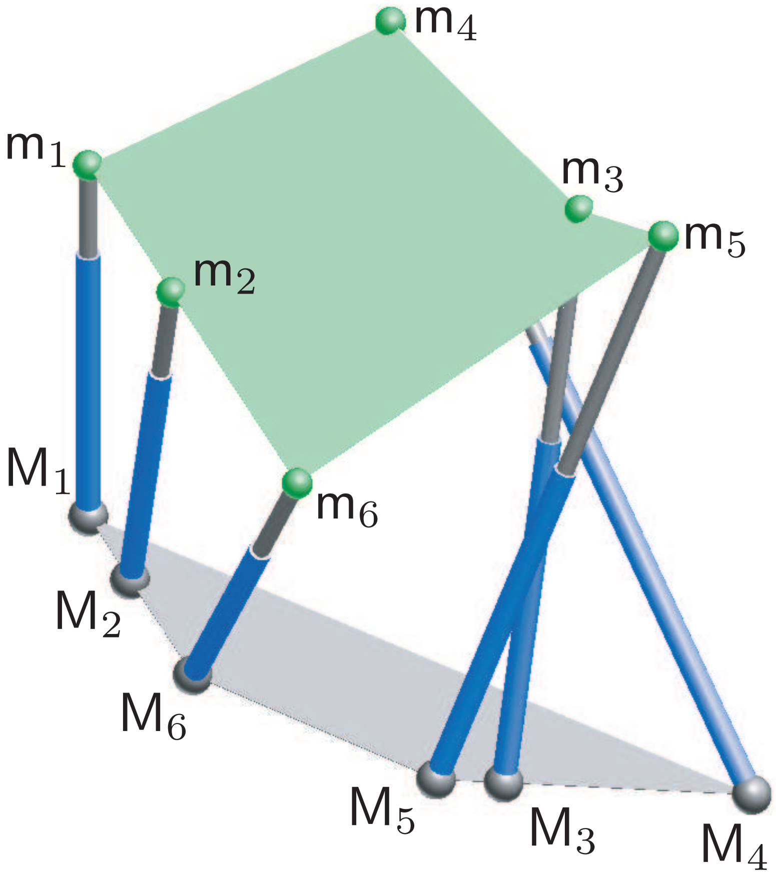
[2] Schönflies-singular SGPs of case (1a)

For the geometric interpretation of this rank condition please see Corollary 1 of **Nawratil [2010,A]**.

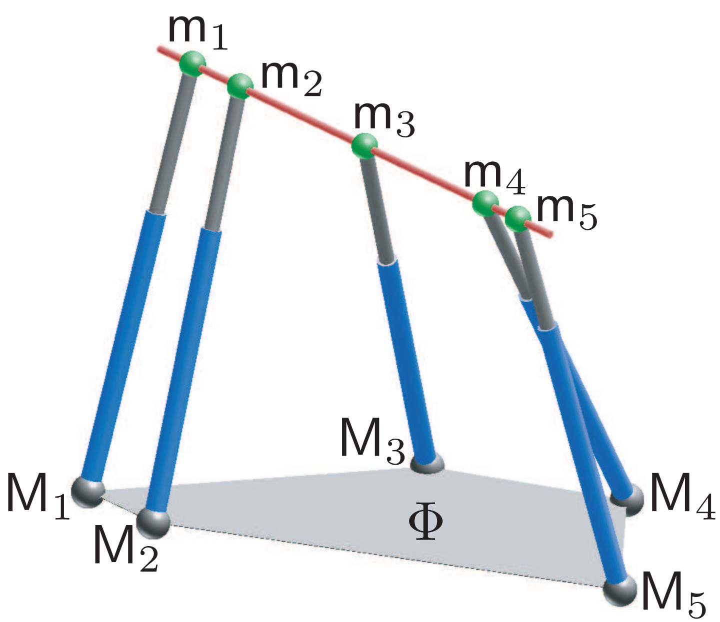
Theorem Nawratil [2010,A]
SGPs of the solution set of case (1a) have the rank property $rk(\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b})_1^6 = 4$.

Due to **Nawratil [2010,C]** this implies:

Theorem Nawratil [2010,A]
SGPs of the solution set of case (1a) have a quadratic singularity surface in the space of translations.



[2] Schönflies-singular SGPs of case (1b)



Degenerated cases of (1a): The 5 legs $l_1, \ldots, l_5$ belong in any configuration with $[m_1, \ldots, m_5] \parallel \Phi$ to a congruence of lines.

Main Theorem Nawratil [2010,B]
X(a)-singular planar SGPs with $\alpha \neq \beta$ and where a is not orthogonal to Φ or φ are necessarily architecturally singular.

Remark: Therefore the SGPs of the solution set of case (1a) are the only non-architecturally singular planar SGPs with $\alpha \neq \beta$, which are X(a)-singular. ◇

[3] Preparatory work and notation

Plücker coordinates of l_i can be written as $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - K\mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$

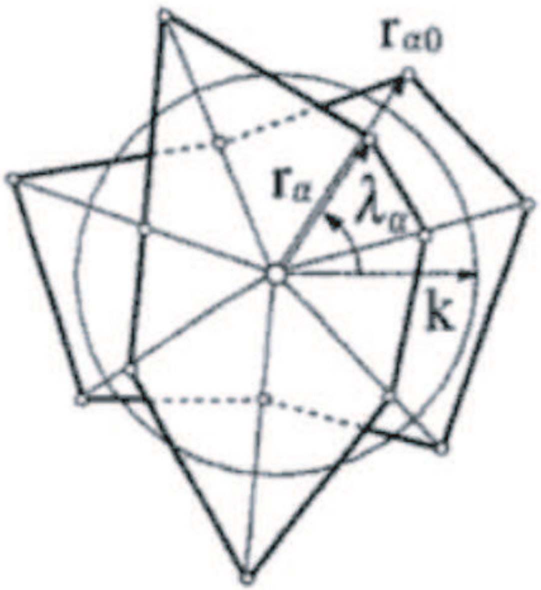
with $\mathbf{R} := (r_{ij}) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\ 2(e_1e_2 - e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 + e_0e_1) \\ 2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}$,

$\mathbf{t} := (t_1, t_2, t_3)^T$ and the homogenizing factor $K := e_0^2 + e_1^2 + e_2^2 + e_3^2$.

Remark: The group SO_3 is parametrized by Euler Parameters (e_0, e_1, e_2, e_3). ◇

l_i belong to a linear line complex $\iff$ $Q := det(\mathbf{Q}) = 0$ with $\mathbf{Q} := \begin{pmatrix} \mathbf{l}_1 & \widehat{\mathbf{l}}_1 \\ \dots & \dots \\ \mathbf{l}_6 & \widehat{\mathbf{l}}_6 \end{pmatrix}$

[4] Case (2a) $\alpha = \beta = \pi/2$



Wohlhart [2000] presented the *polygon platform*, i.e. the anchor points in Φ and φ are related by an inversion.

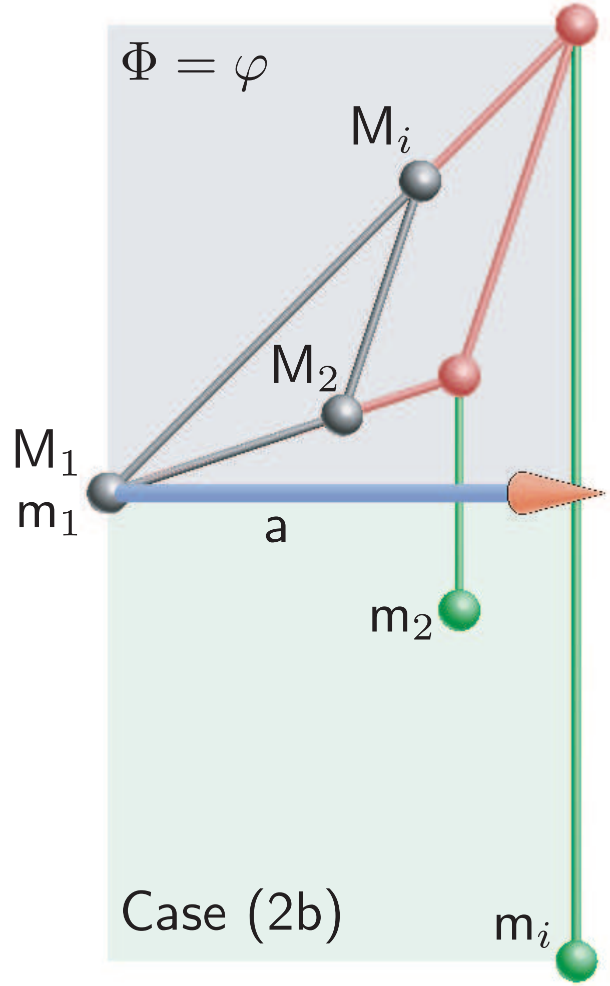
Theorem (2a)
A non-architecturally singular planar SGP, where the axis a is orthogonal to φ and Φ, is X(a)-singular iff $|\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, \mathbf{Ab} - \mathbf{Ba}|_1^6 = 0$ and $|\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, \mathbf{Aa} + \mathbf{Bb}|_1^6 = 0$ are fulfilled.

Proof: For coordinate systems with $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$ and $e_1 = e_2 = 0$ we get $Q = z^3K^2[F_1(e_0^2 - e_3^2) + 2F_2e_0e_3]$. □

Remark: A geometric interpretation of these 2 conditions is still missing. SGPs of the solution set of case (1a) also fulfill $F_1 = F_2 = 0$. ◇

[5] Case (2b) $0 \neq \alpha = \beta \neq \pi/2$

Theorem (2b)
A non-architecturally singular planar SGP with $0 \neq \alpha = \beta \neq \pi/2$ is X(a)-singular iff in a configuration with coinciding carrier planes the anchor points $\{M_i\}$ and $\{m_i\}$ are within an indirect similarity, which is the product of a dilation and the reflection on the orthogonal projection of a onto $\Phi = \varphi$.



Part [A] *No four anchor points are collinear*
We must distinguish two cases, depending on:

$$\left.\begin{matrix} \gamma > \alpha \\ \gamma = \alpha \end{matrix}\right\} \text{with } \gamma := \angle([M_1, M_2], a) \in [0, \pi/2].$$

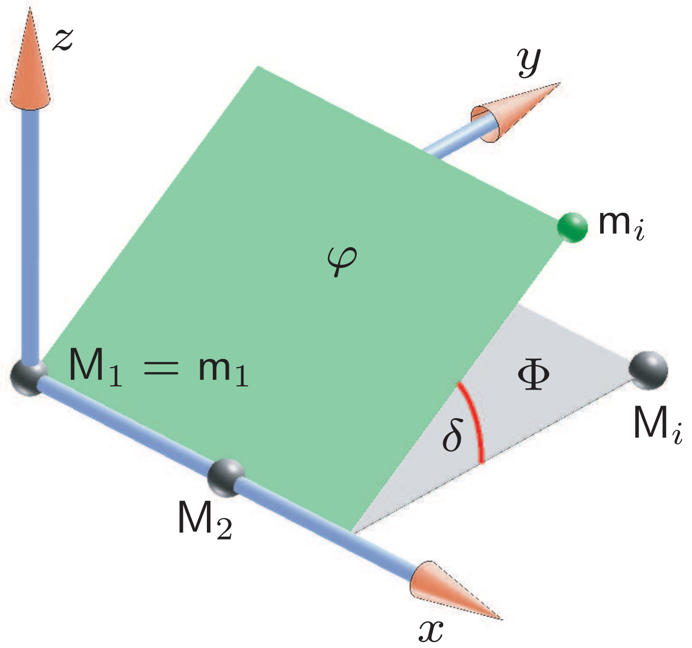
[5] Case (2b) $0 \neq \alpha = \beta \neq \pi/2$

Case $\gamma > \alpha$:

We can rotate φ about a such that the common line s of Φ and φ is parallel to $[\mathsf{M}_1, \mathsf{M}_2]$.

This yields the following coordinatization, which was also used for proving the corresponding part of the Main Theorem:

$\mathbf{M}_i = (A_i, B_i, 0)$, $\mathbf{m}_i = (a_i, b_i \cos\delta, b_i \sin\delta)$ with $A_1 = B_1 = B_2 = a_1 = b_1 = 0$, $\sin\delta \neq 0$.



We set $e_1 = e_4 \cos\mu$, $e_3 = e_4 \sin\mu$, $e_2 = e_4 n$ where e_4 is the homogenizing factor.

Moreover we denote the coefficient of $t_1^i t_2^j t_3^k e_0^u e_4^v$ of Q by Q_{ijk}^{uv}.

[5] Case (2b) $0 \neq \alpha = \beta \neq \pi/2$

Therefore we only have to consider those cases in the proof of the Main Theorem, which yield the contradiction $\alpha = \beta$. There is exactly one such case:

$e_2 \neq 0, \quad b_2 = 0, \quad b_4 = b_3 B_4/B_3, \quad b_5 = b_3 B_5/B_3, \quad K_1 = K_2 = K_4 = 0$ with

$K_1 = |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{a}|_2^6, \; K_2 = |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{b}|_2^6, \; K_4 = |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}|_2^6.$

A detailed study of this case shows (cf. paper) that the Q_{ijk}^{uv}'s can only vanish without contradiction (w.c.) for the solution given in Theorem (2b) or special cases (indirect congruence) of it. □

Case $\gamma = \alpha$:

For this case we can use the same coordinatization as in the case $\gamma > \alpha$, but now we have $e_2 = \delta = 0$. Again a detailed case study (cf. paper) yields the result. □

[5] Case (2b) $0 \neq \alpha = \beta \neq \pi/2$

Part [B] *Four anchor points are collinear*

Similar considerations as for part [A] show that possible solutions of this problem must yield the contradiction $\alpha = \beta$ in the corresponding proof of the Main Theorem. But there does not exist such a contradiction in this proof. But also the only case:

$\mathsf{M}_1, \ldots, \mathsf{M}_4$ and $\mathsf{m}_1, \ldots, \mathsf{m}_4$ collinear with $\alpha = \angle([\mathsf{M}_1, \mathsf{M}_4], \mathsf{a}) = \angle([\mathsf{m}_1, \mathsf{m}_4], \mathsf{a}) = \beta$

which is not covered by the Main Theorem yields a contradiction (cf. paper). □

Remark: The SGPs of the solution set of case (2b) are so-called equiform platforms, which were extensively studied (self-motions, singularities, ...) by **Karger [2001]**. In part [B] of the discussion, we get no solution as an equiform manipulator with four collinear anchor points is already architecturally singular. ◇

[6] Case (2c) $\alpha = \beta = 0$

Theorem (2c)
A non-architecturally singular planar SGP is X(a)-singular, where a is parallel to the x-axes of the fixed and moving system, iff one of the following cases holds:

(1) $rk(\mathbf{1}, \mathbf{b}, \mathbf{B}, \mathbf{Bb})_1^6 = 2$, (2) $rk(\mathbf{1}, \mathbf{b}, \mathbf{B}, \mathbf{Bb}, \mathbf{A} - \mathbf{a})_1^6 = 3$,

(3) $rk(1, \mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{a}, \mathbf{b}, \mathbf{Ab})_1^6 = 5$.

Proof: We can choose coordinate systems such that $A_1 = B_1 = a_1 = b_1 = 0$ hold. We compute Q under consideration of $e_2 = e_3 = 0$ and denotes the coefficient of $t_1^i t_2^j t_3^k e_0^u e_1^v$ of Q by Q_{ijk}^{uv}.

$$Q_{002}^{51} + Q_{002}^{15} + Q_{002}^{33} = 0, \quad Q_{101}^{42} + Q_{101}^{24} = 0 \quad \Longrightarrow \quad K_1 = K_2 = 0$$

In the following, the proof of the necessity splits up into two cases:

[6] Case $rk(\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb})_2^5 = 4$

Under this assumption, we can perform the generalized version of the matrix manipulation given by **Karger [2003]** $\Longrightarrow$ $\underline{\mathbf{l}}_6 = (v_1, v_2, v_3, 0, -w_3, w_2)$ with:

$v_i := r_{i1}K_1 + r_{i2}K_2, \quad w_j := r_{j1}K_3 + r_{j2}K_4$ with $K_3 := |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Aa}|_2^6.$

We compute Q in dependency of K_3 and K_4 which yields $K_4F[1032]$. $K_4 = 0$ yields solution (3). F is fulfilled identically if the following 7 conditions hold:

$$P_1 : Q_{100}^{53} = |\mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{b}|_2^5 = 0 \qquad P_2 : Q_{003}^{40} = |\mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}|_2^5 = 0$$

$$P_3 : Q_{001}^{62} - Q_{001}^{26} = |\mathbf{Ba}, \mathbf{Bb}, \mathbf{b}, \mathbf{a} - \mathbf{A}|_2^5 = 0 \qquad P_4 : Q_{002}^{33} = |\mathbf{A}, \mathbf{Bb}, \mathbf{a}, \mathbf{b}|_2^5 = 0$$

$$P_5 : Q_{001}^{62} + Q_{001}^{26} = |\mathbf{Ba}, \mathbf{Bb}, \mathbf{B}, \mathbf{a} - \mathbf{A}|_2^5 = 0 \qquad P_6 : Q_{101}^{42} = |\mathbf{B}, \mathbf{Bb}, \mathbf{a}, \mathbf{b}|_2^5 = 0$$

$$P_7 : Q_{011}^{42} = |\mathbf{B}, \mathbf{Ba}, \mathbf{b}, \mathbf{a} - \mathbf{A}|_2^5 - |\mathbf{B}, \mathbf{Bb}, \mathbf{A}, \mathbf{a}|_2^5 = 0$$

A case study (cf. paper) shows that the P_i's can only vanish w.c. for solution (2).

[6] Case $rk(\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb})_2^6 < 4$

$\nexists\, i, j, k, l \in \{2, \dots, 6\}$ with $|\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}|_{(i,j,k,l)} \neq 0 \Rightarrow rk(\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb})_2^6 < 4$. A close inspection of Q shows, that it vanishes independently of t_1, t_2, t_3, e_0, e_1 if the following 9 conditions are fulfilled:

$$R_1:\ Q_{003}^{40} - Q_{003}^{04} = |\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{Ab}|_2^6 = 0 \qquad R_2:\ Q_{102}^{31} = |\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{Bb}|_2^6 = 0$$

$$R_3:\ Q_{003}^{04} + Q_{003}^{04} = |\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{B}, \mathbf{Ba}|_2^6 = 0 \qquad R_4:\ Q_{020}^{33} = |\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{Ab}, \mathbf{Bb}|_2^6 = 0$$

$$R_5:\ Q_{101}^{62} + Q_{101}^{26} = |\mathbf{a}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}|_2^6 = 0 \qquad R_6:\ Q_{110}^{33} = |\mathbf{a}, \mathbf{b}, \mathbf{B}, \mathbf{Ab}, \mathbf{Bb}|_2^6 = 0$$

$$R_7:\ Q_{110}^{33} = |\mathbf{b}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}, \mathbf{A} - \mathbf{a}|_2^6 = 0 \qquad R_8:\ Q_{020}^{33} = |\mathbf{b}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}|_2^6 = 0$$

$$R_9:\ Q_{002}^{40} + Q_{002}^{04} = |\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{Ba}, \mathbf{Bb}|_2^6 + |\mathbf{a}, \mathbf{A}, \mathbf{B}, \mathbf{Ab}, \mathbf{Bb}|_2^6 + |\mathbf{b}, \mathbf{B}, \mathbf{Ab}, \mathbf{Ba}, \mathbf{A} - \mathbf{a}|_2^6 = 0$$

A detailed case study (cf. paper) shows that the R_i's can only vanish w.c. for solution (1) and (3), respectively. This closes the proof of the necessity. $\square$

[6] Sufficiency of the conditions

The proof of the sufficiency of the conditions of solution (1) and (2) was done analytically in contrast to the one of solution (3), which was done geometrically according to the method of **Mick and Röschel [1998]**.
For details please see the corresponding technical report **Nawratil [2009]**.

As a byproduct, we get the following geometric characterization of solution (3):

Theorem
Given are two sets of points $\{\mathrm{M}_i\}$ and $\{\mathrm{m}_i\}$ $(i = 1, \ldots, 6)$ in two non-parallel planes Φ and φ. Then the non-architecturally singular planar SGP is X(a)-singular with a $:= (\Phi, \varphi)$ if $\{\mathrm{M}_i, \mathrm{m}_i\}$ are 3-fold conjugate pairs of points with respect to a 2-dim. linear manifold of correlations, whereas the ideal point of a is self-conjugate.

[6] Geometric meaning of the conditions

A short case study (cf. **Nawratil [2009]**) of solution (1) shows that the rank condition $rk(\mathbf{1}, \mathbf{b}, \mathbf{B}, \mathbf{Bb})_1^6 = 2$ corresponds with following two designs:

- $[\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4] \parallel [\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3, \mathsf{m}_4] \parallel [\mathsf{M}_5, \mathsf{M}_6] \parallel [\mathsf{m}_5, \mathsf{m}_6] \parallel$ a,
- $[\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3] \parallel [\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3] \parallel [\mathsf{M}_4, \mathsf{M}_5, \mathsf{M}_6] \parallel [\mathsf{m}_4, \mathsf{m}_5, \mathsf{m}_6] \parallel$ a.

The geometric meaning of the rank condition $rk(\mathbf{1}, \mathbf{b}, \mathbf{B}, \mathbf{Bb}, \mathbf{A} - \mathbf{a})_1^6 = 3$ of solution (2) is still missing.

Remark: Until now, we are only able to identify a geometric meaning with the necessary condition $|\mathbf{1}, \mathbf{b}, \mathbf{B}, \mathbf{Bb}|_{(i,j,k,l)} = 0$. This condition equals $DV(\mathsf{G}_i, \mathsf{G}_j, \mathsf{G}_k, \mathsf{G}_l) = DV(\mathsf{g}_i, \mathsf{g}_j, \mathsf{g}_k, \mathsf{g}_l)$ with $\mathsf{G}_i := [\mathsf{M}_i, \mathsf{U}]$ and $\mathsf{g}_i := [\mathsf{m}_i, \mathsf{U}]$, where DV denotes the cross-ratio and U the ideal point of the axis a. ◇

[7] Cartesian-singular planar SGPs

Definition **Cartesian-singular SGP**
A SGP is called Cartesian-singular (or T(3)-singular) if the manipulator is singular for all transformations from T(3) (applied to the moving part of the SGP).

Due to the Lemma 2.1 of **Mick and Röschel [1998]**, the solution set of case (2c) equals the set of Cartesian-singular planar SGPs, where Φ and φ are not parallel.

Therefore only the case with parallel platform and base is missing, which follows directly from the proof of Theorem (2a) by setting $e_0 = 1$ and $e_3 = 0$.

Theorem
A non-architecturally singular planar SGP, where φ and Φ are parallel, is T(3)-singular iff $|\mathbf{1}, \mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, \mathbf{Ab} - \mathbf{Ba}|_1^6 = 0$ holds.

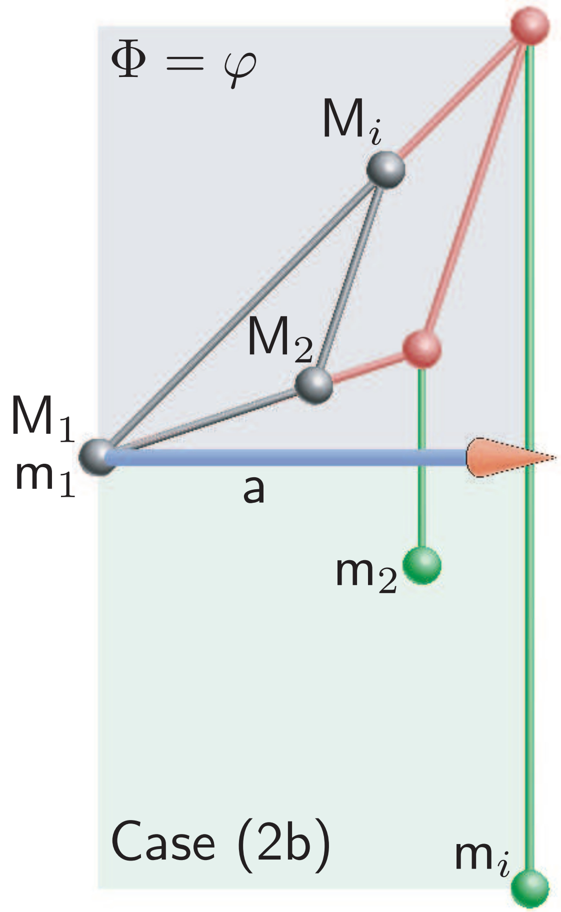
[8] Conclusion

We discussed the special cases (i.e. $\alpha = \beta$) of Schönflies-singular planar SGPs, whereas we distinguished three cases:

(2a) $\alpha = \pi/2$, (2b) $\alpha \in]0, \pi/2[$, (2c) $\alpha = 0$.

As a byproduct, we also characterized all Cartesian-singular planar SGPs.

The determination of the whole set $\mathcal{S}$ of non-planar X(a)-singular SGPs remains open. Note that the degenerated cases of (1a) imply manipulators of $\mathcal{S}$.



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