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HABILITATION THESIS

FLEXIBLE OCTAHEDRA IN THE PROJECTIVE EXTENSION OF THE EUCLIDEAN 3-SPACE AND THEIR APPLICATION

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Chapter 1

Introduction

A polyhedron is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e. in such a way that every face remains congruent to itself during the flex.

All types of flexible octahedra[1] in E^3 were firstly classified by R. BRICARD [3] in 1897. These so-called *Bricard octahedra* are as follows:

type 1 All three pairs of opposite vertices are symmetric with respect to a common line.

type 2 Two pairs of opposite vertices are symmetric with respect to a common plane which passes through the remaining two vertices.

type 3 For a detailed discussion of this asymmetric type we refer to H. STACHEL [15]. We only want to mention that these flexible octahedra possess two flat poses.

In 1978 R. CONNELLY [6] sketched a further algebraic method for the determination of all flexible octahedra in E^3. H. STACHEL [13] presented a new proof which uses mainly arguments from projective geometry beside the converse of *Ivory's Theorem*, which limits this approach to flexible octahedra with finite vertices.

A. KOKOTSAKIS [8] discussed the flexible octahedra as special cases of a sort of meshes named after him (see Fig. 1.1a). As recognized by the author in [10], Kokotsakis' very short and elegant proof for *Bricard octahedra* is also valid for type 3 in the projective extension of E^3 if no two opposite vertices are ideal points.

H. STACHEL [15] also proved the existence of flexible octahedra of type 3 with one vertex at infinity and presented their construction. But up to recent, there are no proofs for Bricard's famous statement known to the author, which enclose the projective extension of E^3 although these flexible structures attracted many prominent mathematicians; e.g. G.T. BENNETT [1], W. BLASCHKE [2], O. BOTTEMA [5], H. LEBESGUE [7] and W. WUNDERLICH [17].

[1] No face degenerates into a line and no two neighboring faces coincide during the flex.

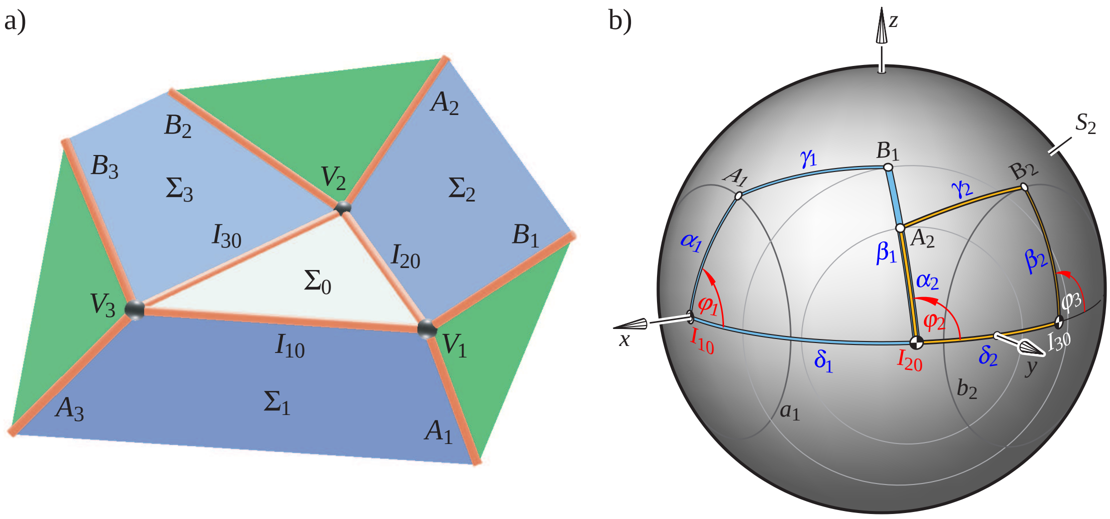

Figure 1.1: a) A *Kokotsakis mesh* is a polyhedral structure consisting of a n-sided central polygon $\Sigma_0 \in E^3$ surrounded by a belt of polygons in the following way: Each side I_{i0} of Σ_0 is shared by an adjacent polygon Σ_i, and the relative motion between cyclically consecutive neighbor polygons is a spherical coupler motion. Therefore each vertex V_i of Σ_0 is the meeting point of four faces. Here the *Kokotsakis* mesh for $n = 3$ which determines an octahedron is given.
b) Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, 2$ (Courtesy of H. STACHEL).

The presented habilitation thesis consisting of the three articles:

[A] NAWRATIL, G., AND STACHEL, H.: *Composition of spherical four-bar-mechanisms*, New Trends in Mechanisms Science (D. Pisla et al. eds.), 99–106, Springer (2010).

[B] NAWRATIL, G.: *Reducible compositions of spherical four-bar linkages with a spherical coupler component*, Mechanism and Machine Theory, in press.

[C] NAWRATIL, G.: *Flexible octahedra in the projective extension of the Euclidean 3-space*, Journal of Geometry and Graphics **14**(2) 147–169 (2010).

closes this gap. Our approach is based on a kinematic analysis of *Kokotsakis meshes* as the composition of spherical coupler motions (see Fig. 1.1b) given by H. STACHEL [16].

The author determined in [B] all cases where the relation between the input angle φ_1 of the arm $I_{10}A_1$ and the output angle φ_3 of $I_{30}B_2$ is reducible and where additionally at least one of these components produces a transmission which equals that of a single spherical coupler. These so-called reducible compositions with a spherical coupler component can be classified into 4 types (cf. Corollary 1 of [B]), whereby the case of the spherical focal mechanism was discussed in more detail under the guidance[2] of H. STACHEL in [A].

[2]The author's contribution to [A] was Lemma 1, the proof of the second part of Lemma 2 and the generation of the given example.

Based on these studies the author of this thesis showed in a first step that there only exist type 2 and type 3 octahedra with one vertex in the plane at infinity[3] (cf. [10]). In a further step the author determined in [C] all octahedra, where at least two vertices are ideal points.

Acknowledgement The research reported in [A,B,C] was supported by Grant No. I 408-N13 of the Austrian Science Fund FWF within the project "Flexible polyhedra and frameworks in different spaces", an international cooperation between FWF and RFBR, the Russian Foundation for Basic Research.

1.1 Application and future research

Analogously to [B] the author prepares a classification of all reducible compositions of spherical four-bar linkages without a spherical coupler component. Based on this study and [B] one can give a complete list of all flexible Kokotsakis meshes with a 4-sided central polygon (a so-called 3×3 complex or Neunflach in German) if *Stachel's conjecture* holds true that all multiply decomposable compounds of spherical four-bars are reducible (with exception of the translatory type and planar-symmetric type).

Such a listing is of great interest because A.I. BOBENKO ET AL. [4] showed that a polyhedral mesh with valence 4 composed of planar quadrilaterals is flexible if and only if all 3×3 complexes are flexible. One possible application scenario is the architectural design of flexible claddings composed of planar quads (cf. H. POTTMANN ET AL. [12]).

Moreover it would be interesting to apply the principle of transference (cf. [14]) to each item of the resulting list of reducible compositions of spherical four-bars in order to study their dual extensions.

A practical application of the author's studies can be found in the field of robotics, because flexible octahedra with one vertex in the plane at infinity correspond with the non-trivial (cf. footnote 1) self-motions of TSSM manipulators with two parallel rotary axes. Moreover the publication [10] also closes the classification of self-motions for parallel manipulators of TSSM type and of 6-3 planar Stewart Gough platforms, respectively.

It should also be noted, that the author showed in a recent work [11], that flexible octahedra also play a central role in the theory of self-motions of general Stewart Gough manipulators with planar platform and planar base.

A further application in robotics could be an open serial chain composed of prisms $\Pi_0, \ldots \Pi_n$ where each pair of neighboring prisms Π_i, Π_{i+1} $(i = 0, \ldots, n-1)$ forms a flexible octahedron, where two opposite vertices are ideal points. Such mechanisms with a constrained motion are also worth to be studied in more detail.

Moreover, if we additionally assume that $\Pi_0 = \Pi_n$ holds, we get a closed serial chain which is in general rigid. It would also be interesting under which geometric conditions such structures are still flexible. Clearly, some aspects of this question are connected with the problem of nR overconstrained linkages (e.g. the spatial $4R$ overconstrained linkage is the Bennett mechanism).

[3] The article [10] can be regarded as the continuation of [9], where a conjecture about the solution of this problem was formulated by the author.

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Chapter 2

Composition of spherical four-bar-mechanisms *(with H. Stachel)*

Abstract We study the transmission by two consecutive four-bar linkages with aligned frame links. The paper focusses on so-called "reducible" examples on the sphere where the 4-4-correspondence between the input angle of the first four-bar and the output-angle of the second one splits. Also the question is discussed whether the components can equal the transmission of a single four-bar. A new family of reducible compositions is the spherical analogue of compositions involved at Burmester's focal mechanism.

Keywords spherical four-bar linkage, overconstrained linkage, Kokotsakis mesh, Burmester's focal mechanism, 4-4-correspondence

2.1 Introduction

Let a spherical four-bar linkage be given by the quadrangle $I_{10}A_1B_1I_{20}$ (see Fig. 2.1) with the frame link $I_{10}I_{20}$, the coupler A_1B_1 and the driving arm $I_{10}A_1$. We use the output angle φ_2 of this linkage as the input angle of a second coupler motion with vertices $I_{20}A_2B_2I_{30}$. The two frame links are assumed in aligned position as well as the driven arm $I_{20}B_1$ of the first four-bar and the driving arm $I_{20}A_2$ of the second one. This gives rise to the following

Questions:
(i) Can it happen that the relation between the input angle φ_1 of the arm $I_{10}A_1$ and the output angle φ_3 of $I_{30}B_2$ is reducible so that the composition admits two one-parameter motions? In this case we call the composition *reducible*.
(ii) Can one of these components produce a transmission which equals that of a single four-bar linkage ?

A complete classification of such reducible compositions is still open, but some examples are known (see Sect. 2.3). For almost all of them exist planar counterparts. We focus on a case where the planar analogue is involved at Burmester's focal mechanism [2, 5, 11, 4] (see Fig. 2.3a). It is not possible to transfer the complete focal mechanism onto the sphere as it is essentially based

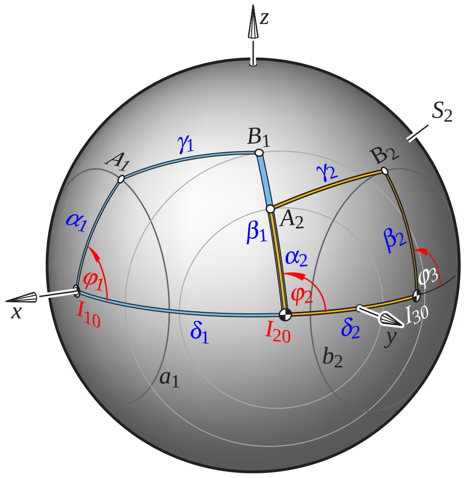

Figure 2.1: Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1,2$

on the fact that the sum of interior angles in a planar quadrangle equals 2π, and this is no longer true in spherical geometry. Nevertheless, algebraic arguments show that the reducibility of the included four-bar compositions can be transferred.

Remark 2.1. The problem under consideration is of importance for the classification of flexible Kokotsakis meshes [7, 1, 10]. This results from the fact that the spherical image of a flexible mesh consists of two compositions of spherical four-bars sharing the transmission $\varphi_1 \mapsto \varphi_3$. All the examples known up to recent [6, 10] are based on reducible compositions. ⋄

The geometry on the unit sphere S^2 contains some ambiguities. Therefore we introduce the following *notations and conventions:*

1. Each point A on S^2 has a diametrically opposed point $\overline{A}$, its *antipode*. For any two points A, B with $B \neq A, \overline{A}$ the *spherical segment* or *bar AB* stands for the shorter of the two connecting arcs on the great circle spanned by A and B. We denote this great circle by $[AB]$.

2. The *spherical distance* $\overline{AB}$ is defined as the arc length of the segment AB on S^2. We require $0 \leq \overline{AB} \leq \pi$ thus including also the limiting cases $B = A$ and $B = \overline{A}$.

3. The *oriented angle* $\sphericalangle ABC$ on S^2 is the angle of the rotation about the axis OB which carries the segment BA into a position aligned with the segment BC. This angle is oriented in the mathematical sense, if looking from outside, and can be bounded by $-\pi < \sphericalangle ABC \leq \pi$.

2.2 Transmission by a spherical four-bar linkage

We start with the analysis of the first spherical four-bar linkage with the frame link $I_{10}I_{20}$ and the coupler A_1B_1 (Fig. 2.1). We set $\alpha_1 = \overline{I_{10}A_1}$ for the length of the driving arm, $\beta_1 = \overline{I_{20}B_1}$ for the output arm, $\gamma_1 := \overline{A_1B_1}$, and $\delta_1 := \overline{I_{10}I_{20}}$. We may suppose

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi\,.$$

The movement of the coupler remains unchanged when A_1 is replaced by its antipode $\overline{A}_1$ and at the same time α_1 and γ_1 are substituted by $\pi-\alpha_1$ and $\pi-\gamma_1$, respectively. The same holds for the other vertices. When I_{10} is replaced by its antipode $\overline{I}_{10}$, then also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of S^2 either at I_{10} or at $\overline{I}_{10}$.

We use a cartesian coordinate frame with I_{10} on the positive x-axis and $I_{10}I_{20}$ in the xy-plane such that I_{20} has a positive y-coordinate (see Fig. 2.1). The input angle φ_1 is measured between $I_{10}I_{20}$ and the driving arm $I_{10}A_1$ in mathematically positive sense. The output angle $\varphi_2 = \sphericalangle \overline{I}_{10}I_{20}B_1$ is the oriented exterior angle at vertex I_{20}. This results in the following coordinates:

$$A_1 = \begin{pmatrix} c\alpha_1 \\ s\alpha_1\, c\varphi_1 \\ s\alpha_1\, s\varphi_1 \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} c\beta_1\, c\delta_1 - s\beta_1\, s\delta_1\, c\varphi_2 \\ c\beta_1\, s\delta_1 + s\beta_1\, c\delta_1\, c\varphi_2 \\ s\beta_1\, s\varphi_2 \end{pmatrix}.$$

Herein s and c are abbreviations for the sine and cosine function, respectively. In these equations the lengths α_1, β_1 and δ_1 are signed. The coordinates would also be valid for negative lengths. The constant length γ_1 of the coupler implies

$$\begin{aligned} &c\alpha_1\, c\beta_1\, c\delta_1 - c\alpha_1\, s\beta_1\, s\delta_1\, c\varphi_2 + s\alpha_1\, c\beta_1\, s\delta_1\, c\varphi_1 \\ &+ s\alpha_1\, s\beta_1\, c\delta_1\, c\varphi_1\, c\varphi_2 + s\alpha_1\, s\beta_1\, s\varphi_1\, s\varphi_2 = c\gamma_1 . \end{aligned} \tag{2.1}$$

In comparison to [3] we emphasize algebraic aspects of this transmission. Hence we express $s\varphi_i$ and $c\varphi_i$ in terms of $t_i := \tan(\varphi_i/2)$ since t_1 is a *projective coordinate* of point A_1 on the circle a_1. The same is true for t_2 and $B_1 \in b_1$. From (2.1) we obtain

$$\begin{aligned} -K_1(1+t_1^2)(1-t_2^2) + L_1(1-t_1^2)(1+t_2^2) + M_1(1-t_1^2)(1-t_2^2) \\ + 4\, s\alpha_1\, s\beta_1\, t_1 t_2 + N_1(1+t_1^2)(1+t_1^2) = 0 , \end{aligned}$$

$$\begin{aligned} K_1 &= c\alpha_1\, s\beta_1\, s\delta_1 , \quad M_1 = s\alpha_1\, s\beta_1\, c\delta_1 , \\ L_1 &= s\alpha_1\, c\beta_1\, s\delta_1 , \quad N_1 = c\alpha_1\, c\beta_1\, c\delta_1 - c\gamma_1 . \end{aligned} \tag{2.2}$$

This biquadratic equation describes a *2-2-correspondence* between points A_1 on circle $a_1 = (I_{10};\alpha_1)$ and B_1 on $b_1 = (I_{20};\beta_1)$. It can be abbreviated by

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \tag{2.3}$$

setting

$$\begin{aligned} c_{00} = -K_1 + L_1 + M_1 + N_1 , \quad c_{11} = 4\, s\alpha_1\, s\beta_1 , \quad c_{02} = K_1 + L_1 - M_1 + N_1 , \\ c_{20} = -K_1 - L_1 - M_1 + N_1 , \qquad c_{22} = K_1 - L_1 + M_1 + N_1 \end{aligned} \tag{2.4}$$

under $c_{11} \neq 0$. Alternative expressions can be found in [10].

Remark 2.2. Also at planar four-bar linkages there is a 2-2-correspondence of type (2.3). ◇

There are two particular cases:

Spherical isogram: Under the conditions $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ opposite sides of the quadrangle $I_{10}A_1B_1I_{20}$ have equal lengths. In this case we have $c_{00} = c_{22} = 0$ in (2.3), and Eq. (2.1) converts into $[s(\alpha_1-\gamma_1)t_2 - (s\alpha_1 + s\gamma_1)t_1]\,[s(\alpha_1-\gamma_1)t_2 - (s\alpha_1 - s\gamma_1)t_1]$ (for details see [10]). *The*

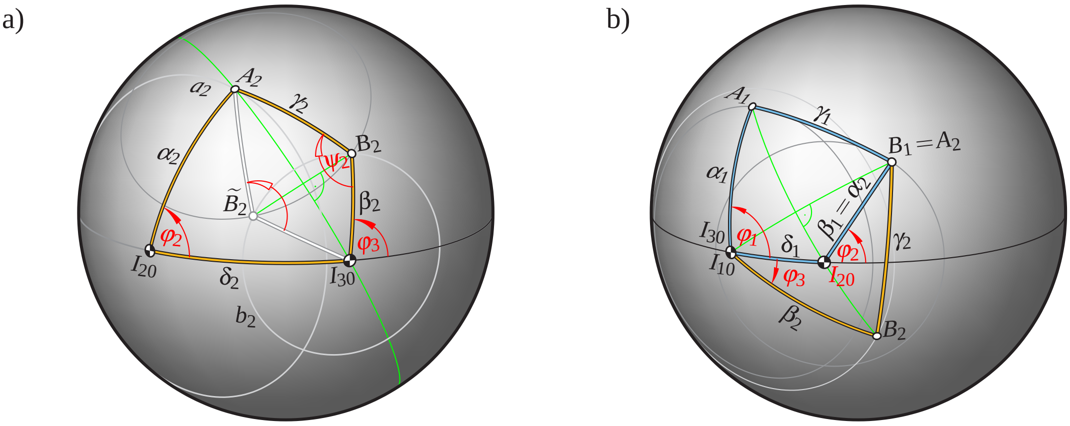

Figure 2.2: a) Opposite angles φ_2 and ψ_2 at the second spherical four-bar $I_{20}A_2B_2I_{30}$.
b) Composition of two orthogonal four-bar linkages with $I_{30} = I_{10}$.

2-2-correspondence splits into two projectivities [1] $t_1 \mapsto t_2 = \frac{\mathrm{s}\alpha_1 \pm \mathrm{s}\gamma_1}{\mathrm{s}(\alpha_1-\gamma_1)}\, t_1$, provided $\alpha_1 \neq \gamma_1, \pi - \gamma_1$. Both projectivities keep $t_1 = 0$ and $t_1 = \infty$ fixed. These parameters belong to the two aligned positions of coupler A_1B_1 and frame link $I_{10}I_{20}$. In these positions a bifurcation is possible between the two one-parameter motions of the coupler against the frame link.

Orthogonal case: For a given point $A_1 \in a_1$ the corresponding $B_1, \widetilde{B}_1 \in b_1$ are the points of intersection between the circles $(A_1;\gamma_1)$ and $b_1 = (I_{20};\beta_1)$ (compare Fig. 2.2a). Hence, the corresponding B_1 and $\widetilde{B}_1$ are located on a great circle perpendicular to the great circle $[A_1I_{20}]$. Under the condition $\cos\alpha_1 \cos\beta_1 = \cos\gamma_1 \cos\delta_1$ which according to [10] is equivalent to $\det\begin{pmatrix} c_{22} & c_{02} \\ c_{20} & c_{00}\end{pmatrix} = 0$, the diagonals of the spherical quadrangle $I_{10}A_1B_1I_{20}$ are orthogonal (Fig. 2.2b) as each of the products equals the products of cosines of the four segments on the two diagonals. Hence, B_1 and $\widetilde{B}_1$ are always aligned with I_{10}, but also conversely, the two points A_1 and $\widetilde{A}_1$ corresponding to B_1 are aligned with I_{20}.

Note that the 2-2-correspondence (2.3) depends only on the ratio of the coefficients $c_{22} : \cdots : c_{00}$. With the aid of a CA-system we can prove:

Lemma 2.1. *For any spherical four-bar linkage the coefficients c_{ik} defined by* (2.4) *obey*

$$c_{11}^6 + 16\,(K^2 + L^2 - 2M^2 - 1)\,c_{11}^4 + 256\left[(M^2 - K^2)(M^2 - L^2) + 2M^2\right] c_{11}^2 - 4096\,M^4 = 0\,.$$

Conversely, in the complex extension any biquadratic equation of type (2.3) *defines the spherical four-bar linkage uniquely — up to replacement of vertices by their antipodes. However, the vertices need not be real.*

At the end of our analysis we focus on opposite angles in the spherical quadrangle $I_{20}A_2B_2I_{30}$: The diagonal A_2I_{30} divides the quadrangle into two triangles, and we inspect the interior angles φ_2 at I_{20} and ψ_2 at B_2 (Fig. 2.2a). Also for non-convex quadrangles, the spherical Cosine Theorem

[1] Since the vertices of the moving quadrangle can be replaced by their antipodes without changing the motion, this case is equivalent to $\beta_1 = \pi - \alpha_1$ and $\delta_1 = \pi - \gamma_1$. We will not mention this in the future but only refer to an 'appropriate choice of orientations' of the hinges.

implies

$$\cos\overline{A_2I_{30}} = \mathrm{c}\beta_2\,\mathrm{c}\gamma_2 + \mathrm{s}\beta_2\,\mathrm{s}\gamma_2\,\mathrm{c}\psi_2 = \mathrm{c}\alpha_2\,\mathrm{c}\delta_2 + \mathrm{s}\alpha_2\,\mathrm{s}\delta_2\,\mathrm{c}\varphi_2\,.$$

Hence there is a linear function

$$\mathrm{c}\psi_2 = k_2 + l_2\mathrm{c}\varphi_2 \quad \text{with} \quad k_2 = \frac{\mathrm{c}\alpha_2\,\mathrm{c}\delta_2 - \mathrm{c}\beta_2\,\mathrm{c}\gamma_2}{\mathrm{s}\beta_2\,\mathrm{s}\gamma_2}\,, \quad l_2 = \frac{\mathrm{s}\alpha_2\,\mathrm{s}\delta_2}{\mathrm{s}\beta_2\,\mathrm{s}\gamma_2}\,. \tag{2.5}$$

For later use it is necessary to define also ψ_2 as an oriented angle, hence

$$\psi_2 = \sphericalangle I_{30}B_2A_2, \quad \varphi_2 = \sphericalangle I_{30}I_{20}A_2 \quad \text{under} \quad -\pi < \psi_2, \varphi_2 \le \pi\,.$$

We note that in general for given φ_2 there are two positions B_2 and $\widetilde{B}_2$ on the circle b_1 obeying (2.5) (Fig. 2.2a). They are placed symmetrically with respect to the diagonal A_2I_{30}; the signs of the corresponding oriented angles ψ_2 are different.

Remark 2.3. Also Eq. (2.5) describes a 2-2-correspondence of type (2.3) between φ_2 and ψ_2, but with $c_{11} = 0$. A parameter count reveals that this 2-2-correspondence does not characterize the underlying four-bar uniquely. ◇

2.3 Composition of two spherical four-bar linkages

Now we use the output angle φ_2 of the first four-bar linkage as input angle of a second coupler motion with vertices $I_{20}A_2B_2I_{30}$ and consecutive side lengths α_2, γ_2, β_2, and δ_2 (Fig. 2.1). The two frame links are assumed in aligned position. In the case $\sphericalangle I_{10}I_{20}I_{30} = \pi$ the length δ_2 is positive, otherwise negative. Analogously, a negative α_2 expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of β_2 means replacing the output angle φ_3 by $\varphi_3 - \pi$. The sign of γ_2 has no influence on the transmission.

Due to (2.3) the transmission between the angles φ_1, φ_2 and the output angle φ_3 of the second four-bar with $t_3 := \tan(\varphi_3/2)$ can be expressed by the two biquadratic equations

$$\begin{aligned} c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0\,, \\ d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0\,. \end{aligned} \tag{2.6}$$

The d_{ik} are defined by equations analogue to Eqs. (2.4) and (2.2). We eliminate t_2 by computing the *resultant* of the two polynomials with respect to t_2 and obtain

$$\det\begin{pmatrix} c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} & 0 \\ 0 & c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} \\ d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00} & 0 \\ 0 & d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00} \end{pmatrix} = 0\,. \tag{2.7}$$

This biquartic equation expresses a *4-4-correspondence* between points A_1 and B_2 on the circles a_1 and b_2, respectively (Fig. 2.1).

Up to recent, to the authors' best knowledge the following examples of reducible compositions are known. Under appropriate notation and orientation these are:

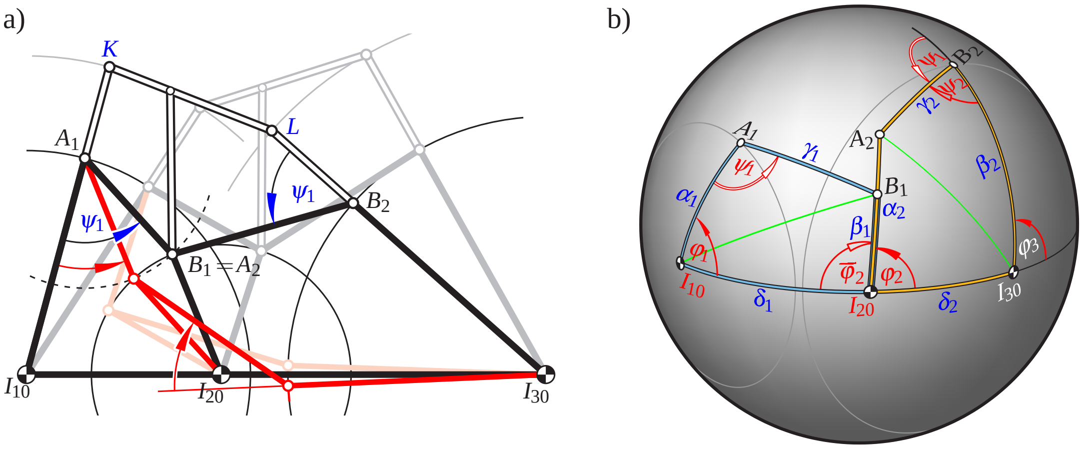

Figure 2.3: a) Burmester's focal mechanism and the second component of a four-bar composition. b) Reducible spherical composition obeying Dixon's angle condition for ψ_1 — equally oriented

1. **Isogonal** type [7, 1]: At each four-bar opposite sides are congruent; the transmission $\varphi_1 \to \varphi_3$ is the product of two projectivities and therefore again a projectivity. Each of the 4 possibilities can be obtained by one single four-bar linkage. This is the spherical image of a flexible octahedron of Type 3 (see, e.g., [8]).

2. **Orthogonal** type [10]: We combine two orthogonal four-bars such that they have one diagonal in common (see Fig. 2.2b), i.e., under $\alpha_2 = \beta_1$ and $\delta_2 = -\delta_1$, hence $I_{30} = I_{10}$. Then the 4-4-correspondence between A_1 and B_2 is the square of a 2-2-correspondence.

3. **Symmetric** type [10]: We specify the second four-bar linkage as mirror of the first one after reflection in an angle bisector at I_{20} (see [Fig. 5b,10]). Thus φ_3 is congruent to the angle opposite to φ_1 in the first quadrangle. Hence the 4-4-correspondence is reducible; the components are expressed by the linear relation $c\varphi_3 = \pm(k_1 + l_1 c\varphi_1)$ in analogy to (2.5).

At the end we present a new family of reducible compositions: In Fig. 2.3a Burmester's focal mechanism is displayed, an overconstrained planar linkage (see [2, 5, 11, 4]). The full lines in this figure show a planar composition of two four-bar linkages with the additional property that the transmission $\varphi_1 \to \varphi_3$ equals that of one single four-bar linkage with the coupler KL. Due to Dixon and Wunderlich this composition is characterized by congruent angles $\psi_1 = \sphericalangle I_{10}A_1B_1$ and $\sphericalangle LB_2A_2$ which is adjacent to $\psi_2 = \sphericalangle I_{30}B_2A_2$.[2]

However, this defines only one component of the full motion of this composition. The second component is defined by $\psi_1 = \sphericalangle I_{10}A_1B_1 = - \sphericalangle LB_2A_2$ (see Fig. 2.3a). For the sake of brevity, we call the overall condition $\sphericalangle I_{10}A_1B_1 = \pm \sphericalangle LB_2A_2$ *Dixon's angle condition* and prove in the sequel that also at the spherical analogue this defines reducible compositions.

[2]This condition is invariant against exchanging the input and the output link. The compositions along the other sides of the four-bar $I_{10}KLI_{30}$ in Fig. 2.3a obey analogous angle conditions.

Lemma 2.2. *For the composition of two spherical four-bars Dixon's angle condition* $\sphericalangle I_{10}A_1B_1 = \pm \sphericalangle \bar{I}_{30}B_2A_2$ *is equivalent to*

$$\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1 : \mathrm{s}\beta_1\,\mathrm{s}\delta_1 : (\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1) = \pm\mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\alpha_2\,\mathrm{c}\delta_2 - \mathrm{c}\beta_2\,\mathrm{c}\gamma_2).$$

In terms of c_{ik} *and* d_{ik} *it is equivalent to proportional polynomials*

$$D_1 = (c_{11}t_2)^2 - 4(c_{22}t_2^2 + c_{20})(c_{02}t_2^2 + c_{00}), \quad D_2 = (d_{11}t_2)^2 - 4(d_{22}t_2^2 + d_{02})(d_{20}t_2^2 + d_{00}).$$

Proof. In the notation of Fig. 2.3b Dixon's angle condition is equivalent to $\mathrm{c}\psi_1 = \mathrm{c}(\pi - \psi_2) = -\mathrm{c}\psi_2 = -k_2 - l_2\,\mathrm{c}\varphi_2$ by (2.5). At the first four-bar we have analogously

$$\mathrm{c}\psi_1 = -k_1 - l_1\,\mathrm{c}\varphi_2\,, \quad k_1 = \frac{\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1}{\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1}\,, \quad l_1 = \frac{\mathrm{s}\beta_1\,\mathrm{s}\delta_1}{\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1}\,. \tag{2.8}$$

Hence, $\mathrm{c}\psi_1 = -\mathrm{c}\psi_2$ for all $\mathrm{c}\varphi_2$ is equivalent to $k_1 = k_2$ and $l_1 = l_2$. This gives the first statement in Lemma 2.2. The $\pm$ results from the fact that changing the sign of γ_2 has no influence on the 2-2-correspondence $\varphi_2 \mapsto \varphi_3$, but replaces ψ_2 by $\psi_2 - \pi$.

If the angle condition holds and $\psi_1 = 0$ or π, the distances $\overline{I_{10}B_1}$ and $\overline{I_{30}A_2}$ are extremal. For the corresponding angles φ_2 there is just one corresponding φ_1 and one φ_3. Hence, when for any t_2 the corresponding t_1-values by (2.3) coincide, then also the corresponding t_3-values by (2.6) are coincident. Hence, the discriminants D_1 and D_2 of the two equations in (2.6) — when solved for t_2 — have the same real or pairwise complex conjugate roots.

Conversely, proportional polynomials D_1 and D_2 have equal zeros. Hence the linear functions in (2.5) and (2.8) give the same $\mathrm{c}\varphi_2$ for $\mathrm{c}\psi_1 = -\mathrm{c}\psi_2 = \pm 1$. Therefore $\mathrm{c}\psi_1 = -\mathrm{c}\psi_2$ is true in all positions, and the composition of the two four-bars fulfills Dixon's angle condition. □

The second characterization in Lemma 2.2 is also valid in the planar case. So, the algebraic essence is the same on the sphere and in the plane. Since in the plane the reducibility is guaranteed, the same must hold on the sphere. This can also be confirmed with the aid of a CA-system: The resultant splits into two biquadratic polynomials like the left hand side in (2.3). By Lemma 2.1 each component equals the transmission by a spherical four-bar, but the length of the frame link differs from the distance $\overline{I_{10}I_{30}}$ because otherwise this would contradict the classification of flexible octahedra. General results on conditions guaranteeing real four-bars have not yet been found. We summarize:

Theorem 2.1. *Any composition of two spherical four-bar linkages obeying Dixon's angle condition* $\psi_1 = \sphericalangle I_{10}A_1B_1 = \pm \sphericalangle \bar{I}_{30}B_2A_2$ *(see Fig. 2.3b) is reducible. Each component equals the transmission* $\varphi_1 \to \varphi_3$ *of a single, but not necessarily real spherical four-bar linkage.*

Example. The data $\alpha_1 = 38.00°$, $\beta_1 = 26.00°$, $\gamma_1 = 41.50°$, $\delta_1 = 58.00°$, $\alpha_2 = -40.0400°$, $\beta_2 = 123.1481°$, $\gamma_2 = -123.3729°$, $\delta_2 = 82.0736°$ yield a reducible 4-4-correspondence according to Theorem 2.1. The components define spherical four-bars with lengths $\alpha_3 = 60.2053°$, $\beta_3 = 53.5319°$, $\gamma_3 = 8.6648°$, $\delta_3 = 14.5330°$ or $\alpha_4 = 24.7792°$, $\beta_4 = 157.1453°$, $\gamma_4 = 160.4852°$, $\delta_4 = 33.8081°$. ◇

2.4 Conclusions

We studied compositions of two spherical four-bar linkages where the 4-4-correspondence between the input angle φ_1 and output angle φ_3 is reducible. We presented a new family of reducible compositions. However, a complete classification is still open. It should also be interesting to apply the principle of transference (e.g., [9]) in order to study dual extensions of these spherical mechanisms.

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Chapter 3

Reducible compositions of spherical four-bar linkages with a spherical coupler component

Abstract We use the output angle of a spherical four-bar linkage $\mathscr{C}$ as the input angle of a second four-bar linkage $\mathscr{D}$ where the two frame links are assumed in aligned position as well as the follower of $\mathscr{C}$ and the input link of $\mathscr{D}$. We determine all cases where the relation between the input angle of the input link of $\mathscr{C}$ and the output angle of the follower of $\mathscr{D}$ is reducible and where additionally at least one of these components produces a transmission which equals that of a single spherical coupler. The problem under consideration is of importance for the classification of flexible 3×3 complexes and for the determination of all flexible octahedra in the projective extension of the Euclidean 3-space.

Keywords Spherical four-bar linkage, Kokotsakis meshes, flexible octahedra, 3×3 complexes

3.1 Introduction

Let a spherical four-bar linkage $\mathscr{C}$ be given by the quadrangle $I_{10}A_1B_1I_{20}$ (see Fig. 3.1) with the frame link $I_{10}I_{20}$, the coupler A_1B_1 and the driving arm $I_{10}A_1$. We use the output angle φ_2 of this linkage as the input angle of a second coupler motion $\mathscr{D}$ with vertices $I_{20}A_2B_2I_{30}$. The two frame links are assumed in aligned position as well as the driven arm $I_{20}B_1$ of $\mathscr{C}$ and the driving arm $I_{20}A_2$ of $\mathscr{D}$.

We want to determine all cases where the relation between the input angle φ_1 of the arm $I_{10}A_1$ and the output angle φ_3 of $I_{30}B_2$ is reducible and where additionally at least one of these components produces a transmission which equals that of a single spherical coupler. Therefore we are looking for all reducible compositions with a so-called spherical coupler component.

The problem under consideration is of importance for the classification of flexible Kokotsakis meshes [1,2,3] with a 4-sided planar central polygon, which are compounds of 3×3 planar

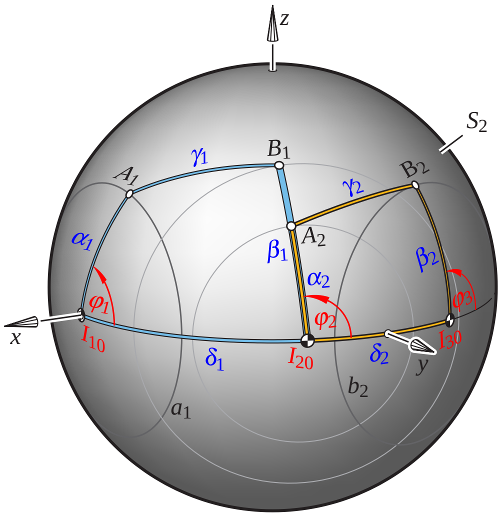

Figure 3.1: Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1,2$ (Courtesy of H. Stachel).

quadrangular plates with hinges between neighboring plates.[1] This results from the fact that the spherical image of a 3×3 complex consists of two compositions of spherical four-bars sharing the transmission $\varphi_1 \mapsto \varphi_3$ (see Fig. 3.1).

Moreover the author also prepares a classification of all reducible compositions of spherical four-bar linkages *without* a spherical coupler component. Based on this ongoing research and the presented article one can give a complete list of all flexible 3×3 complexes if *Stachel's conjecture*[2] holds true that all multiply decomposable compounds of spherical four-bars are reducible (with exception of the translatory type and planar-symmetric type).

Such a listing is of great interest because Bobenko et al. [2] showed that a polyhedral mesh with valence 4 composed of planar quadrilaterals is flexible if and only if all 3×3 complexes are flexible. One possible application scenario is the architectural design of flexible claddings composed of planar quads (cf. Pottmann et al. [4]).

The reducible compositions with a spherical coupler component are of special interest because based on their knowledge one can additionally determine all flexible octahedra in the projective extension of the Euclidean 3-space. This was already done by the author and a full classification of these flexible structures was given in [5,6].

3.1.1 Transmission by a spherical four-bar linkage

We start with the analysis of the first spherical four-bar linkage $\mathscr{C}$ with the frame link $I_{10}I_{20}$ and the coupler A_1B_1 (Fig. 3.1). We set $\alpha_1 := \overline{I_{10}A_1}$ for the spherical length (= arc length) of the driving arm, $\beta_1 := \overline{I_{20}B_1}$ for the output arm, $\gamma_1 := \overline{A_1B_1}$, and $\delta_1 := \overline{I_{10}I_{20}}$. We may suppose

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi\,.$$

[1]Such a structure is also known as 3×3 complex or Neunflach in German.
[2]A proof for this conjecture is in preparation.

The movement of the coupler remains unchanged when A_1 is replaced by its antipode $\overline{A}_1$ and at the same time α_1 and γ_1 are substituted by $\pi - \alpha_1$ and $\pi - \gamma_1$, respectively. The same holds for the other vertices. When I_{10} is replaced by its antipode $\overline{I}_{10}$, then also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of S^2 either at I_{10} or at $\overline{I}_{10}$.

We use a Cartesian coordinate frame with its origin at the spherical center, I_{10} on the positive x-axis and $I_{10}I_{20}$ in the xy-plane such that I_{20} has a positive y-coordinate (see Fig. 3.1). The input angle φ_1 is measured between $I_{10}I_{20}$ and the driving arm $I_{10}A_1$ in mathematically positive sense. The output angle $\varphi_2 = \sphericalangle \overline{I}_{10}I_{20}B_1$ is the oriented exterior angle at vertex I_{20}.

As given in [3] the constant spherical length γ_1 of the coupler implies the following equation:

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \tag{3.1}$$

with $t_i := \tan(\varphi_i/2)$,

$$\begin{gathered} c_{22} = K_1 - L_1 + M_1 + N_1, \quad c_{11} = 4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1 \neq 0, \quad c_{02} = K_1 + L_1 - M_1 + N_1, \\ c_{20} = -K_1 - L_1 - M_1 + N_1, \qquad c_{00} = -K_1 + L_1 + M_1 + N_1, \end{gathered} \tag{3.2}$$

and

$$K_1 = \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1, \quad L_1 = \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1, \quad M_1 = \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1, \quad N_1 = \mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\gamma_1. \tag{3.3}$$

Herein s and c are abbreviations for the sine and cosine function, respectively. In these equations the spherical lengths α_1, β_1 and δ_1 are signed. For a more detailed explanation and alternative expressions of Eq. (3.1) see [3].

Remark 3.1. Note that the 2-2-correspondence (3.1) depends only on the ratio of the coefficients $c_{22} : \cdots : c_{00}$ (cf. Lemma 1 of [7]). ◇

3.1.2 Composition of two spherical four-bar linkages

Now we use the output angle φ_2 of the first four-bar linkage $\mathscr{C}$ as input angle of a second four-bar linkage $\mathscr{D}$ with vertices $I_{20}A_2B_2I_{30}$ and consecutive spherical side lengths α_2, γ_2, β_2, and δ_2 (Fig. 3.1). The two frame links are assumed in aligned position. In the case $\sphericalangle I_{10}I_{20}I_{30} = \pi$ the spherical length δ_2 is positive, otherwise negative. Analogously, a negative α_2 expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of β_2 means replacing the output angle φ_3 by $\varphi_3 - \pi$. The sign of γ_2 has no influence on the transmission.

Due to (3.1) the transmission between the angles φ_1, φ_2 and the output angle φ_3 of the second four-bar with $t_3 := \tan(\varphi_3/2)$ can be expressed by the two biquadratic equations

$$\begin{aligned} \mathsf{C} &:= c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0\,, \\ \mathsf{D} &:= d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0\,. \end{aligned} \tag{3.4}$$

The d_{ik} are defined by equations analogous to Eqs. (3.2) and (3.3). We eliminate t_2 by computing the *resultant* (cf. [8]) of the two polynomials with respect to t_2 and obtain

$$X := \det \begin{pmatrix} c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} & 0 \\ 0 & c_{22}t_1^2 + c_{02} & c_{11}t_1 & c_{20}t_1^2 + c_{00} \\ d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00} & 0 \\ 0 & d_{22}t_3^2 + d_{20} & d_{11}t_3 & d_{02}t_3^2 + d_{00} \end{pmatrix} = 0\,. \tag{3.5}$$

This biquartic equation expresses a *4-4-correspondence* between points A_1 and B_2 on the circles a_1 and b_2, respectively (Fig. 3.1).

Known examples of reducible compositions with a spherical coupler component

Up to recent, to the author's best knowledge the following examples are known. Under appropriate notation and orientation these are:

1. **Isogonal** type [1,2]: At each four-bar opposite sides are congruent; the transmission $\varphi_1 \to \varphi_3$ is the product of two projectivities and therefore again a projectivity. Each of the 4 possibilities can be obtained by one single four-bar linkage. This is the spherical image of a flexible octahedron of Type 3 (see [9]).

2. **Symmetric** type [3]: We specify the second four-bar linkage as mirror of the first one after reflection in an angle bisector at I_{20} (see Fig. 5b of [3]). Thus φ_3 is congruent to the angle opposite to φ_1 in the first quadrangle. Hence the 4-4-correspondence is reducible.

3. **Focal** type [7]: Any composition of two spherical four-bar linkages obeying the angle condition $\psi_1 = \sphericalangle I_{10}A_1B_1 = \pm \sphericalangle \overline{I}_{30}B_2A_2$ (see Fig. 3b of [7]) is reducible. Each component equals the transmission $\varphi_1 \to \varphi_3$ of a single, but not necessarily real spherical four-bar linkage.

Computation of reducible compositions with a spherical coupler component

Given are the two spherical couplers $\mathscr{C}$ and $\mathscr{D}$ and their corresponding transmission equations C and D, respectively (see Eq. (3.4)). In the following we are interested in the conditions the c_{ij}'s and d_{ij}'s have to fulfill such that X of Eq. (3.5) splits up into the product FG with:

- **Symmetric reducible composition:**

$$\begin{aligned} \mathsf{F} &:= f_{22}t_1^2t_3^2 + f_{20}t_1^2 + f_{02}t_3^2 + f_{11}t_1t_3 + f_{00}, \\ \mathsf{G} &:= g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{11}t_1t_3 + g_{00}. \end{aligned} \tag{3.6}$$

  As at least one of the two polynomials F and G should correspond to a spherical coupler $\mathscr{F}$ and $\mathscr{G}$, respectively, we can stop the later done case study (see Section 3.2 and 3.3) if $f_{11} = g_{11} = 0$ holds.

- **First asymmetric reducible composition:**

$$\begin{aligned} \mathsf{F} &:= f_{11}t_1t_3 + f_{00}, \\ \mathsf{G} &:= g_{33}t_1^3t_3^3 + g_{31}t_1^3t_3 + g_{13}t_1t_3^3 + g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{11}t_1t_3 + g_{00}. \end{aligned} \tag{3.7}$$

  As F has to correspond with a spherical coupler component f_{11} cannot vanish. Moreover we can stop the later done case study (see Section 3.5.1, 3.5.2 and 3.5.3) if $g_{33} = g_{31} = g_{13} = 0$ holds, as this yields a special case of the symmetric composition.

- **Second asymmetric reducible composition:**

$$\begin{aligned} \mathsf{F} &:= f_{20}t_1^2 + f_{11}t_1t_3 + f_{00}, \\ \mathsf{G} &:= g_{13}t_1t_3^3 + g_{22}t_1^2t_3^2 + g_{20}t_1^2 + g_{02}t_3^2 + g_{11}t_1t_3 + g_{00}. \end{aligned} \tag{3.8}$$

Again F has to correspond with a spherical coupler component which yields $f_{11} \neq 0$. Moreover we can stop the later done case study (see Section 3.6.1, 3.6.2 and 3.6.3) if $g_{13} = 0$ (special case of the symmetric composition) or $f_{20} = 0$ (special case of the first asymmetric composition) holds.

Lemma 3.1. *The types of compositions given in Eqs. (3.6), (3.7) and (3.8) are all possible compositions with a spherical coupler component.*

Proof: As we can assume without loss of generality (w.l.o.g.) that $f_{11} \neq 0$ holds, we can set $\mathsf{F} := f_{11}t_1t_3 + f_{00}$. Now G can only be of the form given in Eq. (3.7) such that the product FG is a polynomial in t_1 and t_3 with the same structure as X of Eq. (3.5).

Analogous considerations for $\mathsf{F} := f_{20}t_1^2 + f_{11}t_1t_3 + f_{00}$ yields the second asymmetric reducible composition. Choosing $\mathsf{F} := f_{02}t_3^2 + f_{11}t_1t_3 + f_{00}$ implies the same case as the last one (only the variables t_1 and t_3 are interchanged).

Now we can assume $\mathsf{F} := f_{20}t_1^2 + f_{02}t_3^2 + f_{11}t_1t_3 + f_{00}$. For the same reason as above we get G of Eq. (3.6). But this already yields the symmetric reducible composition with $f_{22} = 0$. This finishes the proof. □

As we compute the resultant with respect to t_2 (cf. Eq. (3.5)) the coefficient of t_2^2 in C and D must not vanish. Therefore the two cases $c_{22} = c_{02} = 0$ and $d_{22} = d_{20} = 0$ are excluded. For the discussion of these *excluded cases* we refer to the Sections 3.4, 3.5.4 and 3.6.4, respectively.

In the following we denote the coefficients of $t_1^i t_3^j$ of $Y := \mathsf{FG}$ and X by Y_{ij} and X_{ij}, respectively. By the comparison of these coefficients we get the following 13 equations $Q_{ij} = 0$ with $Q_{ij} := Y_{ij} - X_{ij}$ and

$$(i,j) \in \{(4,4),(4,2),(4,0),(3,3),(3,1),(2,4),(2,2),(2,0),(1,3),(1,1),(0,4),(0,2),(0,0)\},$$

which must be fulfilled. In the following we discuss the solution of this non-linear system of 13 equations for the above given three possible compositions. As there are only 10 (cf. Eq. (3.6) or Eq. (3.7)) resp. 9 (cf. Eq. (3.8)) unknowns f_{ij}'s and g_{ij}'s which have to be determined, there have to be relations between the c_{ij}'s and d_{ij}'s to allow the solution of the whole system. The intention of the following discussion is to determine the subvarieties in the space of design variables c_{ij} and d_{ij}, such that there exists a decomposition of the transmission function where at least one of the resulting functions should correspond to the transmission function of a spherical coupler. Therefore we first eliminate the unknowns f_{ij} and g_{ij} in order to get the equations which only depend on the c_{ij}'s and d_{ij}'s. By solving the resulting equations we obtain the desired relations between these unknowns. In the case of the symmetric reducible composition this is done by the stepwise elimination of the unknowns d_{11} and c_{11}. It turns out that this elimination strategy yields the most compact formulas as it holds up the symmetries between the remaining unknowns.

We show that the three polynomial systems of 13 equations can be solved explicitly by means of resultants. Note that this is a non-trivial task especially in the case of the symmetric reducible composition. In the subsequent elimination process we often use a principle which is explained here at hand of the following simple example (cf. footnote 4 of [10]):

Given are 3 quadratic equations $Q_i = 0$ $i = 1,2,3$ in 3 variables x,y,z and one has to calculate the intersection points of these 3 quadrics. First we eliminate z by computing the resultant R_{ij} of Q_i and Q_j with respect to z. Now $R_{ij} = 0$ is a quartic equation in x,y. Computing again the resultant of e.g. R_{12} and R_{13} with respect to y yields a univariate polynomial of degree 16.

But not all roots of this polynomial are solutions of the intersection problem as it can only have 8 over $\mathbb{C}$ due to *Bezout's Theorem*. To get rid of the 8 pseudo-solutions one can compute more equations as actually necessary, i.e. the resultants of R_{12} and R_{23} as well as R_{13} and R_{23} with respect to y. Now the greatest common divisor (gcd) of these 3 polynomials of degree 16 yields in general the solution-polynomial of degree 8.

The 8 pseudo-solutions stem from the geometric fact that the elimination of the variable z is geometrically equivalent of projecting the intersection curve of the quadrics $Q_i = 0$ and $Q_j = 0$ into the xy-parameter plane. Now 8 intersection points of the two projected intersection curves R_{12} and R_{13} correspond to different points which lie above each other on one projection ray.

3.2 Symmetric reducible composition with $f_{20}g_{02} - f_{02}g_{20} \neq 0$

Under this assumption we can compute f_{22} and g_{22} from the equations $Q_{42} = 0$ and $Q_{24} = 0$, which are both linear in f_{22} and g_{22}. Moreover we can also express f_{00} and g_{00} from the equations $Q_{20} = 0$ and $Q_{02} = 0$ (both linear in f_{00} and g_{00}) and also f_{11} and g_{11} from the equations $Q_{31} = 0$ and $Q_{13} = 0$ (both linear in f_{11} and g_{11}).

3.2.1 The case $g_{20}g_{02} \neq 0$

Under this assumption we can express f_{20} from $Q_{40} = 0$ (linear in f_{20}) and f_{02} from $Q_{04} = 0$ (linear in f_{02}). Now we are left with 5 equations: $Q_{ii} = 0$ with $i = 0, \ldots, 4$. In the following we denote Q_{ii} by Q_i only:

$$Q_0[181], \quad Q_1[94], \quad Q_2[311], \quad Q_3[94], \quad Q_4[181],$$

where the number in the square brackets gives the number of terms of the numerator. Moreover it should be noted that we can factor out $c_{11}d_{11} \neq 0$ from Q_1 and Q_3.

In the following we compute the resultant of Q_i and Q_j with respect to g_{20}, which is denoted by R_{ij}. In the next step we compute the resultant of R_{13} and R_{ij} with respect to d_{11}, which is denoted by T_{ij}. Now it can easily be seen that the six expressions

$$T_{34}, \quad T_{01}, \quad T_{14}, \quad T_{03}, \quad T_{23}, \quad T_{12},$$

have the factors $g_{02}c_{11}W_1W_2W_3W_4W_5W_6$ in common with

$$\begin{aligned}
&W_1 := c_{02}c_{22}d_{00}d_{02} - c_{00}c_{20}d_{20}d_{22}, \quad W_2 := c_{00}c_{22}d_{00}d_{22} - c_{20}c_{02}d_{20}d_{02},\\
&W_3 := d_{00}d_{22} - d_{20}d_{02}, \quad W_4 := d_{00}c_{22} - c_{20}d_{20}, \quad W_5 := d_{02}c_{02} - c_{00}d_{22},\\
&W_6 := c_{11}^4 - 8c_{11}^2(c_{00}c_{22} + c_{20}c_{02}) + 16(c_{00}c_{22} - c_{20}c_{02})^2.
\end{aligned}$$

It should be noted that $W_4 = 0$ implies $f_{20} = 0$ and that $W_5 = 0$ implies $f_{02} = 0$. Moreover $W_3 = 0$ implies that $\mathscr{D}$ is an orthogonal spherical four-bar mechanism.[3] The condition $W_6 = 0$ can be rewritten in terms of $\alpha_1, \beta_1, \gamma_1$ and δ_1 as:

$$W_6 = 256\sin(\alpha_1)^2\sin(\beta_1)^2\sin(\gamma_1)^2\sin(\delta_1)^2.$$

[3] The diagonals of the spherical quadrangle $I_{20}A_2B_2I_{30}$ are orthogonal (cf. [3,7]).

Therefore this factor cannot vanish, as at least the spherical length of one spherical bar equals 0 (or π). Clearly, also the corresponding factor which is obtained by substituting c_{ij} by d_{ij} in W_6 cannot vanish without contradiction (w.c.). Therefore we can assume for the rest of this article that these two factors are different from zero.

Lemma 3.2. *Under the assumption* $g_{20}g_{02}(f_{20}g_{02} - f_{02}g_{20})W_1W_2W_3W_4W_5 \neq 0$ *there does not exist a symmetric reducible composition with a spherical coupler component.*

Proof: We factor $g_{02}c_{11}W_1W_2W_3W_4W_5W_6$ out from T_{ij} and denote the resulting polynomial by L_{ij}. Now the polynomials L_{34}, L_{01}, L_{14} and L_{03} have still the factor $H := 4W_4W_5W_7 + c_{11}^2W_2$ with

$$W_7 := c_{22}c_{00} - c_{20}c_{02}$$

in common. We distinguish two cases:

I. $H = 0$: In this case the greatest common divisor of L_{12} and L_{23} can only vanish w.c. for $c_{00}c_{02}c_{20}c_{22}d_{00}d_{02}d_{20}d_{22}M[14]W_7$. As $W_7 = 0$ implies together with $H = 0$ that $W_2 = 0$ must hold (a contradiction) we can assume $W_7 \neq 0$. Then the resultant of M and H with respect to c_{11} cannot vanish w.c..

   Therefore we are only left with the possibility $c_{ij}d_{ij} = 0$.

II. For the case $H \neq 0$ we proceed as follows: Beside T_{34} and T_{14} we also compute the resultant of R_{14} and R_{34} with respect to d_{11}, which is denoted by T_{13}. The common factors of T_{34}, T_{14} and T_{13} are given by:

$$c_{22}c_{20}d_{22}d_{02}g_{02}c_{11}W_1W_2W_3W_4W_5W_6H.$$

   Alternatively the same procedure can be done by denoting the resultant of R_{01} and R_{03} with respect to d_{11} by T_{13}. The common factors of T_{01}, T_{03} and T_{13} are given by:

$$c_{00}c_{02}d_{00}d_{20}g_{02}c_{11}W_1W_2W_3W_4W_5W_6H.$$

Due to I and II there can only be a reducible composition if $c_{ij}d_{ij} = 0$ holds. In the following we show that these cases also yield contradictions:

1. $c_{20}c_{22}d_{02}d_{22} = 0$: In all 4 cases $R_{24} = 0$ yields the contradiction.
2. $c_{00}c_{02}d_{20}d_{00} = 0$: In all 4 cases $R_{02} = 0$ yields the contradiction. □

We can even prove a stronger statement:

Lemma 3.3. *For the case* $g_{20}g_{02}(f_{20}g_{02} - f_{02}g_{20})W_4W_5 \neq 0$ *the condition* $W_1 = 0$ *is a necessary condition for a symmetric reducible composition with a spherical coupler component.*

Proof: Due to Lemma 3.2 we have to show that $W_2 = 0$ and $W_3 = 0$ do not yield a solution for $W_1 \neq 0$. We start by a rough discussion of the cases $W_2 = 0$ and $W_3 = 0$ and then we go into detail:

- $W_2 = 0$: Firstly we discuss the special cases. $W_2 = 0$ holds in the following 6 cases (without contradicting $W_4W_5 \neq 0$) if two variables out of $\{c_{ij}, d_{ij}\}$ are equal to zero:
$$c_{00} = c_{20} = 0, \quad d_{00} = d_{02} = 0, \tag{3.9}$$
$$c_{00} = d_{20} = 0, \quad d_{02} = c_{22} = 0, \quad d_{00} = c_{02} = 0, \quad d_{22} = c_{20} = 0. \tag{3.10}$$
It is very easy to see that the R_{ij} cannot vanish for both cases of Eq. (3.9). For the other cases we get:

  i. $c_{00} = d_{20} = 0$ or $d_{00} = c_{02} = 0$: In both cases Q_0 and Q_1 cannot vanish w.c..
  ii. $d_{02} = c_{22} = 0$ or $d_{22} = c_{20} = 0$: In both cases Q_3 and Q_4 cannot vanish w.c..

  We proceed with the general case. Due to the done discussion of the special cases we can set $d_{02} := Ac_{00}c_{22}d_{22}$ and $d_{00} := Ac_{02}c_{20}d_{20}$ with $A \in \mathbb{R} \setminus \{0\}$ w.l.o.g.. Now all R_{ij} contain the factor W_8 with
$$W_8 := Ad_{22}d_{20}c_{11}^2 - d_{11}^2 = 0.$$
For the case $W_8 \neq 0$ we consider the polynomials R_{ij}, R_{ik}, R_{jk}. Then we compute all three possible resultants of these polynomials (after factoring out W_8) with respect to d_{11} and calculate the greatest common divisor gcd_{ijk} (for $i, j, k \in \{0, 1, 3, 4\}$ and i, j, k pairwise distinct).

  The polynomials gcd_{014} and gcd_{034} have the following factors in common:
$$(Ac_{02}c_{22} - 1)(Ac_{02}c_{22} + 1)W_3W_6$$
where $(Ac_{02}c_{22} - 1) = 0$ yields $W_4W_5 = 0$, a contradiction. The vanishing of the remaining factor $Ac_{02}c_{22} + 1 = 0$ yields $W_1 = 0$, a contradiction.

  These are all solutions because beside $(Ac_{02}c_{22} - 1)(Ac_{02}c_{22} + 1)W_3W_6 = 0$ the expressions gcd_{014} and gcd_{034} can only vanish for $c_{02} = c_{22} = 0$, a contradiction. Therefore in the case $W_2 = 0$ one of the factors W_3 or W_8 must vanish.

- $W_3 = 0$: Now $d_{00}d_{22} - d_{20}d_{02} = 0$ must hold and $\mathscr{D}$ is an orthogonal coupler. For the discussion we can set $d_{00} := Ad_{02}$ and $d_{20} := Ad_{22}$ with $A \in \mathbb{R} \setminus \{0\}$ w.l.o.g.. After factoring out $g_{02}d_{11}W_4W_5$ from R_{14}, R_{34} and R_{13} we can compute T_{34}, T_{14} and T_{13} with
$$gcd(T_{13}, T_{14}, T_{34}) := c_{20}c_{22}W_1W_2W_6.$$
The alternative way of computation yields
$$gcd(T_{01}, T_{03}, T_{13}) := c_{00}c_{02}W_1W_2W_6.$$
As all possibilities of $c_{20}c_{22} = 0$ and $c_{00}c_{02} = 0$ imply $W_1 = 0$ or $W_2 = 0$, the factor W_2 must vanish.

We proceed with the detailed discussion of the open cases:

⋆ $W_2 = W_3 = 0$: Due to $W_3 = 0$ we set $d_{00} := Ad_{02}$ and $d_{20} := Ad_{22}$ with $A \in \mathbb{R} \setminus \{0\}$. Then W_2 splits up into $Ad_{02}d_{22}W_7$ and therefore $W_7 = 0$ must hold ($\Rightarrow \mathscr{C}$ and $\mathscr{D}$ are orthogonal). As a consequence we set $c_{00} := Bc_{02}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. Note that we can assume $c_{02}c_{22}d_{02}d_{22} \neq 0$ due to Eqs. (3.9) and (3.10). Now all R_{ij} with $i, j \in \{0,1,3,4\}$ and $i \neq j$ equal

$$g_{02}d_{02}d_{22}ABW_4W_5c_{11}d_{11}(Bd_{22} + d_{02})W_9W_{10}$$

with

$$W_9 := Ad_{02}d_{22}c_{11}^2 - Bc_{02}c_{22}d_{11}^2, \quad W_{10} := 4W_4W_5 + Ad_{02}d_{22}c_{11}^2 + Bc_{02}c_{22}d_{11}^2.$$

As $Bd_{22} + d_{02} = 0$ yields $W_1 = 0$ two cases remain:

a. $W_9 = 0$: The computation of the resultant of W_9 and Q_i with respect to d_{11} is denoted by U_i. Now it can easily be seen that U_0 and U_4 can only vanish for $(Ac_{22}g_{02} - g_{20}c_{02})F[19] = 0$, U_1 and U_3 for $G[10] = 0$ and U_2 for $Bc_{02}c_{22}(Ac_{22}g_{02} - g_{20}c_{02})H[34] = 0$. As $Ac_{22}g_{02} - g_{20}c_{02} = 0$ yields $f_{02}g_{20} - f_{20}g_{02} = 0$, a contradiction, we proceed with the case $F = G = H = 0$.

   We compute the resultant of G and H with respect to g_{20} and the resultant of G and F with respect to g_{20}. It can easily be seen that these two resultants cannot vanish w.c..

b. $W_{10} = 0$, $W_9 \neq 0$: For both possible solutions of $W_{10} = 0$ for c_{11} the resultant R_{12} cannot vanish w.c..

⋆ $W_2 = W_8 = 0$: In this case we have $d_{02} := Ac_{00}c_{22}d_{22}$ and $d_{00} := Ac_{02}c_{20}d_{20}$ with $A \in \mathbb{R} \setminus \{0\}$. Moreover, the resultant of W_8 and Q_i with respect to d_{11} is denoted by V_i. We can factor $c_{20}d_{20}g_{02} - g_{20}c_{00}d_{22}$ out from V_0, V_2 and V_4 because its vanishing yields $f_{02}g_{20} - f_{20}g_{02} = 0$, a contradiction. In the following we denote the resultant of V_i and V_j with respect to g_{20} by P_{ij}. We compute P_{13}, P_{03} and P_{01}. Then P_{13} can only vanish w.c. for $4(c_{22}c_{00} + c_{20}c_{02}) - c_{11}^2$. Now the two resultants of this factor with P_{03} and P_{01}, respectively, with respect to c_{11} cannot vanish w.c.. This finishes the proof of Lemma 3.3. □

The case $W_1 = 0$, $W_4W_5 \neq 0$

Special cases: $W_1 = 0$ holds only in the following 8 cases (without contradicting $W_4W_5 \neq 0$) if two variables out of the set $\{c_{ij}, d_{ij}\}$ are equal to zero:

i. $c_{22} = d_{22} = 0$ or $c_{20} = d_{02} = 0$: Now $Q_3 = 0$ and $Q_4 = 0$ are fulfilled identically. Then the resultant of R_{02} and R_{12} with respect to d_{11} cannot vanish w.c..

ii. $c_{00} = d_{00} = 0$ or $c_{02} = d_{20} = 0$: Now $Q_0 = 0$ and $Q_1 = 0$ are fulfilled identically. Then the resultant of R_{23} and R_{24} with respect to d_{11} cannot vanish w.c..

We remain with the following 4 cases:

$$c_{00} = c_{22} = 0, \quad c_{02} = c_{20} = 0, \quad d_{00} = d_{22} = 0, \quad d_{02} = d_{20} = 0. \tag{3.11}$$

In all 4 cases the conditions are already sufficient for a reducible composition. In each case we end up with one homogeneous quadratic equation in g_{20}, g_{02}:

1. $c_{00} = c_{22} = 0$: The equation equals: $(g_{20}c_{02}d_{02} + g_{02}c_{20}d_{20})^2 - g_{20}g_{02}d_{20}d_{02}c_{11}^2$.
2. $c_{02} = c_{20} = 0$: The equation equals: $(g_{20}c_{00}d_{22} + g_{02}c_{22}d_{00})^2 - g_{20}g_{02}d_{00}d_{22}c_{11}^2$.
3. $d_{00} = d_{22} = 0$: The equation equals: $(g_{20}c_{02}d_{02} + g_{02}c_{20}d_{20})^2 - g_{20}g_{02}c_{20}c_{02}d_{11}^2$.
4. $c_{02} = c_{20} = 0$: The equation equals: $(g_{20}c_{00}d_{22} + g_{02}c_{22}d_{00})^2 - g_{20}g_{02}c_{00}c_{22}d_{11}^2$.

The general case: Due to the discussed special cases we can set $d_{02} := Ac_{00}c_{20}d_{22}$ and $d_{20} := Ac_{02}c_{22}d_{00}$ with $A \in \mathbb{R} \setminus \{0\}$ w.l.o.g.. Then the equation $R_{13} = 0$ is fulfilled identically. Moreover $R_{34} = 0$ implies $R_{14} = 0$. The factors of R_{34} are:

$$c_{20}g_{02}AW_4W_5W_{11}W_{12}W_{13}$$

with

$$W_{11} := 4d_{00}d_{22}(Ac_{00}c_{22} - 1)(Ac_{20}c_{02} - 1) + Ad_{00}d_{22}c_{11}^2 - d_{11}^2, \quad W_{12} := d_{11}^2 + Ad_{00}d_{22}c_{11}^2,$$
$$W_{13} := 4AW_7^2d_{11}^2 + 4d_{00}d_{22}(A^2c_{00}c_{22}c_{20}c_{02} - 1)^2c_{11}^2 - (Ac_{00}c_{22} + 1)(Ac_{20}c_{02} + 1)c_{11}^2d_{11}^2.$$

Therefore one of the factors $W_{11}W_{12}W_{13}$ must vanish.

<u>$W_{11} = 0$</u>**:** The computation of the resultant of W_{11} and Q_i with respect to d_{11} is denoted by U_i. Now it can easily be seen that U_0 and U_4 can only vanish for $(d_{00}c_{22}g_{02} - g_{20}c_{00}d_{22})F[32] = 0$, U_1 and U_3 for $G[20] = 0$ and U_2 for $(d_{00}c_{22}g_{02} - g_{20}c_{00}d_{22})H[70] = 0$. As $d_{00}c_{22}g_{02} - g_{20}c_{00}d_{22} = 0$ yields $f_{02}g_{20} - f_{20}g_{02} = 0$, a contradiction, we proceed with the case $F = G = H = 0$.

We compute the resultant R_{FG} of F and G with respect to g_{20}, and analogously the resultants R_{FH} and R_{GH}. Then R_{FG} can only vanish w.c. for $M_1M_2 = 0$ with

$$M_1 := 2(Ac_{20}c_{02} - 1)(Ac_{22}c_{00} - 1) + Ac_{11}^2,$$
$$M_2 := 4(Ac_{22}c_{00} - 1)W_7 - (Ac_{22}c_{00} + 1)c_{11}^2.$$

- $M_1 = 0$: We compute the resultant of M_1 and R_{FH} resp. R_{GH} with respect to c_{11}. Then the greatest common divisor of the resulting expressions can only vanish w.c. for $Ac_{20}c_{02} + 1 = 0$ (which implies $W_2 = 0$). W.l.o.g. we can solve this equation for A. Then for both solutions of $M_1 = 0$ for c_{11} the resulting expression of F cannot vanish w.c..
- $M_2 = 0$: Analogous considerations as for the case $M_1 = 0$ yield the following necessary conditions: $W_7(Ac_{20}c_{02} + 1)(A^2c_{00}c_{20}c_{02}c_{22} - 1) = 0$ where $A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$ implies $W_3 = 0$. Together with $M_2 = 0$ this yields $W_7 = Ac_{20}c_{02} + 1 = 0$ or $W_7 = A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$. The latter can be seen by computing the resultant of $A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$ and M_2 with respect to A. Therefore only two cases remain:
  - ⋆ $W_7 = Ac_{20}c_{02} + 1 = 0$: As $W_7 = 0$ holds we can set $c_{00} := Bc_{02}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$ w.l.o.g.. Moreover we can express A from $Ac_{20}c_{02} + 1 = 0$. Then H can only vanish w.c. for $c_{11}^2 - 16Bc_{02}c_{22} = 0$. For both solutions with respect to c_{11} the polynomial F cannot vanish w.c..
  - ⋆ $W_7 = A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$: We set again $c_{00} := Bc_{02}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. For both solutions of $A^2c_{00}c_{20}c_{02}c_{22} - 1 = 0$ with respect to A none of the resultants R_{FG}, R_{FH} and R_{GH} can vanish w.c..

<u>$W_{12} = 0$**,** $W_{11} \neq 0$**:**</u> W.l.o.g. we can express A from $W_{12} = 0$. The only R_{ij}'s which are not fulfilled are R_{2i} with $i \in \{0,1,3,4\}$. It can easily be seen that these 4 expressions can only vanish w.c. for $c_{20}c_{02}d_{11}^2 - d_{00}d_{22}c_{11}^2 = 0$. For both solutions of this equation with respect to d_{11} the equations $Q_1 = 0$ and $Q_3 = 0$ can only vanish w.c. for $W_7 = 0$. Therefore we set $c_{00} := Bc_{02}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. Now the resultant of Q_2 and Q_0 (or Q_4) with respect to g_{20} cannot vanish w.c..

<u>$W_{13} = 0$**,** $W_{11}W_{12} \neq 0$**:**</u> Assuming $(Ac_{00}c_{22}+1)(Ac_{20}c_{02}+1)c_{11}^2 - 4AW_7^2 \neq 0$ we can compute d_{11} from $W_{13} = 0$. For both possible solutions the only R_{ij}'s which are not fulfilled are again R_{2i} with $i \in \{0,1,3,4\}$. It can easily be shown that these 4 expressions cannot vanish w.c..

Now we set $(Ac_{00}c_{22}+1)(Ac_{20}c_{02}+1)c_{11}^2 - 4AW_7^2 = 0$: Assuming $(Ac_{00}c_{22}+1)(Ac_{20}c_{02}+1) \neq 0$ we can solve the condition for c_{11}. For both solutions W_{13} cannot vanish w.c.. Both remaining special cases $(Ac_{00}c_{22}+1)(Ac_{20}c_{02}+1) = 0$ imply $W_7 = 0$ and therefore we set $c_{00} := Bc_{02}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. Now $W_{13} = 0$ is already fulfilled identically. For both special cases the resultants R_{02} and R_{24} cannot vanish w.c..
We sum up the results of this case study in the following theorem:

Theorem 3.1. *For the case* $g_{20}g_{02}(f_{20}g_{02} - f_{02}g_{20}) \neq 0$ *there only exists a symmetric reducible composition with a spherical coupler component if and only if the spherical coupler* $\mathscr{C}$ *or* $\mathscr{D}$ *is a spherical isogram.*

Proof: Due to the case study of Subsection 3.2.1 we have to show that the 4 special cases of Eq. (3.11) imply spherical isograms.

1. $c_{00} = c_{22} = 0$: It was already shown in [3] that $c_{00} = c_{22} = 0$ is equivalent with the conditions $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$, i.e. $\mathscr{C}$ is a spherical isogram.

2. $c_{20} = c_{02} = 0$: Substituting of the angle expressions yields:

$$c_{02} + c_{20} = 2\cos(\alpha_1 + \beta_1)\cos(\delta_1) - 2\cos(\gamma_1), \quad c_{02} - c_{20} = 2\sin(\alpha_1 + \beta_1)\sin(\delta_1).$$

   Under consideration of $0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi$ we get the following solution: $\beta_1 = \pi - \alpha_1$ and $\delta_1 = \pi - \gamma_1$. The couplers of item 1 and item 2 have the same movement because we get item 2 by replacing I_{20} of item 1 by its antipode $\bar{I}_{20}$. Clearly, the same holds for the coupler $\mathscr{D}$. Therefore all four special cases correspond with *spherical isograms*.

Now it remains to show that $W_4W_5 = 0$ only yields contradictions if we assume that none of the couplers is a spherical isogram. W.l.o.g. we set $W_4 = 0$. It is an easy task to show that no reducible composition exists (the proof is left to the reader) for the four special cases

$$d_{00} = c_{20} = 0, \quad d_{00} = d_{20} = 0, \quad c_{22} = c_{20} = 0, \quad c_{22} = d_{20} = 0.$$

Here we only discuss the general case: W.l.o.g. we can set $d_{00} := Ac_{20}$ and $d_{20} := Ac_{22}$ with $A \in \mathbb{R} \setminus \{0\}$ and assume $d_{00}c_{20}d_{20}c_{22} \neq 0$. Now the greatest common divisor of T_{34}, T_{14} and T_{13} can only vanish w.c. for $d_{02}c_{22} - c_{20}d_{22} = 0$. Therefore we can set w.l.o.g. $d_{02} := Bd_{22}$ and $c_{20} := Bc_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. Moreover we can assume $d_{22} \neq 0$ because otherwise $Q_4 = 0$ yields the contradiction. After factoring out all factors of Q_1, Q_3 and Q_4 which cannot vanish w.c. we compute R_{34} and R_{14}. Finally the resultant of these two expressions with respect to c_{11} yields the contradiction. □

Clearly, due to the symmetry of the equations also the following theorem holds:

Theorem 3.2. *For the case* $f_{20}f_{02}(f_{20}g_{02}-f_{02}g_{20})\neq 0$ *there only exists a symmetric reducible composition with a spherical coupler component if and only if the spherical coupler* $\mathscr{C}$ *or* $\mathscr{D}$ *is a spherical isogram.*

3.2.2 The case $g_{20}g_{02}=0$

Due to Theorems 3.1 and 3.2 we only have to discuss those cases for which $g_{20}g_{02}=0$, $f_{20}f_{02}=0$ and $f_{20}g_{02}-f_{02}g_{20}\neq 0$ hold. There are only the following two symmetric cases: $f_{02}=g_{20}=0$ or $f_{20}=g_{02}=0$. W.l.o.g. we set $f_{02}=g_{20}=0$. Then Q_{40} and Q_{04} can only vanish for $W_4=W_5=0$.

Special cases: First of all we discuss the special cases, where we distinguish four groups:

1. $\mathscr{C}$ and $\mathscr{D}$ are spherical isograms:

$$d_{00}=d_{22}=c_{02}=c_{20}=0,\qquad c_{00}=c_{22}=d_{02}=d_{20}=0.$$

2. $\mathscr{C}$ is a spherical isogram and $\mathscr{D}$ not:

$$c_{00}=c_{22}=d_{02}=c_{20}=0,\qquad c_{02}=c_{20}=d_{00}=c_{00}=0.$$

3. $\mathscr{D}$ is a spherical isogram and $\mathscr{C}$ not:

$$d_{00}=d_{22}=c_{20}=d_{02}=0,\quad d_{02}=d_{20}=c_{00}=d_{00}=0.$$

4. $\mathscr{C}$ and $\mathscr{D}$ are no spherical isograms:

$$d_{00}=c_{20}=d_{02}=c_{00}=0,\quad d_{00}=d_{20}=c_{02}=c_{00}=0,\quad c_{22}=c_{20}=d_{02}=d_{22}=0.$$

It can easily be verified that all 9 special cases yield a contradiction. The proof is left to the reader. In the next step we check the semispecial[4] cases:

1. $d_{00}=Ac_{20}$, $d_{20}=Ac_{22}$ with $A\in\mathbb{R}\setminus\{0\}$ and $c_{20}c_{22}\neq 0$:
   a. $c_{00}=c_{02}=0$: In this case Q_{22} splits up into two factors. In both cases we can compute f_{20} w.l.o.g.. Then Q_{33} can only vanish w.c. for $c_{22}d_{02}-d_{22}c_{20}=0$ which yields together with $Q_{44}=0$ the contradiction.
   b. $d_{02}=d_{22}=0$: Analogous considerations as in the last case also yield the contradiction.
   c. $c_{00}=d_{02}=0$: In this case $Q_{00}=0$ and $Q_{44}=0$ imply $d_{22}=c_{02}=0$. Then $Q_{22}=0$ yields the contradiction.
   d. $d_{22}=c_{02}=0$: Analogous considerations as in the last case also yield the contradiction.

2. $d_{22}=Bc_{02}$, $d_{02}=Bc_{00}$ with $B\in\mathbb{R}\setminus\{0\}$ and $c_{00}c_{02}\neq 0$:
   a. $d_{00}=c_{20}=0$ or $c_{22}=d_{20}=0$: Analogous considerations as in the semispecial case 1c yield the contradiction.
   b. $d_{00}=d_{20}=0$ or $c_{22}=c_{20}=0$: Analogous considerations as in the semispecial case 1a yield the contradiction.

[4]The term semispecial is used for those cases yielding $W_4=W_5=0$, where only two variables out of the set $\{c_{ij},d_{ij}\}$ are equal to zero.

The general case: Finally we can discuss the general case. Due to the discussed special cases we can set $d_{00} = Ac_{20}$, $d_{20} = Ac_{22}$, $d_{22} = Bc_{02}$, $d_{02} = Bc_{00}$ for $A, B \in \mathbb{R} \setminus \{0\}$. As for $W_7 = 0$ the expression Q_{00} cannot vanish w.c. we can compute f_{20} from $Q_{00} = 0$ w.l.o.g.. Then the resultant of Q_{11} and Q_{22} with respect to d_{11} already yields the contradiction.

We sum up the results of this subsection in the following theorem:

Theorem 3.3. *For the case $g_{20}g_{02} = 0$, $f_{20}f_{02} = 0$ and $f_{20}g_{02} - f_{02}g_{20} \neq 0$ there does not exist a symmetric reducible composition with a spherical coupler component.*

3.3 Symmetric reducible composition with $f_{20}g_{02} - f_{02}g_{20} = 0$

3.3.1 Very special case of $f_{20}g_{02} - f_{02}g_{20} = 0$

This case is very special as the equation $f_{20}g_{02} - f_{02}g_{20} = 0$ is trivially fulfilled for $f_{20} = f_{02} = g_{20} = g_{02} = 0$. In this case the equations $Q_{40} = 0$, $Q_{31} = 0$, $Q_{13} = 0$ and $Q_{04} = 0$ imply:

$$d_{00}c_{22} = 0, \quad c_{00}d_{22} = 0, \quad d_{02}c_{02} = 0, \quad d_{20}c_{20} = 0.$$

Now we have to discuss all possible non-contradicting combinatorial cases which can again be grouped into four classes:

1. $\mathscr{C}$ and $\mathscr{D}$ are spherical isograms:

$$d_{00} = d_{22} = c_{02} = c_{20} = 0, \qquad c_{00} = c_{22} = d_{02} = d_{20} = 0.$$

We only discuss the first case (for the second case we refer to analogy). As at least f_{11} or g_{11} must be different from zero we can assume $f_{11} \neq 0$ w.l.o.g.. Then we can express g_{22} from $Q_{33} = 0$ and g_{00} from $Q_{11} = 0$.

a. $d_{02} \neq 0$: We can compute f_{11} from $Q_{44} = 0$ w.l.o.g.. Now Q_{00} splits up into:

$$(c_{00}d_{20}f_{22} - f_{00}c_{22}d_{02})[c_{11}d_{11}c_{00}d_{20}d_{02}c_{22} + g_{11}(d_{02}c_{22}f_{00} + c_{00}d_{20}f_{22})].$$

The first factor can always be solved for f_{00}. Then only one equation $Q_{22} = 0$ remains which can be solved for g_{11} w.l.o.g..

For the second factor the same procedure also holds if we assume $g_{11} \neq 0$. For $g_{11} = 0$ the second factor can only vanish w.c. for $c_{00} = 0$. Then the remaining equation $Q_{22} = 0$ can be solved for f_{00} w.l.o.g..

b. $d_{02} = 0$: Now Q_{44} can only vanish for $f_{22}g_{11} = 0$.

i. For $f_{22} = 0$ we can solve $Q_{22} = 0$ for g_{11} and one equation remains where $c_{00} = 0$ factors out. For $c_{00} \neq 0$ the remaining factor can be solved for f_{11} w.l.o.g..

ii. For $g_{11} = 0$ we are left with $Q_{22} = 0$ and $Q_{00} = 0$. Both equations are fulfilled for $c_{00} = 0$. For $c_{00} \neq 0$ we can solve $Q_{00} = 0$ for f_{00} and $Q_{22} = 0$ for f_{22} w.l.o.g..

2. $\mathscr{C}$ is a spherical isogram and $\mathscr{D}$ not:

$$c_{00} = c_{22} = d_{02} = c_{20} = 0, \qquad c_{02} = c_{20} = d_{00} = c_{00} = 0.$$

   In the following we only discuss the first case in detail (for the second case we refer to analogy). Due to item 1 we can assume w.l.o.g. $c_{02}d_{20} \neq 0$. Then $Q_{20} = 0$ implies $d_{00} = 0$. As at least f_{11} or g_{11} must be different from zero we can assume w.l.o.g. that $f_{11} \neq 0$ holds. Under this assumption we can express g_{22} from $Q_{33} = 0$ and g_{00} from $Q_{11} = 0$. Now Q_{00} and Q_{44} can only vanish w.c. for $f_{00} = f_{22} = 0$ but then $Q_{22} = 0$ yields the contradiction.

3. $\mathscr{D}$ is a spherical isogram and $\mathscr{C}$ not:

$$d_{00} = d_{22} = c_{20} = d_{02} = 0, \quad d_{02} = d_{20} = c_{00} = d_{00} = 0.$$

   Analogously considerations as in item 2 yield the contradiction.

4. $\mathscr{C}$ and $\mathscr{D}$ are no spherical isograms:

$$d_{00} = c_{20} = d_{02} = c_{00} = 0, \quad d_{00} = d_{20} = c_{02} = c_{00} = 0, \quad c_{22} = c_{20} = d_{02} = d_{22} = 0.$$

   We start by discussing the case $d_{00} = c_{20} = d_{02} = c_{00} = 0$: Due to the above discussed cases we can assume $c_{02}c_{22}d_{20}d_{22} \neq 0$. As at least f_{11} or g_{11} must be different from zero we can assume $f_{11} \neq 0$ w.l.o.g.. Then we can express g_{22} from $Q_{33} = 0$ and g_{00} from $Q_{11} = 0$. Now Q_{00} and Q_{44} can only vanish w.c. for $f_{00} = f_{22} = 0$ but then $Q_{22} = 0$ yields the contradiction.

   In the remaining two cases we get the contradiction much more easier, because Q_{20} and Q_{42}, respectively, cannot vanish w.c..

We sum up the results of this subsection in the following theorem:

Theorem 3.4. *For any symmetric reducible composition with a spherical coupler component and* $g_{20} = g_{02} = f_{20} = f_{02} = 0$ *the couplers* $\mathscr{C}$ *and* $\mathscr{D}$ *are spherical isograms.*

In the following we formulate the main theorem for the symmetric reducible composition:

Theorem 3.5. *If a symmetric reducible composition with a spherical coupler component is given, then it is one of the following cases or a special case of them, respectively:*

1. *One spherical coupler is a spherical isogram,*

2. *the spherical couplers are forming a spherical focal mechanism which is analytically given by:*

$$\begin{aligned} c_{00}c_{20} = \lambda d_{00}d_{02}, \quad c_{22}c_{02} = \lambda d_{22}d_{20}, \quad \textit{with} \quad \lambda \in \mathbb{R}\setminus\{0\} \\ \textit{and} \quad c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = \lambda[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})], \end{aligned} \tag{3.12}$$

3. *both spherical couplers are orthogonal with* $c_{22} = c_{02} = d_{00} = d_{02} = 0$ *resp.* $d_{22} = d_{20} = c_{00} = c_{20} = 0$.

Proof: The Theorems 3.1 and 3.2 yield item 1 of Theorem 3.5. Moreover Theorem 3.4 implies a special case of item 1. Now the discussion of the special cases, the general case and the excluded case is missing. It turns out that the corresponding case studies only yield solutions which are one of the three cases of Theorem 3.5 or special cases of them, respectively. The detailed discussion of cases is performed in Subsection 3.3.2 and 3.3.3 and Section 3.4. Moreover it should be noted that Eq. (3.12) is the algebraic characterization of the *focal* type of Subsection 3.1.2.

3.3.2 Special cases of $f_{20}g_{02} - f_{02}g_{20} = 0$

Due to the last subsection we can discuss the following four special cases

$$f_{20} = f_{02} = 0, \quad f_{20} = g_{20} = 0, \quad f_{02} = g_{02} = 0, \quad g_{20} = g_{02} = 0,$$

under the assumption, that not all elements of $\{f_{02}, f_{20}, g_{02}, g_{20}\}$ are equal to zero. Due to the symmetry of the conditions only 2 of these 4 cases must be discussed (for the other cases we can refer to analogy).

The case $f_{20} = f_{02} = 0$

For the discussion we can assume w.l.o.g. that $g_{20} \neq 0$ holds. Under this assumption we can compute f_{22} from $Q_{42} = 0$, f_{00} from $Q_{20} = 0$ and f_{11} from $Q_{31} = 0$. Then Q_{40} and Q_{04} can only vanish for $W_4 = W_5 = 0$.

Special cases: We start again with the discussion of the special cases:

1. $\mathscr{C}$ and $\mathscr{D}$ are spherical isograms:

   $$d_{00} = d_{22} = c_{02} = c_{20} = 0, \qquad c_{00} = c_{22} = d_{02} = d_{20} = 0.$$

   These two cases yield easy contradictions as all f_{ij} vanish.

2. $\mathscr{C}$ is a spherical isogram and $\mathscr{D}$ not: The case $c_{02} = c_{20} = d_{00} = c_{00} = 0$ yields an easy contradiction as all f_{ij} vanish. For the remaining case $c_{00} = c_{22} = d_{02} = c_{20} = 0$ there exists the following reducible composition which is a special case of item 1 of Theorem 3.5: Now $f_{11} = f_{22} = 0$ holds. W.l.o.g. we can assume $d_{20}d_{00} \neq 0$ because otherwise $f_{00} = 0$ holds. Now the remaining three equations $Q_{22} = 0$, $Q_{11} = 0$ and $Q_{00} = 0$ can be solved w.l.o.g. for g_{22}, g_{11} and g_{00}, respectively.

3. $\mathscr{D}$ is a spherical isogram and $\mathscr{C}$ not: The case $d_{00} = d_{22} = c_{20} = d_{02} = 0$ yields an easy contradiction as all f_{ij} vanish. For the remaining case $d_{02} = d_{20} = c_{00} = d_{00} = 0$ there exists a analogous reducible composition as in the last case.

4. $\mathscr{C}$ and $\mathscr{D}$ are no spherical isograms: The case $d_{00} = c_{20} = d_{02} = c_{00} = 0$ yields an easy contradiction as all f_{ij} vanish. Two cases remain:

   $$d_{00} = d_{20} = c_{02} = c_{00} = 0, \qquad c_{22} = c_{20} = d_{02} = d_{22} = 0.$$

   We start discussing the first case: Now $f_{11} = f_{00} = 0$ holds. W.l.o.g. we can assume $c_{22}c_{20} \neq 0$ because otherwise $f_{22} = 0$ holds. Then $Q_{22} = 0$ implies $g_{00} = 0$. Now the remaining three equations $Q_{44} = 0$, $Q_{33} = 0$ and $Q_{24} = 0$ can be solved w.l.o.g. for g_{22}, g_{11} and g_{02}, respectively. This yields a special spherical focal mechanism (item 2 of Theorem 3.5).

   For the second case we get $f_{11} = f_{22} = 0$. W.l.o.g. we can assume $d_{00}d_{20} \neq 0$ because otherwise $f_{00} = 0$ holds. Then $Q_{22} = 0$ implies $g_{22} = 0$. Now the remaining three equations $Q_{11} = 0$, $Q_{02} = 0$ and $Q_{00} = 0$ can be solved w.l.o.g. for g_{11}, g_{02} and g_{00}, respectively. This yields a special spherical focal mechanism (item 2 of Theorem 3.5).

Semispecial cases: In the next step we check the semispecial cases (cf. footnote 4):

1. $d_{00} = Ac_{20}$, $d_{20} = Ac_{22}$ with $A \in \mathbb{R} \setminus \{0\}$ and $c_{20}c_{22} \neq 0$: The following 4 cases are again special spherical focal mechanisms (item 2 of Theorem 3.5):

   a. $c_{00} = c_{02} = 0$: It can easily be verified that there only exists a reducible composition if and only if $d_{02} = d_{22} = 0$ holds.

   b. $d_{02} = d_{22} = 0$: It can easily be verified that there only exists a reducible composition if and only if $c_{00} = c_{02} = 0$ holds.

   c. $c_{00} = d_{02} = 0$: It can easily be verified that there only exists a reducible composition if and only if $Ad_{22}c_{11}^2 - c_{02}d_{11}^2 = 0$ holds.

   d. $d_{22} = c_{02} = 0$: It can easily be verified that there only exists a reducible composition if and only if $Ad_{02}c_{11}^2 - c_{00}d_{11}^2 = 0$ holds.

2. $d_{22} = Bc_{02}$, $d_{02} = Bc_{00}$ with $B \in \mathbb{R} \setminus \{0\}$ and $c_{00}c_{02} \neq 0$:

   a. $d_{00} = c_{20} = 0$: We get a contradiction as all f_{ij} vanish.

   b. $c_{22} = d_{20} = 0$: We get a contradiction as all f_{ij} vanish.

   c. $d_{00} = d_{20} = 0$: In this case $f_{00} = f_{11} = 0$ holds. W.l.o.g. we can assume $c_{20}c_{22} \neq 0$ because otherwise $f_{22} = 0$ holds and all f_{ij} would vanish. Then Q_{13} cannot vanish w.c..

   d. $c_{22} = c_{20} = 0$: In this case $f_{22} = f_{11} = 0$ holds. W.l.o.g. we can assume $d_{20}d_{00} \neq 0$ because otherwise $f_{00} = 0$ holds and all f_{ij} would vanish. Then Q_{13} cannot vanish w.c..

The general case: Finally we can discuss the general case. Due to the discussed special cases we can set $d_{00} = Ac_{20}$, $d_{20} = Ac_{22}$, $d_{22} = Bc_{02}$, $d_{02} = Bc_{00}$ for $A, B \in \mathbb{R} \setminus \{0\}$. Now we can express g_{02} from $Q_{13} = 0$, g_{22} from $Q_{44} = 0$, g_{00} from $Q_{00} = 0$, g_{11} from $Q_{33} = 0$ and one equation remains: $ABc_{11}^2 - d_{11}^2 = 0$. This case yields also a special spherical focal mechanism (item 2 of Theorem 3.5).

The case $f_{02} = g_{02} = 0$

W.l.o.g. we can assume $f_{20}g_{20} \neq 0$ because otherwise we would get a special case of $f_{20} = f_{02} = 0$ or of its symmetric case $g_{20} = g_{02} = 0$. Therefore we can compute f_{22} from $Q_{42} = 0$, f_{00} from $Q_{20} = 0$, f_{11} from $Q_{31} = 0$ and f_{20} from $Q_{40} = 0$ w.l.o.g.. Now f_{20} can only vanish for $W_4 = 0$. Moreover Q_{13} and Q_{04} can only vanish w.c. for $c_{00}d_{22} = 0$ and $c_{02}d_{02} = 0$. We get the following combinatorial cases:

$$c_{00} = c_{02} = 0, \quad c_{00} = d_{02} = 0, \quad d_{22} = c_{02} = 0, \quad d_{22} = d_{02} = 0.$$

<u>$c_{00} = c_{02} = 0 \Rightarrow c_{22} \neq 0$:</u> Now Q_{24} can only vanish w.c. for $d_{22}d_{02} = 0$:

1. $d_{22} = 0 \Rightarrow d_{20} \neq 0$: Due to $Q_{00} = 0$ we have to distinguish two cases:

   a. $g_{00} = 0$: Due to $Q_{11} = 0$ we have to distinguish further two cases:

i. $d_{00}=0$: Assuming $2c_{20}d_{20}g_{11}+c_{11}d_{11}g_{20}\neq 0$ we can express g_{22} from $Q_{33}=0$. Then one equation remains:

$$d_{02}c_{11}^2g_{20}^2+c_{20}g_{11}c_{11}d_{11}g_{20}+c_{20}^2g_{11}^2d_{20}=0.$$

This yields a spherical focal mechanism where $\mathscr{D}$ is additionally a spherical isogram (items 1 and 2 of Theorem 3.5).
For $2c_{20}d_{20}g_{11}+c_{11}d_{11}g_{20}=0$ we can express g_{11} from this equation w.l.o.g.. Then $Q_{33}=0$ cannot vanish w.c..

ii. $g_{11}=0$, $d_{00}\neq 0$: Now we can compute g_{22} from $Q_{22}=0$ w.l.o.g.. Then the remaining two equations can only vanish w.c. for $c_{20}=0$ ($\mathscr{C}$ is a spherical isogram; item 1 of Theorem 3.5) or $d_{02}=0$, which yields a special spherical focal mechanism (item 2 of Theorem 3.5).

b. $g_{00}\neq 0$: In this case we solve the remaining factor of Q_{00} for g_{00}. This can be done w.l.o.g.. Moreover we can compute g_{11} from $Q_{11}=0$ and g_{22} from $Q_{22}=0$ w.l.o.g.. Then the remaining two equations can only vanish w.c. for $c_{20}=0$ ($\mathscr{C}$ is a spherical isogram; item 1 of Theorem 3.5) or $d_{02}=0$, which yields a special spherical focal mechanism (item 2 of Theorem 3.5).

2. $d_{02}=0$, $d_{22}\neq 0$: Due to $Q_{00}=0$ we have to distinguish two cases:

a. $g_{00}=0$: Due to $Q_{11}=0$ we have to distinguish further two cases:

i. $d_{00}=0$: Assuming $2c_{20}d_{20}g_{11}+c_{11}d_{11}g_{20}\neq 0$ we can express g_{22} from $Q_{33}=0$. Now it can easily be seen that the remaining two equations $Q_{44}=0$ and $Q_{22}=0$ cannot vanish w.c..
For $2c_{20}d_{20}g_{11}+c_{11}d_{11}g_{20}=0$ we can express g_{11} from this equation w.l.o.g.. Then $Q_{33}=0$ cannot vanish w.c..

ii. $d_{20}=0$, $d_{00}\neq 0$: Assuming $2c_{22}d_{00}g_{11}+c_{11}d_{11}g_{20}\neq 0$ we can express g_{22} from $Q_{33}=0$. Then one equation remains:

$$d_{22}c_{11}^2g_{20}^2+c_{22}g_{11}c_{11}d_{11}g_{20}+c_{22}^2g_{11}^2d_{00}=0.$$

This yields a spherical focal mechanism where $\mathscr{D}$ is additionally a spherical isogram (items 1 and 2 of Theorem 3.5).
For $2c_{22}d_{00}g_{11}+c_{11}d_{11}g_{20}=0$ we can express g_{11} from this equation w.l.o.g.. Then $Q_{22}=0$ yields the contradiction.

iii. $g_{11}=0$, $d_{00}d_{20}\neq 0$: Now we can compute g_{22} from $Q_{22}=0$ w.l.o.g.. Then the remaining two equations cannot vanish w.c..

b. $g_{00}\neq 0$: In this case we solve the remaining factor of Q_{00} for g_{00}. This can be done w.l.o.g.. Moreover we can compute g_{11} from $Q_{11}=0$ and g_{22} from $Q_{22}=0$ w.l.o.g.. Then Q_{22} cannot vanish w.c..

$d_{22}=d_{02}=0\Rightarrow d_{20}\neq 0$: For this case we refer to analogy. It can be done similarly to the last case if the variables are substituted as follows:

$$c_{00}\leftrightarrow d_{02},\quad c_{02}\leftrightarrow d_{22},\quad c_{20}\leftrightarrow d_{00},\quad c_{22}\leftrightarrow d_{20},\quad c_{11}\leftrightarrow d_{11}.$$

$c_{00} = d_{02} = 0$: In the following we distinguish two cases:

1. Assuming $g_{20}c_{11}d_{11}(d_{00}c_{22} + c_{20}d_{20}) + 2(d_{00}c_{22} - c_{20}d_{20})^2 g_{11} \neq 0$ we can express g_{00} from $Q_{11} = 0$ and g_{22} from $Q_{33} = 0$. Then we compute the resultants R_{ij} of Q_{ii} and Q_{jj} with respect to g_{11} for $i, j \in \{0, 2, 4\}$ and $i \neq j$. The greatest common divisor of these three resultants can only vanish w.c. for:

   a. $c_{22} = 0$: Now $\mathscr{C}$ is a spherical isogram (item 1 of Theorem 3.5). It can easily be seen that the remaining two equations $Q_{00} = 0$ and $Q_{22} = 0$ can only vanish w.c. for:

   $$g_{11}^2 d_{20}^2 c_{20} + g_{20}^2 d_{11}^2 c_{02} + d_{11} c_{11} g_{20} g_{11} d_{20} = 0.$$

   b. $d_{20} = 0$, $c_{22} \neq 0$: Now $\mathscr{D}$ is a spherical isogram (item 1 of Theorem 3.5). It can easily be seen that the remaining two equations $Q_{44} = 0$ and $Q_{22} = 0$ can only vanish w.c. for:

   $$g_{20}^2 c_{11}^2 d_{22} + g_{11}^2 c_{22}^2 d_{00} + g_{11} c_{22} d_{11} c_{11} g_{20} = 0.$$

   c. $4d_{22}c_{02}(d_{00}c_{22} - c_{20}d_{20}) + d_{22}d_{20}c_{11}^2 - c_{02}d_{11}^2 c_{22} = 0$, $c_{22}d_{20} \neq 0$: We distinguish again three cases:

      i. Assuming $d_{22}c_{02} \neq 0$ we can compute c_{20} from this equation. Now it can easily be seen that the remaining 3 equations can only vanish w.c. if a homogeneous quadratic equation in g_{20} and g_{11} (with 10 terms) is fulfilled. This equation can be solved w.l.o.g. for g_{11}. This yields a special spherical focal mechanism (item 2 of Theorem 3.5).

      ii. $d_{22} = 0$: Then the equation can only vanish for $c_{02} = 0$. Now the remaining equations can only vanish w.c. if a homogeneous linear equation in g_{20} and g_{11} (with 5 terms) is fulfilled. This equation can be solved w.l.o.g. for g_{11}. This yields a special spherical focal mechanism (item 2 of Theorem 3.5).

      iii. $c_{02} = 0$, $d_{22} \neq 0$: Then the equation cannot vanish w.c..

   Now we assume that the greatest common divisor of the resultants R_{02}, R_{04}, R_{24} is different from zero. Moreover we can set $c_{20}d_{00} \neq 0$ because both cases imply a contradiction. Therefore we can express c_{22} from the only non-contradicting factor of R_{24}. Then $R_{02} = 0$ implies $d_{22}d_{00}c_{11}^2 + c_{02}c_{20}d_{11}^2 = 0$ which can be solved for d_{22} w.l.o.g.. Now Q_{00} and Q_{44} cannot vanish w.c..

2. $g_{20}c_{11}d_{11}(d_{00}c_{22} + c_{20}d_{20}) + 2(d_{00}c_{22} - c_{20}d_{20})^2 g_{11} = 0$: W.l.o.g. we can solve this equation for g_{11}. Now we proceed as follows: $Q_{00} = 0$ is a homogeneous quadratic equation in g_{20}, g_{00} and $Q_{44} = 0$ is a homogeneous quadratic equation in g_{20}, g_{22}. Moreover, $Q_{22} = 0$ is also a homogeneous quadratic equation in g_{20}, g_{22}, g_{00} where g_{00} and g_{22} appear only linear. From these 3 equations we eliminate g_{00} and g_{22} by applying the resultant method. We compute the resultant R of Q_{44} and Q_{22} with respect to g_{22}. Then we compute the resultant of R and Q_{00} with respect to g_{00}. This resultant can only vanish w.c. if a homogeneous factor $F[22]$ of degree 16 in the unknowns c_{ij}, d_{ij} is fulfilled. Moreover Q_{11} and Q_{33} can only vanish w.c. for:

$$d_{00}d_{20}(4c_{02}c_{20}^2 d_{20} - 4c_{02}d_{00}c_{22}c_{20} - d_{00}c_{11}^2 c_{22} - d_{20}c_{11}^2 c_{20}) = 0, \tag{3.13}$$

$$c_{20}c_{22}(4d_{22}d_{00}^2 c_{22} - 4d_{22}c_{20}d_{00}d_{20} - c_{22}d_{11}^2 d_{00} - c_{20}d_{11}^2 d_{20}) = 0. \tag{3.14}$$

It can easily be seen that the cases $d_{00}d_{20}c_{20}c_{22} = 0$ only imply contradictions. Therefore we can assume w.l.o.g. $d_{00}d_{20}c_{20}c_{22} \neq 0$. Moreover we distinguish two cases:

a. $d_{00}c_{22} + c_{20}d_{20} \neq 0$: Under this assumption we can compute c_{11} and d_{11} from the remaining factors of Eqs. (3.13) and (3.14), respectively. For all four branches F is fulfilled identically. In all cases we end up with a special spherical focal mechanism (item 2 of Theorem 3.5).

b. $d_{00}c_{22} + c_{20}d_{20} = 0$: W.l.o.g. we can solve this equation for d_{00}. Then Q_{11} and Q_{33} can only vanish w.c. for $c_{02} = 0$ and $d_{22} = 0$. Again F is fulfilled identically. This yields a special spherical focal mechanism (item 2 of Theorem 3.5).

$d_{22} = c_{02} = 0 \Rightarrow d_{20}c_{22} \neq 0$**:** For this case we refer to analogy. It can be done similarly to the last case if the variables are substituted as follows:

$$c_{00} \leftrightarrow d_{22}, \quad c_{02} \leftrightarrow d_{02}, \quad c_{20} \leftrightarrow d_{20}, \quad c_{22} \leftrightarrow d_{00}, \quad c_{11} \leftrightarrow d_{11}.$$

3.3.3 General case of $f_{20}g_{02} - f_{02}g_{20} = 0$

Due to the discussion of the last two subsections we can assume w.l.o.g. that $f_{20}g_{02}f_{02}g_{20} \neq 0$ holds. Therefore we can set $f_{20} = Ag_{20}$ and $f_{02} = Ag_{02}$ with $A \in \mathbb{R} \setminus \{0\}$.

In the next step we compute g_{20} from $Q_{40} = 0$ which yields $\pm W_4/\sqrt{A}$. Moreover we can express g_{02} from $Q_{04} = 0$ which yields $\pm W_5/\sqrt{A}$. Therefore we have to distinguish the following cases:

The case $g_{20} = -W_4/\sqrt{A}$**,** $g_{02} = W_5/\sqrt{A}$ **or** $g_{20} = W_4/\sqrt{A}$**,** $g_{02} = -W_5/\sqrt{A}$

W.l.o.g. we can compute g_{22} and g_{00} from $Q_{42} = 0$ and $Q_{02} = 0$. In the following we distinguish two cases:

1. $g_{11}A - f_{11} \neq 0$: Now we can express f_{00} and f_{22} from $Q_{11} = 0$ and $Q_{33} = 0$ w.l.o.g.. Moreover we can compute f_{11} from $Q_{31} = 0$ w.l.o.g.. Then Q_{13} can only vanish w.c. for $W_1 = 0$.

   Special cases: First of all we discuss again the special cases: $W_1 = 0$ holds only in the following 8 cases (without contradicting $W_4W_5 \neq 0$) if two variables out of the set $\{c_{ij}, d_{ij}\}$ are equal to zero:

   a. $c_{22} = d_{22} = 0$: It can easily be seen that the following expression has to vanish in order to get a reducible composition:

   $$4c_{20}d_{02}(c_{00}d_{20} - c_{02}d_{00}) + c_{11}^2 d_{00}d_{02} - c_{20}d_{11}^2 c_{00}.$$

   b. $c_{20} = d_{02} = 0$: It can easily be seen that the following expression has to vanish in order to get a reducible composition:

   $$4c_{22}d_{22}(c_{00}d_{20} - c_{02}d_{00}) - c_{11}^2 d_{22}d_{20} + c_{02}d_{11}^2 c_{22}.$$

c. $c_{02} = d_{20} = 0$: It can easily be seen that the following expression has to vanish in order to get a reducible composition:

$$4c_{00}d_{00}(d_{22}c_{20} - c_{22}d_{02}) + c_{11}^2 d_{00}d_{02} - c_{20}d_{11}^2 c_{00}.$$

d. $c_{00} = d_{00} = 0$: It can easily be seen that the following expression has to vanish in order to get a reducible composition:

$$4d_{20}c_{02}(d_{22}c_{20} - c_{22}d_{02}) - c_{11}^2 d_{22}d_{20} + c_{02}d_{11}^2 c_{22}.$$

e. $c_{00} = c_{22} = 0$: In this case $Q_{24} = 0$ and $Q_{20} = 0$ imply $d_{00} = d_{22} = 0$, which already yields a reducible composition.

f. $d_{00} = d_{22} = 0$: In this case $Q_{24} = 0$ and $Q_{20} = 0$ imply $c_{00} = c_{22} = 0$, which already yields a reducible composition.

g. $c_{20} = c_{02} = 0$: In this case $Q_{24} = 0$ and $Q_{20} = 0$ imply $d_{20} = d_{02} = 0$, which already yields a reducible composition.

h. $d_{20} = d_{02} = 0$: In this case $Q_{24} = 0$ and $Q_{20} = 0$ imply $c_{20} = c_{02} = 0$, which already yields a reducible composition.

The cases a-d imply special spherical focal mechanisms (item 2 of Theorem 3.5). In the cases e-h we get spherical focal mechanisms where both couplers are spherical isograms (items 1 and 2 of Theorem 3.5).

The general case: Due to the discussed special cases we can set $c_{00} := Bc_{22}d_{02}c_{02}$ and $d_{00} := Bd_{22}c_{20}d_{20}$ with $B \in \mathbb{R} \setminus \{0\}$. Then Q_{24} and Q_{20} can only vanish w.c. if their common factor

$$4d_{20}c_{02}(c_{22}d_{02} - d_{22}c_{20})(Bd_{22}c_{22} - 1) + c_{22}c_{02}d_{11}^2 - d_{22}c_{11}^2 d_{20} \tag{3.15}$$

vanishes. This condition is already sufficient for a reducible composition and it yields the general spherical focal mechanism case given in item 2 of Theorem 3.5.

2. $g_{11}A - f_{11} = 0$: W.l.o.g. we can set $f_{11} = g_{11}A$. Moreover we can compute g_{11} from $Q_{31} = 0$ w.l.o.g.. Then Q_{13} can only vanish w.c. for $W_1 = 0$.

   In the 8 special cases of $W_1 = 0$ it can easily be seen that the remaining equations cannot vanish w.c. (the proof is left to the reader).

   Therefore we only discuss the general case in more detail: W.l.o.g. we can set $c_{00} := Bc_{22}d_{02}c_{02}$ and $d_{00} := Ad_{22}c_{20}d_{20}$ with $B \in \mathbb{R} \setminus \{0\}$. Now Q_{11} and Q_{33} can only vanish w.c. for:

$$4d_{20}(c_{22}Bd_{22} - 1)(d_{02} - c_{20}d_{22}^2 B) + d_{11}^2(1 + c_{22}Bd_{22}) = 0.$$

   Moreover Q_{24} and Q_{20} can only vanish w.c. for Eq. (3.15). If these two conditions are fulfilled then we already get a reducible composition. Clearly this yields a special spherical focal mechanism (item 2 of Theorem 3.5).

The case $g_{20} = W_4/\sqrt{A}$, $g_{02} = W_5/\sqrt{A}$ or $g_{20} = -W_4/\sqrt{A}$, $g_{02} = -W_5/\sqrt{A}$

W.l.o.g. we can compute g_{22} and g_{00} from $Q_{42} = 0$ and $Q_{02} = 0$.

1. $g_{11}A - f_{11} \neq 0$: Under this assumption we can express f_{00} and f_{22} from $Q_{11} = 0$ and $Q_{33} = 0$ w.l.o.g.. Moreover we can compute f_{11} from $Q_{31} = 0$ w.l.o.g.. Then Q_{13} can only vanish w.c. for $W_2 = 0$. First of all we discuss again the special cases: $W_2 = 0$ holds only in the 6 cases given in Eqs. (3.9) and (3.10) (without contradicting $W_4W_5 \neq 0$) if two variables out of the set $\{c_{ij}, d_{ij}\}$ are equal to zero. It is very easy to verify that these cases do not yield a solution (the proof is left to the reader).

   Here we only discuss the general case in more detail: W.l.o.g. we can set $d_{00} := Bc_{02}c_{20}d_{20}$ and $d_{02} := Ac_{22}c_{00}d_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. Now Q_{24} and Q_{20} can only vanish w.c. for $d_{20}Bc_{11}^2d_{22} - d_{11}^2 = 0$, which can be solved for d_{22} w.l.o.g.. Then the resultant of Q_{00} and Q_{44} with respect to g_{11} can only vanish w.c. in the following two cases:

   a. $Bc_{02}c_{22} + 1 = 0$: W.l.o.g. we can solve this equation for B. Now the resultant of Q_{22} and the only non-contradicting factor of Q_{00} and Q_{44} with respect to g_{11} can only vanish w.c. for $W_7 = 0$. Therefore we set $c_{00} = Lc_{02}$ and $c_{20} = Lc_{22}$ with $L \in \mathbb{R} \setminus \{0\}$. Then $Q_{00} = 0$ and $Q_{44} = 0$ imply the contradiction, as $g_{11} = 0$ yields $f_{11} = 0$.
   b. $W_7 = 0$, $Bc_{02}c_{22} + 1 \neq 0$: W.l.o.g. we set $c_{00} = Lc_{02}$ and $c_{20} = Lc_{22}$ with $L \in \mathbb{R} \setminus \{0\}$. Now the resultant of Q_{22} and the only non-contradicting factor of Q_{00} and Q_{44} with respect to g_{11} cannot vanish w.c..

2. $g_{11}A - f_{11} = 0$: W.l.o.g. we can set $f_{11} = g_{11}A$. Moreover we can compute g_{11} from $Q_{31} = 0$ w.l.o.g.. Then Q_{13} can only vanish w.c. for $W_2 = 0$.

   For the six special cases given in Eqs. (3.9) and (3.10) it can easily be seen that the remaining equations cannot vanish w.c. (the proof is left to the reader).

   Therefore we only discuss the general case in more detail: W.l.o.g. we can set $d_{00} := Bc_{02}c_{20}d_{20}$ and $d_{02} := Bc_{22}c_{00}d_{22}$ with $B \in \mathbb{R} \setminus \{0\}$. Now Q_{24} and Q_{20} can only vanish w.c. for $d_{20}Bc_{11}^2d_{22} - d_{11}^2 = 0$, which can be solved for d_{22} w.l.o.g.. Then we can compute c_{20} from $Q_{33} = 0$ w.l.o.g.. Finally Q_{11} cannot vanish w.c..

3.4 Excluded cases of the symmetric reducible composition

3.4.1 The case $c_{22} = c_{02} = 0$

As we compute the resultant X of C and D with respect to t_2 the coefficient of t_2^2 in D must not vanish. Moreover, as still at least one of the two polynomials F and G should correspond to a spherical coupler, we can stop the discussion if $d_{22} = d_{20} = 0$ or $f_{11} = g_{11} = 0$ holds.

Due to $Q_{44} = 0$, $Q_{04} = 0$ and $Q_{24} = 0$ either $f_{22} = f_{02} = 0$ or $g_{22} = g_{02} = 0$ must hold. W.l.o.g. we can assume $f_{22} = f_{02} = 0$. Then two cases have to be distinguished:

1. $f_{11} = 0$: As a consequence we can assume $g_{11} \neq 0$ w.l.o.g.. Therefore we can express f_{20} and f_{00} from $Q_{31} = 0$ and $Q_{11} = 0$, respectively. Moreover due to $c_{11}d_{11} \neq 0$ we can compute g_{22} from $Q_{42} = 0$, g_{20} from $Q_{40} = 0$, g_{02} from $Q_{02} = 0$ and g_{00} from $Q_{00} = 0$. Now the remaining

two expressions Q_{22} and Q_{20} can only vanish w.c. for $d_{00} = d_{02} = 0$ (item 3 of Theorem 3.5). From the other excluded case we get the second possibility $d_{22} = d_{20} = c_{00} = c_{20} = 0$ given in item 3 of Theorem 3.5.

2. $f_{11} \neq 0$: From $Q_{33} = 0$ and $Q_{13} = 0$ we get $g_{22} = g_{02} = 0$. Moreover we can compute g_{20} from $Q_{31} = 0$, g_{11} from $Q_{22} = 0$ and g_{00} from $Q_{11} = 0$ w.l.o.g.. Then we distinguish again two cases:

   a. $d_{22} = 0$: Now only the three conditions $Q_{40} = 0$, $Q_{20} = 0$ and $Q_{00} = 0$ remain. It can easily be seen that there exists a reducible composition if $d_{00} = 0$ holds ($\Rightarrow \mathscr{D}$ is a spherical isogram; item 1 of Theorem 3.5) or if $d_{02}d_{00}c_{11}^2 + 4d_{02}c_{20}c_{00}d_{20} - c_{00}d_{11}^2c_{20} = 0$ with $d_{02} \neq 0$ holds ($\Rightarrow$ special spherical focal mechanism; item 2 of Theorem 3.5).

   b. $d_{22} \neq 0$: Then Q_{42} and Q_{02} can only vanish w.c. for $c_{00} = c_{20} = 0$ ($\Rightarrow \mathscr{C}$ degenerates into a special spherical isogram as $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = \pi/2$ hold). The remaining expressions Q_{40}, Q_{20} and Q_{00} can only vanish w.c. for:

      i. $d_{00} = d_{02} = 0$: This is a special case of item 1 and item 3 of Theorem 3.5.

      ii. $d_{00} = f_{20} = f_{00} = 0$: We get a special case of item 1 of Theorem 3.5.

For the discussion of the second excluded case $d_{22} = d_{20} = 0$ we refer to analogy.

3.4.2 The special case $c_{22} = c_{02} = d_{22} = d_{20} = 0$

In this special case the resultant yields:

$$X := c_{11}d_{00}t_1 - c_{00}d_{11}t_3 + c_{11}d_{02}t_1t_3^2 - c_{20}d_{11}t_3t_1^2. \tag{3.16}$$

This expression cannot have a reducible composition of the form $X = \mathsf{F}\mathsf{G}$ with F and G of Eq. (3.6). This finishes the proof of Theorem 3.5. □

In order that we must not discuss this special case again for the two asymmetric reducible compositions given in the following two sections, we investigate Eq. (3.16) for any reducible compositions with a spherical coupler. It can easily be seen that Eq. (3.16) has the desired property if and only if $d_{00} = 0$ or $c_{00} = 0$ holds. Therefore at least one of the couplers has to be a spherical isogram (item 1 of Theorem 3.5). For the special case $d_{00} = c_{00} = 0$ we get a spherical focal mechanism where $\mathscr{C}$ and $\mathscr{D}$ are spherical isograms (items 1 and 2 of Theorem 3.5).

3.5 First asymmetric reducible composition

Computation of $Q_{04} = 0$ and $Q_{40} = 0$ shows that $W_4 = W_5 = 0$ must hold. First of all we discuss the special cases:

3.5.1 Special cases

We distinguish the following 4 groups:

1. $\mathscr{C}$ and $\mathscr{D}$ are spherical isograms:

$$d_{00} = d_{22} = c_{02} = c_{20} = 0, \qquad c_{00} = c_{22} = d_{02} = d_{20} = 0.$$

   We only discuss the first case (for the other we refer to analogy): Due to $Q_{42} = 0$ and $Q_{24} = 0$ we get $g_{31} = g_{13} = 0$. Moreover we get $g_{20} = g_{02} = 0$ from $Q_{31} = 0$ and $Q_{13} = 0$, respectively. Then we can express g_{33} from $Q_{44} = 0$, g_{22} from $Q_{33} = 0$, g_{11} from $Q_{22} = 0$ and g_{00} from $Q_{11} = 0$ w.l.o.g.. Then the equation $Q_{00} = 0$ remains, which is a homogeneous quartic equation in f_{00} and f_{11}. This equation can be solved w.l.o.g. for f_{00}. This yields a spherical focal mechanism where $\mathscr{C}$ and $\mathscr{D}$ are spherical isograms (items 1 and 2 of Theorem 3.5).

2. $\mathscr{C}$ is a spherical isogram and $\mathscr{D}$ not: For $c_{00} = c_{22} = d_{02} = c_{20} = 0$ the equations $Q_{44} = 0$, $Q_{42} = 0$ and $Q_{24} = 0$ imply $g_{31} = g_{13} = g_{33} = 0$, a contradiction.

   For the second possibility $c_{02} = c_{20} = d_{00} = c_{00} = 0$ we get $g_{31} = 0$ from $Q_{44} = 0$. Moreover we can compute g_{33} from $Q_{44} = 0$, g_{22} from $Q_{33} = 0$, g_{11} from $Q_{22} = 0$, g_{20} from $Q_{31} = 0$ and g_{13} from $Q_{24} = 0$ w.l.o.g.. Then we distinguish 2 cases:

   a. $f_{00} = 0$: The remaining two equations $Q_{13} = 0$ and $Q_{11} = 0$ imply $g_{02} = g_{00} = 0$ (item 1 of Theorem 3.5).

   b. $f_{00} \neq 0$: $Q_{02} = 0$ and $Q_{00} = 0$ imply $g_{02} = g_{00} = 0$. Then Q_{13} cannot vanish w.c..

3. $\mathscr{D}$ is a spherical isogram and $\mathscr{C}$ not: This can be done analogously to 2.

4. $\mathscr{C}$ and $\mathscr{D}$ are no spherical isograms: For the cases

$$d_{00} = c_{20} = d_{02} = c_{00} = 0, \quad c_{22} = c_{20} = d_{02} = d_{22} = 0,$$

   we immediately get a contradiction as $Q_{44} = 0$, $Q_{42} = 0$ and $Q_{24} = 0$ imply $g_{31} = g_{13} = g_{33} = 0$.

   For the third case $d_{00} = d_{20} = c_{02} = c_{00} = 0$ we distinguish two cases:

   a. $f_{00} = 0$: Now we get $g_{00} = g_{02} = g_{11} = g_{20} = 0$ from $Q_{31} = 0$, $Q_{22} = 0$, $Q_{13} = 0$ and $Q_{11} = 0$. The remaining four equations can be solved for g_{33}, g_{31}, g_{22}, and g_{13} w.l.o.g.. This yields a special spherical focal mechanism (item 2 of Theorem 3.5).

   b. $f_{00} \neq 0$: Due to $Q_{20} = 0$, $Q_{02} = 0$ and $Q_{00} = 0$ we get $g_{20} = g_{02} = g_{00} = 0$. Then $Q_{31} = 0$, $Q_{13} = 0$ and $Q_{11} = 0$ imply $g_{31} = g_{13} = g_{11} = 0$. Moreover $Q_{42} = 0$, $Q_{24} = 0$ and $Q_{22} = 0$ can only vanish w.c. for $g_{22} = c_{20} = d_{02} = 0$. Now Q_{44} and Q_{33} cannot vanish w.c..

3.5.2 Semispecial cases

The meaning of semispecial cases is the same one as given in footnote 4. We distinguish the following cases:

1. In the first part we set $d_{00} = Ac_{20}$ and $d_{20} = Ac_{22}$ with $A \in \mathbb{R} \setminus \{0\}$ and $c_{20}c_{22} \neq 0$. For the following 4 cases we can assume $f_{00} \neq 0$ w.l.o.g. because for $f_{00} = 0$ the equation $Q_{20} = 0$ cannot vanish w.c..

a. $c_{00} = c_{02} = 0$: We get $g_{00} = g_{02} = 0$ from $Q_{02} = 0$ and $Q_{00} = 0$, respectively. Moreover we can compute g_{33} from $Q_{44} = 0$, g_{31} from $Q_{42} = 0$, g_{22} from $Q_{33} = 0$, g_{20} from $Q_{31} = 0$, g_{13} from $Q_{24} = 0$ and g_{11} from $Q_{22} = 0$. Then Q_{20} can only vanish w.c. for $f_{00} = -Ac_{11}f_{11}/d_{11}$. Then the remaining two equations can only vanish w.c. for $d_{22} = d_{02} = 0$. This corresponds to a special spherical focal mechanism (item 2 of Theorem 3.5).

b. $d_{22} = d_{02} = 0$: This case can be done analogously. Finally in this case we get the conditions $c_{00} = c_{02} = 0$.

c. $c_{00} = d_{02} = 0$: We get $g_{13} = 0$ from $Q_{24} = 0$ and $g_{02} = 0$ from $Q_{02} = 0$. Moreover we can compute g_{33} from $Q_{44} = 0$, g_{31} from $Q_{42} = 0$, g_{22} from $Q_{33} = 0$, g_{20} from $Q_{31} = 0$ and g_{11} from $Q_{22} = 0$. Then $Q_{20} = 0$ and $Q_{00} = 0$ imply $f_{00} = -Ac_{11}f_{11}/d_{11}$ and $g_{00} = -Ad_{11}c_{20}^2c_{02}^2/(c_{11}f_{11})$, respectively. Finally one equation remains: $Ad_{22}c_{11}^2 - c_{02}d_{11}^2 = 0$. This also yields a special spherical focal mechanism (item 2 of Theorem 3.5).

d. $d_{22} = c_{02} = 0$: This case can be done analogously. Finally in this case we end up with the equation $Ad_{02}c_{11}^2 - c_{00}d_{11}^2 = 0$.

2. For the second semispecial case $d_{22} = Bc_{02}$, $d_{02} = Bc_{00}$ with $B \in \mathbb{R} \setminus \{0\}$ and $c_{00}c_{02} \neq 0$ we refer to analogy.

3.5.3 General case

W.l.o.g. we can set $d_{00} = Ac_{20}$, $d_{20} = Ac_{22}$, $d_{22} = Bc_{02}$ and $d_{02} = Bc_{00}$ with $A, B \in \mathbb{R} \setminus \{0\}$. We can compute g_{33} from $Q_{44} = 0$, g_{31} from $Q_{42} = 0$, g_{22} from $Q_{33} = 0$, g_{20} from $Q_{31} = 0$, g_{13} from $Q_{24} = 0$ and g_{11} from $Q_{22} = 0$. As for $f_{00} = 0$ the equation $Q_{20} = 0$ cannot vanish w.c. we can assume $f_{00} \neq 0$. Therefore we can express g_{02} from $Q_{02} = 0$ and g_{00} from $Q_{00} = 0$ w.l.o.g.. Then $Q_{20} = 0$ implies $f_{00} = -Ac_{11}f_{11}/d_{11}$. Finally one equation remains, namely: $ABc_{11}^2 - d_{11}^2 = 0$, which indicates a special spherical focal mechanism (item 2 of Theorem 3.5).

3.5.4 Excluded cases

Now we discuss the case $c_{22} = c_{02} = 0$. The equations $Q_{44} = 0$ and $Q_{24} = 0$ imply $g_{33} = g_{13} = 0$. Then the equations $Q_{33} = 0$ and $Q_{13} = 0$ yield $g_{22} = g_{02} = 0$. Moreover we can assume $c_{20}d_{22} \neq 0$ because otherwise $Q_{42} = 0$ yields $g_{31} = 0$, a contradiction. Therefore the equations $Q_{40} = 0$ and $Q_{02} = 0$ imply $d_{20} = c_{00} = 0$ ($\Rightarrow \mathscr{C}$ is a spherical isogram). We proceed by expressing g_{31} from $Q_{42} = 0$, g_{20} from $Q_{31} = 0$ and g_{11} from $Q_{22} = 0$. Then we distinguish two cases:

1. $f_{00} = 0$: In this case the remaining two equations can only vanish w.c. for $g_{00} = d_{00} = 0$. This yields a spherical focal mechanism where $\mathscr{C}$ is a spherical isogram (items 1 and 2 of Theorem 3.5).

2. $f_{00} \neq 0$: Then $Q_{00} = 0$ implies $g_{00} = 0$ and from $Q_{11} = 0$ we get $d_{02} = 0$. Finally a homogeneous quadratic equation in f_{00}, f_{11} remains which can be solved for f_{00} w.l.o.g.. This also yields a spherical focal mechanism where $\mathscr{C}$ is a spherical isogram (items 1 and 2 of Theorem 3.5).

For the discussion of the second excluded case $d_{22} = d_{20} = 0$ we refer to analogy.

3.6 Second asymmetric reducible composition

Computation of $Q_{44} = 0$ and $Q_{40} = 0$ shows that $c_{22}d_{02} - d_{22}c_{20} = 0$ and $W_5 = 0$, respectively, must hold. First of all we discuss the special cases; i.e. all cases where $c_{22}d_{02} - d_{22}c_{20} = 0$ and $W_5 = 0$ hold due to the vanishing of some c_{ij}'s and d_{ij}'s.

3.6.1 Special cases

It can easily be seen that there only exists two possible special cases:

1. $d_{22} = d_{02} = 0$: In this case Q_{24} cannot vanish w.c..
2. $d_{02} = c_{00} = c_{20} = 0$: Again $Q_{24} = 0$ yields the contradiction.

3.6.2 Semispecial cases

Here the term semispecial is used for those cases yielding $c_{22}d_{02} - d_{22}c_{20} = W_5 = 0$, where only two variables out of the set $\{c_{ij}, d_{ij}\}$ are equal to zero. In this asymmetric case there is only one possible semispecial case, namely: $d_{02} := Ac_{20}$, $d_{22} := Ac_{22}$ and $c_{02} = c_{00} = 0$ with $A \in \mathbb{R} \setminus \{0\}$ and $c_{20}c_{22} \neq 0$. W.l.o.g. we can compute g_{22} from $Q_{42} = 0$ and g_{13} from $Q_{24} = 0$. In the following we distinguish two cases:

1. $f_{00} = 0$: Then $Q_{13} = 0$ and $Q_{11} = 0$ imply $g_{02} = g_{00} = 0$. Now we can compute g_{20} from $Q_{40} = 0$ and g_{11} from $Q_{22} = 0$ w.l.o.g.. Moreover we get $f_{20} = -d_{11}f_{11}/(Ac_{11})$ from $Q_{33} = 0$. Finally, the remaining two equations can only vanish w.c. for $d_{00} = d_{20} = 0$. This yields a special case ($B = 0$) of the later given Theorem 3.6.
2. $f_{00} \neq 0$: $Q_{02} = Q_{00} = 0$ imply $g_{02} = g_{00} = 0$. Then Q_{13} cannot vanish w.c..

3.6.3 General case

Due to the last two subsections we can set $c_{20} := Ad_{02}$, $c_{22} := Ad_{22}$, $c_{02} := Bd_{22}$ and $c_{00} := Bd_{02}$ with $A, B \in \mathbb{R} \setminus \{0\}$ and $d_{02}d_{22} \neq 0$. W.l.o.g. we can compute g_{22} from $Q_{42} = 0$, g_{20} from $Q_{40} = 0$ and g_{13} from $Q_{24} = 0$. Then Q_{33} can only vanish w.c. for $f_{20} = -Ad_{11}f_{11}/c_{11}$. Moreover we can express g_{02} from $Q_{13} = 0$. Now $Q_{02} = 0$ implies $f_{00} = -Bd_{11}f_{11}/c_{11}$ and from $Q_{22} = 0$ we can compute g_{11} w.l.o.g.. Then $Q_{11} = 0$ yields $g_{00} = 0$ and Q_{31} and Q_{00} can only vanish w.c. for $W_3 = 0$. It can easily be seen that W_3 and the last remaining expression Q_{20} can only vanish w.c. for $d_{00} = d_{20} = 0$. This yields a new reducible composition (cf. Theorem 3.6).

3.6.4 Excluded cases

The case $c_{22} = c_{02} = 0$ does not yield a solution as Q_{24} cannot vanish w.c.. Therefore we consider the other excluded case $d_{22} = d_{02} = 0$. Now $Q_{42} = 0$ implies $g_{22} = 0$. Then Q_{33} cannot vanish w.c.. End of all cases.

We sum up the results of Section 3.5, Section 3.6 and the last paragraph of Section 3.4.2 into the following theorem:

Theorem 3.6. *Beside special cases of the isogram type and focal type of Theorem 3.5 there exists one special asymmetric reducible composition with a spherical coupler component, namely the following:*
$c_{20} := Ad_{02}$, $c_{22} := Ad_{22}$, $c_{02} := Bd_{22}$, $c_{00} := Bd_{02}$, $d_{00} = d_{20} = 0$, $d_{02}d_{22} \neq 0$ *resp.* $d_{02} := Ac_{20}$, $d_{22} := Ac_{22}$, $d_{20} := Bc_{22}$, $d_{00} := Bc_{20}$, $c_{00} = c_{02} = 0$, $c_{20}c_{22} \neq 0$ *with* $A \in \mathbb{R}\setminus\{0\}$ *and* $B \in \mathbb{R}$.

3.7 Conclusion and final remarks

The results of this article are summed up in the following corollary:

Corollary 3.1. *If a reducible composition of two spherical four-bar linkages with a spherical coupler component is given, then it is one of the following cases or a special case of them, respectively:*

a. One of the following four cases hold:

$$c_{00} = c_{22} = 0, \quad , d_{00} = d_{22} = 0, \quad c_{20} = c_{02} = 0, \quad d_{20} = d_{02} = 0,$$

b. The following algebraic conditions hold for $\lambda \in \mathbb{R}\setminus\{0\}$*:*

$$c_{00}c_{20} = \lambda d_{00}d_{02}, \quad c_{22}c_{02} = \lambda d_{22}d_{20}, \quad c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = \lambda[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})],$$

c. One of the following two cases hold:

$$c_{22} = c_{02} = d_{00} = d_{02} = 0, \quad d_{22} = d_{20} = c_{00} = c_{20} = 0,$$

d. One of the following two cases hold for $A \in \mathbb{R}\setminus\{0\}$ *and* $B \in \mathbb{R}$*:*

$$\star\, c_{20} := Ad_{02},\; c_{22} := Ad_{22},\; c_{02} := Bd_{22},\; c_{00} := Bd_{02},\; d_{00} = d_{20} = 0,\; d_{02}d_{22} \neq 0,$$

$$\star\, d_{02} := Ac_{20},\; d_{22} := Ac_{22},\; d_{20} := Bc_{22},\; d_{00} := Bc_{20},\; c_{00} = c_{02} = 0,\; c_{20}c_{22} \neq 0.$$

A comparison of Corollary 3.1 with the known examples of reducible compositions with a spherical coupler component given in Subsection 3.1.2 shows that we have found 3 new cases; namely items a, c and d of Corollary 3.1. The determination and geometric interpretation of the corresponding flexible 3×3 complexes implied by these new cases of reducible compositions is dedicated to future research.

We close the paper with the following concluding remarks:

- Note that the spherical coupler components of the given reducible compositions must not correspond to real spherical four-bars. For example, the spherical four-bars given by

$$\alpha_1 = 38^\circ, \qquad \beta_1 = 26^\circ, \qquad \gamma_1 = 41.5^\circ, \qquad \delta_1 = 58^\circ, \tag{3.17}$$

$$\alpha_2 = 158.4394^\circ, \qquad \beta_2 = 137.3509^\circ, \qquad \gamma_2 = 28.4922^\circ, \qquad \delta_2 = 53.2701^\circ, \tag{3.18}$$

form a spherical focal mechanism, but it shows up that both spherical coupler components F and G do not correspond with real spherical four-bar linkages.[5]

Moreover it should be noted that an example of a spherical focal mechanism where both components F and G correspond with real spherical four-bar linkages is given in [7]. Clearly, there also exist spherical focal mechanisms where only one of the components F and G corresponds with a real spherical four-bar linkage, like the following example given by:

$$\alpha_2 = 42.4420^\circ, \qquad \beta_2 = 60^\circ, \qquad \gamma_2 = 34.9019^\circ, \qquad \delta_2 = 42^\circ,$$

and $\alpha_1, \ldots, \delta_1$ of Eq. (3.17). Now one component corresponds with the following spherical four-bar linkage:

$$\alpha_3 = 59.9608^\circ, \qquad \beta_3 = 52.2807^\circ, \qquad \gamma_3 = 37.0987^\circ, \qquad \delta_3 = 82.1748^\circ.$$

- Moreover, it was shown by the author in [5] that a second possibility is hidden in the algebraic characterization of the spherical focal mechanisms (cf. Eq. (3.12)) beside the one given in [7]. It can easily be seen that the *symmetric* type (cf. Subsection 3.1.2) is a special case of this second *focal* type.

- Clearly, the *isogonal* type given in Subsection 3.1.2 is a special case of the *isogram* type (item a of Corollary 3.1) but also of the *focal* type (item b of Corollary 3.1).

- Beside the compositions given in Subsection 3.1.2 also the *orthogonal* type [3] is known which is as follows: Two orthogonal four-bars are combined such that they have one diagonal in common (see Fig. 5a of [3]), i.e. under $\alpha_2 = \beta_1$ and $\delta_2 = -\delta_1$, hence $I_{30} = I_{10}$. Then the 4-4-correspondence between A_1 and B_2 is the square of a 2-2-correspondence of the form

$$s_{21}t_1^2t_3 + s_{12}t_1t_3^2 + s_{10}t_1 + s_{01}t_3 = 0$$

(cf. [3]) and therefore this component cannot produce a transmission which equals that of a single spherical coupler. As already mentioned in Section 3.1, a paper on such reducible compositions without a spherical coupler component is in preparation. At this point it should only be noted that the given *orthogonal* type can be generalized as follows: $d_{00} := Ac_{20}$, $d_{20} := Ac_{22}$, $d_{22} := Bc_{02}$, $d_{02} := Bc_{00}$ with $A, B \in \mathbb{R}$ and $\mathscr{C}$ being an orthogonal coupler.

- Finally it should be noted that also the transmission function of a planar four-bar mechanism can be written in the form of Eq. (3.1). Therefore also the following statement holds (see also [5]):

Corollary 3.2. *If a reducible composition of two planar four-bar linkages with aligned frame links is given and the transmission equals that of a single planar four-bar mechanism (i.e. a planar coupler component), then one of the algebraic conditions characterizing the four items a-d of Corollary 3.1 is fulfilled.*

[5] General results on conditions guaranteeing real four-bars have not yet been found.

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Chapter 4

Flexible octahedra in the projective extension of the Euclidean 3-space

Abstract In this paper we complete the classification of flexible octahedra in the projective extension of the Euclidean 3-space. If all vertices are Euclidean points then we get the well known Bricard octahedra. All flexible octahedra with one vertex on the plane at infinity were already determined by the author in the context of self-motions of TSSM manipulators with two parallel rotary axes. Therefore we are only interested in those cases where at least two vertices are ideal points. Our approach is based on Kokotsakis meshes and reducible compositions of two four-bar linkages.

Keywords Flexible octahedra, Kokotsakis meshes, Bricard octahedra

4.1 Introduction

A polyhedron is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e. every face remains congruent to itself during the flex.

4.1.1 Review

In 1897 R. BRICARD [5] proved that there are three types of flexible octahedra [1] in the Euclidean 3-space E^3. These so-called *Bricard octahedra* are as follows:

type 1 All three pairs of opposite vertices are symmetric with respect to a common line.

type 2 Two pairs of opposite vertices are symmetric with respect to a common plane which passes through the remaining two vertices.

type 3 For a detailed discussion of this type we refer to [23]. We only want to mention that these flexible octahedra possess two flat poses.

[1] No face degenerates into a line and no two neighboring faces coincide permanently during the flex.

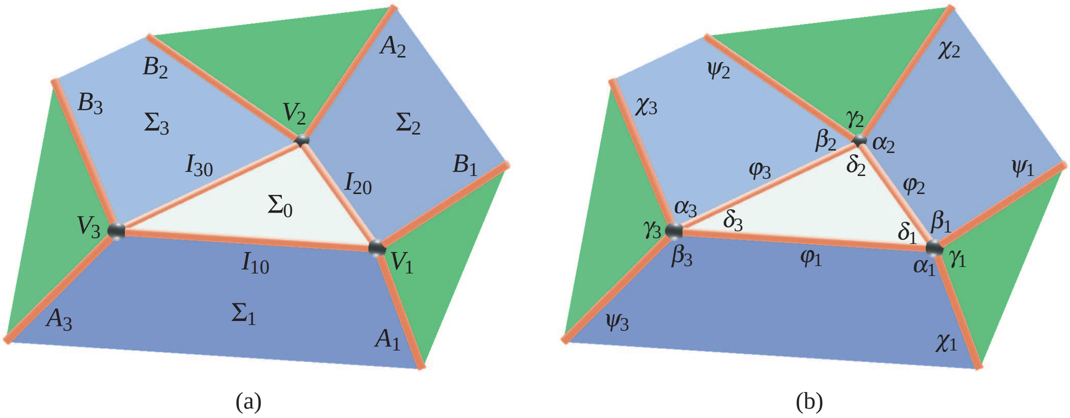

Figure 4.1: A *Kokotsakis mesh* is a polyhedral structure consisting of a n-sided central polygon $\Sigma_0 \in E^3$ surrounded by a belt of polygons in the following way: Each side I_{i0} of Σ_0 is shared by an adjacent polygon Σ_i, and the relative motion between cyclically consecutive neighbor polygons is a spherical coupler motion. Here the *Kokotsakis* mesh for $n = 3$ which determines an octahedron is given. φ_i, χ_i and ψ_i denote the angles enclosed by neighboring faces.

Due to *Cauchy's theorem* [8] all three types are non-convex, but they have even self-intersections.

As I.K. SABITOV [20] proved the *Bellows Conjecture*, every flexible polyhedron in E^3 keeps also its volume constant during the flex. Especially for *Bricard octahedra* it was shown by R. CONNELLY [9] that all three types have a vanishing volume. R. CONNELLY [10] also constructed the first flexible polygonal embedding of the 2-sphere into E^3. A simplified flexing sphere was presented by K. STEFFEN [26]. Note that both flexing spheres are compounds of *Bricard octahedra*.

R. ALEXANDER [1] has shown that every flexible polyhedron in E^3 preserves its total mean curvature during the flex (see also I. PAK [19, p. 264]). Recently V. ALEXANDROV [2] showed that the *Dehn invariants* (cf. [12]) of any *Bricard octahedron* remain constant during the flex and that the *Strong Bellows Conjecture* (cf. [11]) holds true for the *Steffen polyhedron*.

H. STACHEL [24] proved that all *Bricard octahedra* are also flexible in the hyperbolic 3-space. Moreover H. STACHEL [22] presented flexible cross-polytopes of the Euclidean 4-space.

4.1.2 Related work and overview

As already mentioned all types of flexible octahedra in E^3 were firstly classified by R. BRICARD [5]. His proof presented in [6] is based on properties of a *strophoidal* spatial cubic curve. In 1978 R. CONNELLY [9] sketched a further algebraic method for the determination of all flexible octahedra in E^3. H. STACHEL [21] presented a new proof which uses mainly arguments from projective geometry beside the converse of *Ivory's Theorem*, which limits this approach to flexible octahedra with finite vertices.

A. KOKOTSAKIS [14] discussed the flexible octahedra as special cases of a sort of meshes named after him (see Fig. 4.1). As recognized by the author in [18] Kokotsakis very short and elegant proof for *Bricard octahedra* is also valid for type 3 in the projective extension $E^\star$ of E^3 if no two opposite vertices are ideal points. H. STACHEL [23] also proved the existence of flexible octahedra of type 3 with one vertex at infinity and presented their construction. Moreover the author determined in [18] all flexible octahedra where one vertex is an ideal point.

Up to recent, there are no proofs for Bricard's famous statement known to the author which enclose the projective extension of E^3 although these flexible structures attracted many prominent mathematicians; e.g.; G.T. BENNETT [3], W. BLASCHKE [4], O. BOTTEMA [7], H. LEBESGUE [13] and W. WUNDERLICH [27]. The presented article together with [18] closes this gap.

Our approach is based on a kinematic analysis of *Kokotsakis meshes* as the composition of spherical coupler motions given by H. STACHEL [25], which is repeated in more detail in Section 4.2. In Section 4.3 we determine all flexible octahedra where no pair of opposite vertices are ideal points. The remaining special cases are treated in Section 4.4.

4.2 Notation and related results

We inspect a *Kokotsakis mesh* for $n=3$ (see Fig. 4.1). If we intersect the planes adjacent to the vertex V_i with a sphere S^2 centered at this point, the relative motion Σ_i/Σ_{i+1} (mod 3) is a spherical coupler motion.

4.2.1 Transmission by a spherical four-bar mechanism

We start with the analysis of the first spherical four-bar linkage $\mathscr{C}$ with the frame link $I_{10}I_{20}$ and the coupler A_1B_1 according to H. STACHEL [25] (see Fig. 4.1 and 4.2).

We set $\alpha_1 := \overline{I_{10}A_1}$ for the spherical length of the driving arm, $\beta_1 := \overline{I_{20}B_1}$ for the output arm, $\gamma_1 := \overline{A_1B_1}$ and $\delta_1 := \overline{I_{10}I_{20}}$. We may suppose $0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi$.

The coupler motion remains unchanged when A_1 is replaced by its antipode $\overline{A}_1$ and at the same time α_1 and γ_1 are substituted by $\pi-\alpha_1$ and $\pi-\gamma_1$, respectively. The same holds for the other vertices. When I_{10} is replaced by its antipode $\overline{I}_{10}$, then also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of S^2 either at I_{10} or at $\overline{I}_{10}$.

We use a cartesian coordinate frame with I_{10} on the positive x-axis and $I_{10}I_{20}$ in the xy-plane such that I_{20} has a positive y-coordinate (see Fig. 4.2). The input angle φ_1 is measured between $I_{10}I_{20}$ and the driving arm $I_{10}A_1$ in mathematically positive sense. The output angle $\varphi_2 = \sphericalangle \overline{I}_{10}I_{20}B_1$ is the oriented exterior angle at vertex I_{20}. As given in [25] the constant spherical length γ_1 of the coupler implies the following equation

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \tag{4.1}$$

with $t_i = \tan(\varphi_i/2)$, $c_{11} = 4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1 \neq 0$,

$$\begin{aligned} c_{00} &= N_1 - K_1 + L_1 + M_1, \quad & c_{02} &= N_1 + K_1 + L_1 - M_1,\\ c_{20} &= N_1 - K_1 - L_1 - M_1, \quad & c_{22} &= N_1 + K_1 - L_1 + M_1, \end{aligned} \tag{4.2}$$

$$K_1 = \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1, \quad L_1 = \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1, \quad M_1 = \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1, \quad N_1 = \mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\gamma_1. \tag{4.3}$$

In this equation s and c are abbreviations for the sine and cosine function, respectively, and the spherical lengths α_1, β_1 and δ_1 are signed.

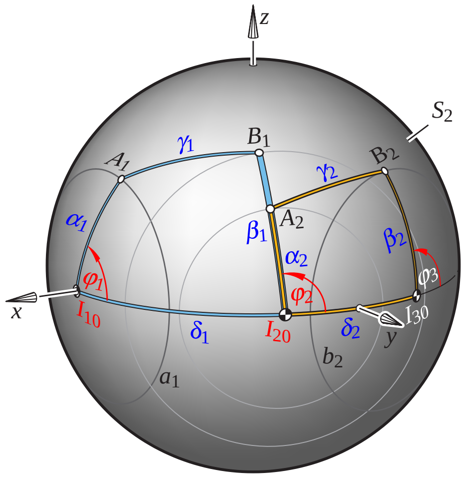

Figure 4.2: Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1,2$ (Courtesy of H. Stachel).

Note that the biquadratic equation Eq. (4.1) describes a *2-2-correspondence* between points A_1 on the circle $a_1 = (I_{10}; \alpha_1)$ and B_1 on $b_1 = (I_{20}; \beta_1)$ (see Fig. 4.2). Moreover, this 2-2-correspondence only depends on the ratio of the coefficients $c_{22} : \cdots : c_{00}$ (cf. Lemma 1 of [16]).

4.2.2 Composition of two spherical four-bar linkages

Now we use the output angle φ_2 of the first four-bar linkage $\mathscr{C}$ as input angle of a second four-bar linkage $\mathscr{D}$ with vertices $I_{20}A_2B_2I_{30}$ and consecutive spherical side lengths α_2, γ_2, β_2 and δ_2 (Fig. 4.2). The two frame links are assumed in aligned position. In the case $\sphericalangle I_{10}I_{20}I_{30} = \pi$ the spherical length δ_2 is positive, otherwise negative. Analogously, a negative α_2 expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of β_2 means replacing the output angle φ_3 by $\varphi_3 - \pi$. The sign of γ_2 has no influence on the transmission and therefore we can assume without loss of generality (w.l.o.g.) that $\gamma_2 > 0$ holds.

Due to (4.1) the transmission between the angles φ_1, φ_2 and the output angle φ_3 of the second four-bar with $t_3 := \tan(\varphi_3/2)$ can be expressed by the two biquadratic equations

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0, \quad d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0. \quad (4.4)$$

The d_{ik} are defined by equations analogue to Eqs. (4.2) and (4.3).

The author already determined in [17] all cases where the relation between the input angle φ_1 of the arm $I_{10}A_1$ and the output angle φ_3 of $I_{30}B_2$ is reducible and where additionally at least one of these components produces a transmission which equals that of a single spherical four-bar linkage $\mathscr{R}$ (= spherical quadrangle $I_{10}I_{r0}B_3A_3$). These so-called reducible compositions with a spherical coupler component can be summarized as follows (cf. Theorem 5 and 6 of [17]):

Theorem 4.1. *If a reducible composition of two spherical four-bar linkages with a spherical coupler component is given, then it is one of the following cases:*

(a) *One spherical coupler is a spherical isogram which happens in one of the following four cases:*
$$c_{00}=c_{22}=0,\ d_{00}=d_{22}=0,\ c_{20}=c_{02}=0,\ d_{20}=d_{02}=0,$$

(b) *the spherical couplers are forming a spherical focal mechanism which is analytically given for* $F\in\mathbb{R}\setminus\{0\}$ *by:*

$$\begin{aligned} c_{00}c_{20}&=Fd_{00}d_{02},\quad c_{22}c_{02}=Fd_{22}d_{20},\\ c_{11}^2-4(c_{00}c_{22}+c_{20}c_{02})&=F[d_{11}^2-4(d_{00}d_{22}+d_{20}d_{02})], \end{aligned} \tag{4.5}$$

(c) $c_{22}=c_{02}=d_{00}=d_{02}=0$ *resp.* $d_{22}=d_{20}=c_{00}=c_{20}=0$,

(d) $c_{20}=Ad_{02}$, $c_{22}=Ad_{22}$, $c_{02}=Bd_{22}$, $c_{00}=Bd_{02}$, $d_{00}=d_{20}=0$, $d_{02}d_{22}\neq 0$ *resp.* $d_{02}=Ac_{20}$, $d_{22}=Ac_{22}$, $d_{20}=Bc_{22}$, $d_{00}=Bc_{20}$, $c_{00}=c_{02}=0$, $c_{20}c_{22}\neq 0$ *with* $A\in\mathbb{R}\setminus\{0\}$ *and* $B\in\mathbb{R}$.

4.2.3 Geometric aspects of Theorem 4.1

Spherical isogram: Now we point out the geometric difference between the two spherical isograms given by $c_{00}=c_{22}=0$ and $c_{20}=c_{02}=0$, respectively.

(i) It was already shown in [25] that $c_{00}=c_{22}=0$ is equivalent with the conditions $\beta_1=\alpha_1$ and $\delta_1=\gamma_1$ which determines a spherical isogram.

(ii) $c_{20}=c_{02}=0$ is equivalent with the conditions $\beta_1=\pi-\alpha_1$ and $\delta_1=\pi-\gamma_1$ (cf. [17]). Note that the couplers of both isograms have the same movement because we get item (ii) by replacing either I_{10} or I_{20} of item (i) by its antipode.

Moreover it should be noted that the cosines of opposite angles in the spherical isograms (of both types) are equal (cf. §8 of [14]).

Spherical focal mechanism: Here also two cases can be distinguished:

(i) In [16] it was shown that the characterization of the spherical focal mechanism given in Theorem 4.1 is equivalent with the condition

$$\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1:\mathrm{s}\beta_1\,\mathrm{s}\delta_1:(\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1-\mathrm{c}\beta_1\,\mathrm{c}\delta_1)=\mathrm{s}\beta_2\,\mathrm{s}\gamma_2:\mathrm{s}\alpha_2\,\mathrm{s}\delta_2:(\mathrm{c}\alpha_2\,\mathrm{c}\delta_2-\mathrm{c}\beta_2\,\mathrm{c}\gamma_2).$$

Moreover it should be noted that in this case always $\mathrm{c}\chi_1=-\mathrm{c}\psi_2$ holds with $\chi_1=\sphericalangle I_{10}A_1B_1$ and $\psi_2=\sphericalangle I_{30}B_2A_2$.

(ii) But in the algebraic characterization of the spherical focal mechanism (4.5) also a second possibility is hidden, namely:

$$\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1:\mathrm{s}\beta_1\,\mathrm{s}\delta_1:(\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1-\mathrm{c}\beta_1\,\mathrm{c}\delta_1)=\mathrm{s}\beta_2\,\mathrm{s}\gamma_2:\mathrm{s}\alpha_2\,\mathrm{s}\delta_2:(\mathrm{c}\beta_2\,\mathrm{c}\gamma_2-\mathrm{c}\alpha_2\,\mathrm{c}\delta_2).$$

In this case always $\mathrm{c}\chi_1=\mathrm{c}\psi_2$ holds. Note that we get this case from the first one by replacing either I_{30} or I_{10} by its antipode.

4.3 The general case of flexible octahedra in $E^\star$

In this section we assume that no pair of opposite vertices of the octahedron are ideal points. As a consequence there exists at least one face of the octahedron where all three vertices are in E^3. This face corresponds to Σ_0 in Fig. 4.1. Now the *Kokotsakis mesh* for $n = 3$ is flexible if and only if the transmission of the composition of the two spherical four-bar linkages $\mathscr{C}$ and $\mathscr{D}$ equals that of a single spherical four-bar linkage $\mathscr{R}$ with $I_{r0} = I_{30}$.

It was shown in [18] that the items (c) and (d) of Theorem 4.1 as well as the spherical focal mechanism of type (i) do not yield a solution for this problem. Moreover it should be noted that the composition of two spherical isograms of any type also forms a spherical focal mechanism as Eq. (4.5) holds, and then the spherical four-bar linkage $\mathscr{R}$ also has to be a spherical isogram. This implies the following necessary conditions already given in [18]:

Lemma 4.1. *If an octahedron in the projective extension of E^3 is flexible where no two opposite vertices are ideal points, then its spherical image is a composition of spherical four-bar linkages $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{R}$ of the following type:*

1. *$\mathscr{C}$ and $\mathscr{D}$, $\mathscr{C}$ and $\mathscr{R}$ as well as $\mathscr{D}$ and $\mathscr{R}$ are forming a spherical focal mechanism of type (ii),*
2. *$\mathscr{C}$ and $\mathscr{D}$ are forming a spherical focal mechanism of type (ii) and $\mathscr{R}$ is a spherical isogram,*
3. *$\mathscr{C}$, $\mathscr{D}$ and hence also $\mathscr{R}$ are spherical isograms.*

4.3.1 Flexible octahedra of type 3 with vertices at infinity

In contrast to the proof for type 1 and type 2 A. KOKOTSAKIS showed without any limiting argumentation with respect to $E^\star$ that the third case of Lemma 4.1 corresponds with the *Bricard octahedron* of type 3 if no two opposite vertices are ideal points. Therefore the following angle conditions given in [14] also have to hold in our case:

$$\delta_i = \gamma_i, \quad \alpha_i = \beta_i, \quad \delta_3 + \gamma_3 = \pi, \quad \alpha_i + \beta_i = \pi, \quad \text{for} \quad i = 1,2, \tag{4.6}$$

where the angles are denoted according to Fig. 4.1(b). For $\beta_1 + \alpha_2 = \pi$ and $\beta_2 + \alpha_3 = \pi$ two of the remaining 3 vertices are ideal points. These conditions already imply $\beta_3 + \alpha_1 = \pi$ and therefore also the third remaining vertex has to be an ideal point. Moreover all three vertices are collinear which follows directly from the existence of the two flat poses. This already yields a contradiction (cf. footnote 1). Together with Theorem 2 of [18] this proves the following statement:

Theorem 4.2. *A flexible octahedron of type 3 with one finite face can have not more than one vertex at infinity.*

<u>*Remark*</u> 4.1. For the construction of these flexible octahedra see H. STACHEL [23]. ⋄

4.3.2 Flexible octahedra with a face or an edge at infinity

We can even generalize the observation that if three vertices are ideal points then they have to be collinear in order to get a flexible structure:

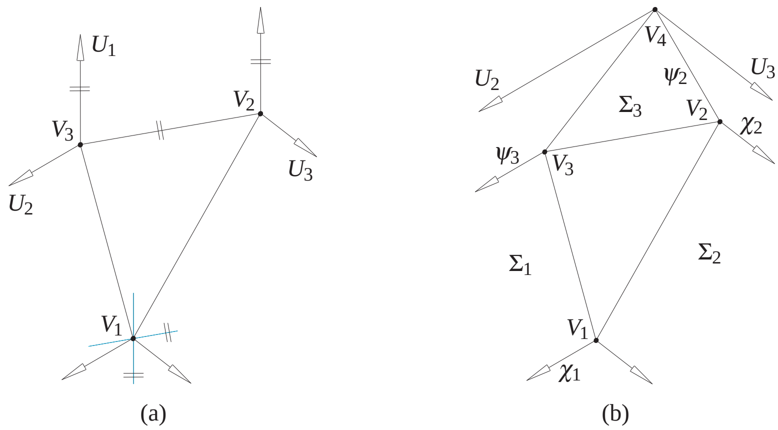

Figure 4.3: Both figures can be seen as a parallel projection of a spatial structure but on the other hand also as a planar configuration, because such structures has to possess two flat poses.

Theorem 4.3. *In the projective extension of E^3 there do not exist flexible octahedra where one face is at infinity if the other 3 vertices are finite.*

Proof: Given are the finite vertices V_1, V_2, V_3 and the three ideal points U_1, U_2, U_3 (see Fig. 4.3(a)). W.l.o.g. we can assume that the face $[V_1, U_2, U_3]$ is fixed. Since $[U_1, U_2, U_3]$ is a face of the octahedron, also the direction of U_1 is fixed.

Now the points V_2 and V_3 have to move on circles about their footpoints F_2 and F_3 with respect to $[V_1, U_3]$ and $[V_1, U_2]$, respectively. Note that F_2, V_2, V_3, F_3 can also be seen as an RSSR mechanism (cf. [15]) with intersecting rotary axes in V_1. We split up the vector V_2V_3 in a component $\mathbf{u}$ in direction U_1 and in a component orthogonal to it. Now the octahedron is flexible if the length of the component $\mathbf{u}$ is constant during the RSSR motion. It can easily be seen that a spherical motion of $[V_2, V_3]$ with center V_1 and this distance property can only be a composition of a rotation about a parallel to $[V_2, V_3]$ through V_1 and a rotation about $[V_1, U_1]$.

Then we consider one of the two possible configurations where V_1, V_2, V_3, U_1 are coplanar. Due to our considerations the velocity vectors of V_2 and V_3 with respect to the fixed system are orthogonal to this plane as they can only be a linear combination of the velocity vectors implied by the rotation about $[V_1, U_1]$ or about a parallel to $[V_2, V_3]$ through V_1. In order to guarantee that these vectors are tangent to the circles of the RSSR mechanism, the two rotary axes also have to lie within the plane V_1, V_2, V_3, U_1. Therefore U_1, U_2, U_3 are collinear and this again contradicts the definition of an octahedron. □

Moreover we can also prove the following theorem:

Theorem 4.4. *In the projective extension of E^3 there do not exist flexible octahedra with a finite face and one edge at infinity.*

Proof: We assume that $V_1, \ldots, V_4$ are finite and that U_2, U_3 are ideal points. We consider again V_1, U_2, U_3 as the fixed system. Now we split up the octahedron into two parts: in a mechanism which consists of V_1, U_2, V_3, V_4 and in one which is determined by V_1, V_2, U_3, V_4 (see Fig. 4.3(b)). Note that both mechanisms have the kinematic structure of a serial 2R chain.

We consider the configuration where the 2R chain V_1, U_2, V_3, V_4 is singular, i.e. these four points are coplanar where the carrier plane is denoted by ε. Now this mechanism can only induce a velocity to V_4 which is orthogonal to ε. The other 2R chain also implies a velocity to V_4 and its direction is orthogonal to U_3. In order to guarantee that the directions of the two velocities in V_4 are fitting together (which is a necessary condition for the flexibility) the point U_3 has to be located in ε. Therefore the points V_1, U_2, U_3, V_3, V_4 are within ε which equals the plane of the fixed system.

In the following we show that also the point V_2 has to lie in ε if the octahedron is of type 1 or type 2, respectively:

type 1 In this case the spherical image of the motion transmission from Σ_1 to Σ_2 via V_3 and V_2 is a spherical focal mechanism of type (ii). Therefore the condition $c\chi_2 = c\psi_3$ (see Fig. 4.3(b)) holds which implies that also the other 2R chain has to be in a singular configuration.

type 2 We have to distinguish three subcases depending on the vertices V_i $(i = 1, 2, 3)$ in which the spherical image of the motion transmission corresponds to a spherical isogram:

- $i = 1$: Now the spherical image of the motion transmission from Σ_1 to Σ_2 via V_3 and V_2 is a spherical focal mechanism of type (ii) which equals the above discussed case.
- $i = 3$: Now the spherical image of the motion transmission from Σ_1 to Σ_3 via V_1 and V_2 is a spherical focal mechanism of type (ii) which implies $c\chi_1 = c\psi_2$. As χ_1 equals 0 or π this already yields that all 6 vertices are coplanar.
- $i = 2$: This case can be done analogously as the above one if we start with a singular configuration of the 2R chain V_1, V_2, U_3, V_4.

Moreover, as there always exist two singular configurations of a 2R chain, a flexible octahedron where one edge is an ideal line has to have two flat poses.

In order to admit two flat poses, either V_4 has to be an ideal point (cf. Theorem 4.3) or V_2, U_2, V_3, U_3 have to be located on a line which again yields a contradiction as U_2 coincides with U_3. This already finishes the proof. □

<u>*Remark*</u> 4.2. The two geometric/kinematic proofs of Theorem 4.3 and 4.4 demonstrate the power of geometry in the context of flexibility because purely algebraic proofs for these statements seem to be a complicated task. ◇

4.4 Special cases of flexible octahedra in $E^\star$

In the first part of this section we determine all flexible octahedra with at least three vertices on the plane at infinity. These so called trivial flexible octahedra are the content of the next theorem:

Theorem 4.5. *In the projective extension of E^3 any octahedron is flexible where at least two edges are ideal lines but no face coincides with the plane at infinity.*

Proof: Under consideration of footnote 1 there are only two types of octahedra fulfilling the requirements of this theorem. These two types are as follows:

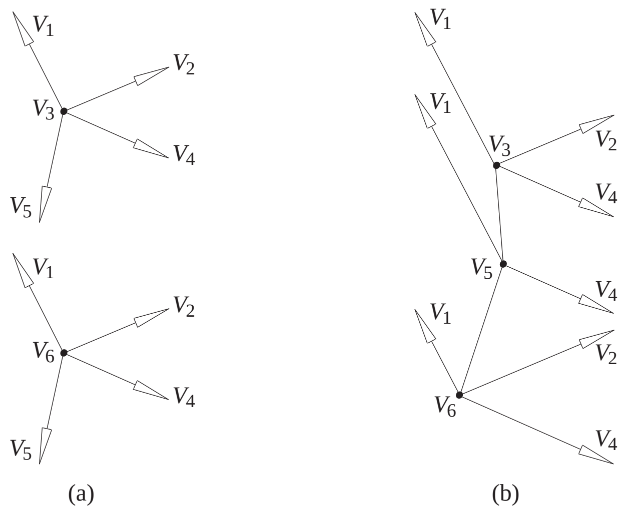

Figure 4.4: The degenerated flexible octahedra of type (a) have a 4-parametric self-motion in contrast to those of type (b) which possess a constrained one.

a. two pairs of opposite vertices are ideal points,

b. three vertices are ideal points where two of them are opposite ones.

It can immediately be seen from Fig. 4.4(a) and (b), that these two degenerated cases are flexible. A detailed proof is left to the reader. □

Due to the Theorems 4.3, 4.4 and 4.5 the only open problem is the determination of all flexible octahedra where only one pair of opposite vertices are ideal points. For the discussion of these octahedra we need some additional considerations which are prepared in the next two subsections.

4.4.1 Central triangles with one ideal point

Given is an octahedron where two opposite vertices are ideal points and the remaining four vertices are in E^3. The four faces through an ideal point built a 4-sided prism where the motion transmission between opposite faces equals the one of the corresponding planar four-bar mechanism (orthogonal cross section of the prism). It can easily be seen that the input angle φ_1 and the output angle φ_2 of a planar four-bar linkage (see Fig. 4.5) are related by:

$$p_{22}t_1^2t_2^2 + p_{20}t_1^2 + p_{02}t_2^2 + p_{11}t_1t_2 + p_{00} = 0 \tag{4.7}$$

with $t_i := \tan(\varphi_i/2)$, $p_{11} = -8ab$ and

$$\begin{aligned} p_{22} &= (a-b+c+d)(a-b-c+d), \quad & p_{20} &= (a+b+c+d)(a+b-c+d), \\ p_{02} &= (a+b+c-d)(a+b-c-d), \quad & p_{00} &= (a-b+c-d)(a-b-c-d). \end{aligned} \tag{4.8}$$

W.l.o.g. we can assume $a,b,c,d > 0$. Moreover in [18] the following lemma was proven:

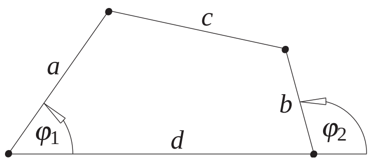

Figure 4.5: Planar four-bar mechanism with driving arm a, follower b, coupler c and base d.

Lemma 4.2. *If a reducible composition of one planar and one spherical four-bar linkage with a spherical coupler component is given, then one of the algebraic conditions characterizing the four cases of Theorem 4.1 is fulfilled.*

A closer study of the items (a)-(d) of Theorem 4.1 with respect to Lemma 4.2 was also done in [18], where we assumed that V_1 denotes the ideal point. In the following we sum up the achieved results:

ad (a) The conditions $c_{00} = c_{22} = 0$ imply $a = b$ and $c = d$, i.e. the planar four-bar mechanism is a parallelogram or an antiparallelogram. Note that opposite angles in the parallelogram and in the antiparallelogram are equal.

In contrast, $c_{20} = c_{02} = 0$ has no solution under the assumption $a,b,c,d > 0$.

ad (b) In this case we only get a solution if the relation

$$2ac : 2bd : (a^2 - b^2 + c^2 - d^2) = \mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\beta_2\,\mathrm{c}\gamma_2 - \mathrm{c}\alpha_2\,\mathrm{c}\delta_2)$$

holds. Moreover this condition implies $\mathrm{c}\chi_1 = \mathrm{c}\psi_2$.

ad (c) The case $d_{22} = d_{20} = c_{00} = c_{20} = 0$ does not yield a solution because $c_{00} = c_{20} = 0$ cannot be fulfilled for $a,b,c,d > 0$.

The other case $d_{00} = d_{02} = c_{22} = c_{02} = 0$ implies $\mathrm{c}\varphi_1 = \mathrm{c}\psi_1$ and $\mathrm{c}\chi_2 = \mathrm{c}\varphi_3$ or as second possibility $\mathrm{c}\varphi_1 = \mathrm{c}\psi_1$ and V_2,V_3,V_5,V_6 are coplanar.

ad (d) The case $c_{20} = Ad_{02}$, $c_{22} = Ad_{22}$, $c_{02} = Bd_{22}$, $c_{00} = Bd_{02}$, $d_{00} = d_{20} = 0$, $d_{02}d_{22} \neq 0$ does not yield a solution.

The other case $d_{02} = Ac_{20}$, $d_{22} = Ac_{22}$, $d_{20} = Bc_{22}$, $d_{00} = Bc_{20}$, $c_{00} = c_{02} = 0$, $c_{20}c_{22} \neq 0$ implies the relations $\mathrm{c}\varphi_2 = \mathrm{c}\chi_1$ and $\mathrm{c}\varphi_1 = \mathrm{c}\chi_3$.

4.4.2 Preparatory lemmata

In order to give the proof of the main theorem in the Section 4.4.3 in a compact form we prove the following two preparatory lemmata:

Lemma 4.3. *A planar base polygon of a 4-sided prism[2] remains planar during the flex if and only if one of the following cases hold:*

[2]We exclude those prisms where always two pairs of neighboring sides coincide during the flex, as they are not of interest for the problem under consideration (cf. footnote 1).

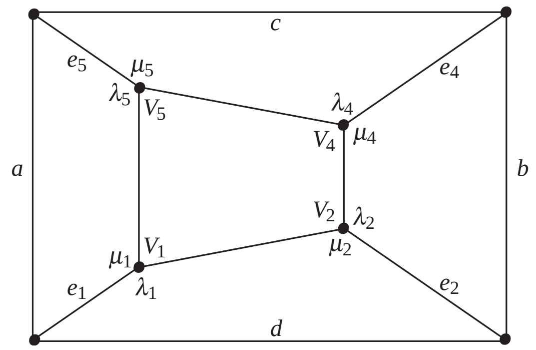

Figure 4.6: Perspective view of an orthogonal cross section of the prism (= four-bar linkage a,b,c,d) and of its four coplanar vertices V_1,V_2,V_4,V_5. Note that the dihedral angles along the prism edges e_i are denoted by τ_i. Moreover the face angles of the prism at V_i are denoted by μ_i and λ_i, respectively.

1. *The edges of the prism are orthogonal to the planar base,*
2. *the planar quadrilateral is a deltoid and the edges are orthogonal to the deltoid's line of symmetry,*
3. *the planar quadrilateral is an antiparallelogram and its plane of symmetry is parallel to the edges of the prism,*
4. *the planar quadrilateral is a parallelogram.*

Proof: We consider the orthogonal cross section of a prism which is an ordinary four-bar mechanism as given in Fig. 4.5. We denote with s and l the shortest and longest bar, respectively, and with p and q the length of the remaining bars. As item 1 is trivial we assume that the edges of the prism are not orthogonal to the planar base. For the used notation of the following case study please see Fig. 4.6:

1. $s+l<p+q$: Due to *Grashof's theorem* we get a double-crank mechanism if we fix the shortest bar s. Considering all four poses where the sides coincide with the frame link already imply the contradiction.

2. $s+l>p+q$: If we fix any of the four bars we always get a double-rocker mechanism. W.l.o.g. we can assume that d is the longest bar. As a consequence the following inequalities hold:

$$d+a>b+c \quad \text{and} \quad d+b>a+c. \tag{4.9}$$

   Therefore there exists a configuration where the edges e_1,e_2,e_5 are coplanar ($\tau_1=0 \Rightarrow e_5$ is between e_1 and e_2). This implies that the points V_1,V_2,V_5 have to be collinear which is the case if $\lambda_1=\mu_1$ and $\lambda_5=\mu_2$ hold. The analogous consideration for the edges e_1,e_2,e_4 yield $\lambda_2=\mu_2$ and $\lambda_1=\mu_4$.

   Now $\lambda_1=\mu_1$, $\lambda_2=\mu_2$ and the coplanarity condition of V_1,V_2,V_4,V_5 yield that $\tau_1=0$ implies $\tau_2=0$. Therefore there exists a flat pose which contradicts our assumption $s+l>p+q$.

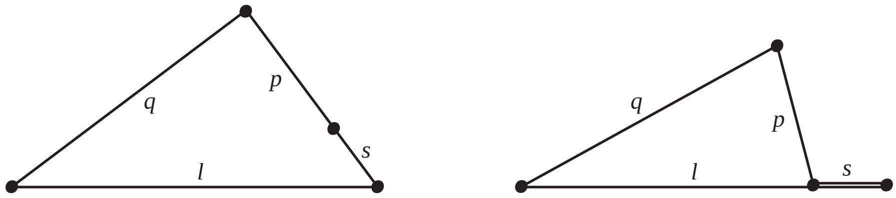

Figure 4.7: Special poses of the four-bar linkage ($l = 5$, $s = 1$, $p = 2$, $q = 4$) where l and s are neighboring bars.

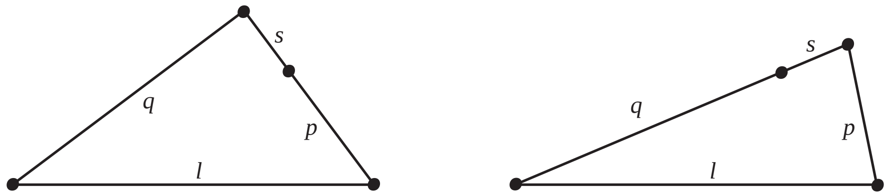

Figure 4.8: Special poses of the four-bar linkage ($l = 5$, $s = 1$, $p = 2$, $q = 4$) where l and s are opposite bars.

3. $s + l = p + q$: Here we assume that the prism only has one flat position. In this case we have to distinguish two subcases:

   a. l and s are neighboring bars: W.l.o.g. we set $l = d$, $s = b$, $p = c$ and $q = a$. Due to the inequalities
   $$l + q > s + p \quad \text{and} \quad p + q > l - s, \tag{4.10}$$
   there exist the following two special poses of the prism illustrated in Fig. 4.7. These two poses imply $\lambda_2 = \lambda_4$, $\mu_4 = \mu_5$ and $\lambda_2 = \mu_2$, $\lambda_1 = \mu_4$, respectively. Together with the coplanarity condition of V_1, V_2, V_4, V_5 these conditions yield one of the following three cases:

      i. V_1, V_2, V_4 are always collinear which contradicts footnote 2,
      ii. V_2, V_4, V_5 are always collinear which contradicts footnote 2,
      iii. $[V_1, V_2]$ and $[V_4, V_5]$ are parallel. This already yields the contradiction as a four-bar mechanism where two opposite bars are always parallel during the motion, can only be a parallelogram.

   b. l and s are opposite bars: W.l.o.g. we set $l = d$, $s = c$, $p = a$ and $q = b$. Due to the inequalities
   $$l + p > s + q \quad \text{and} \quad l + q > s + p, \tag{4.11}$$
   there exist the following two special poses of the prism illustrated in Fig. 4.8. These two poses imply $\lambda_2 = \lambda_4$, $\mu_4 = \mu_5$ and $\lambda_4 = \lambda_5$, $\mu_1 = \mu_5$, respectively. Together with the coplanarity condition of V_1, V_2, V_4, V_5 these conditions yield one of the following three cases:

      i. V_2, V_4, V_5 are always collinear which contradicts footnote 2,
      ii. V_1, V_4, V_5 are always collinear which contradicts footnote 2,
      iii. $[V_1, V_5]$ and $[V_2, V_4]$ are parallel. This yields the same contradiction as the corresponding case given above.

4. $s+l=p+q$: Now we assume that the prism has two flat positions. Then a,b,c,d can only form a deltoid, a parallelogram or an antiparallelogram. For these three cases we show by the following short computation that the base remains planar during the flex if and only if item 2, 3 or 4 of Lemma 4.3 holds.

   W.l.o.g. we can assume that the prism has z-parallel edges and that V_1 coincides with the origin. Then the remaining points have coordinates:

$$V_2 = \begin{pmatrix} d \\ 0 \\ h_2 \end{pmatrix}, \quad V_4 = \begin{pmatrix} d+bc\varphi_2 \\ bs\varphi_2 \\ h_4 \end{pmatrix}, \quad V_5 = \begin{pmatrix} ac\varphi_1 \\ as\varphi_1 \\ h_5 \end{pmatrix}, \tag{4.12}$$

   with $a,b,c,d>0$. Therefore beside Eq. (4.7) the coplanarity condition $det(V_2,V_3,V_4)=0$ has to hold, which can be written under consideration of $t_i := \tan(\varphi_i/2)$ for $i=1,2$ as follows:

$$a[dh_4+h_2(b-d)]t_1t_2^2 + a[dh_4-h_2(d+b)]t_1 + b(ah_2-dh_5)t_2 - b(ah_2+dh_5)t_1^2t_2 = 0. \tag{4.13}$$

   Moreover due to footnote 2 we can assume that t_1 or t_2 is not constant zero during the flex. In the next step we compute the resultant R of Eq. (4.7) and Eq. (4.13) with respect to t_1.

   - Deltoid: W.l.o.g. we can set $a=d$ and $b=c$. Moreover we can assume $c \neq d$ because otherwise we get a rhombus which is discussed later on as a special case of the parallelogram case. Now R can only vanish without contradiction for $(h_5-h_2)[q_1(c-d)t_2^2+q_2(d+c)]=0$ with

$$q_1 = (h_2+h_5)c+(2h_4-h_2-h_5)d, \quad q_2 = (h_2+h_5)c-(2h_4-h_2-h_5)d.$$

     Therefore we have to distinguish two cases:

     ⋆ $q_1=q_2=0$: This factors can only vanish without contradiction for $h_2=-h_5$ and $h_4=0$ which already yields item 2 of Lemma 4.3.

     ⋆ $h_2=h_5$: Now Eq. (4.7) and Eq. (4.13) have the common factor $dt_1 \neq 0$. Then the resultant of the remaining factors wit respect to t_1 can only vanish without contradiction (w.c.) for $\overline{q}_1(c-d)t_2^2+\overline{q}_2(d+c)=0$ with

$$\overline{q}_1 = h_5c+(h_4-h_5)d, \quad \overline{q}_2 = h_5c-(h_4-h_5)d.$$

     $\overline{q}_1=\overline{q}_2=0$ implies $h_4=h_5=0$, a contradiction.

   - Parallelogram/antiparallelogram: Now we set $a=b$ and $c=d$. Then R can only vanish w.c. for $(h_5-h_4+h_2)[\hat{q}_1(b-d)t_2^2+\hat{q}_2(b+d)]=0$ with

$$\hat{q}_1 = (h_5-h_4-h_2)b+(h_2-h_4-h_5)d, \quad \hat{q}_2 = (h_5-h_4-h_2)b-(h_2-h_4-h_5)d.$$

     Therefore we have to distinguish three cases:

     ⋆ $b=d$ and $\hat{q}_2=0$: This implies $h_2=h_5$. Now Eq. (4.7) and Eq. (4.13) have the common factor $d^2t_1 \neq 0$. Then the resultant of the remaining factors with respect to t_1 can only vanish w.c. for $h_4=2h_2$. Then the common factor of Eq. (4.7) and Eq. (4.13) yields t_1-t_2 which implies a special solution of item 4 of Lemma 4.3.

- ⋆ $\hat{q}_1 = \hat{q}_2 = 0$: This two conditions already imply $h_4 = 0$ and $h_2 = h_5$. As the common factor of Eq. (4.7) and Eq. (4.13) equals $t_1(b+d) - t_2(b-d)$ this case yields item 3 of Lemma 4.3.
- ⋆ $h_5 - h_4 + h_2 = 0$: If this condition is fulfilled the common factor of Eq. (4.7) and Eq. (4.13) equals $t_1 - t_2$ and therefore we get item 4 of Lemma 4.3. □

Lemma 4.4. *A planar four-bar mechanism with $l + s \geq p + q$ which is no parallelogram or antiparallelogram always has a configuration with parallel arms if l is one of these arms. Moreover such a four-bar mechanism has also a configuration where the coupler is parallel to the base. These two configurations coincide (⇒ folded pose) if and only if the four-bar linkage is a deltoid.*

Proof: For the proof we use the notation of the four-bar mechanism from Fig. 4.5. Now there are the following two possibilities such that the arms a, b are parallel:

1. They are located on the same side with respect to the base-line d. Therefore $\varphi_1 = \varphi_2$ holds and the corresponding equation of Eq. (4.7) reads as:

$$(a-b-c+d)(a-b+c+d)t_1^2 + (a-b+c-d)(a-b-c-d) = 0 \tag{4.14}$$

   As a consequence we get a real solution of the problem if

$$-(a-b-c+d)\underbrace{(a-b+c+d)}_{>0}(a-b+c-d)\underbrace{(a-b-c-d)}_{<0} \geq 0 \tag{4.15}$$

   holds.[3] Therefore we get a solution in one of the following four cases:

$$(i)\quad a+d > b+c \quad \text{and} \quad a+c > b+d, \qquad (iii)\quad a+d = b+c, \tag{4.16}$$

$$(ii)\quad a+d < b+c \quad \text{and} \quad a+c < b+d, \qquad (iv)\quad a+c = b+d. \tag{4.17}$$

   Now one of the cases (i) or (ii) is fulfilled if one of the arms a, b is the longest bar of the mechanism. Clearly, we can also assume in the special cases (iii) and (iv) w.l.o.g. that one of the arms a, b is the longest bar of the mechanism. This proves the first part of the lemma.

2. They are not located on the same side with respect to the base-line d, hence $\varphi_1 = \varphi_2 + \pi$. Now the corresponding equation of Eq. (4.7) reads as:

$$(a+b-c+d)(a+b+c+d)t_1^2 + (a+b-c-d)(a+b+c-d) = 0 \tag{4.18}$$

   Therefore we get a real solution of the problem if

$$-\underbrace{(a+b-c+d)}_{>0}\underbrace{(a+b+c+d)}_{>0}(a+b-c-d)\underbrace{(a+b+c-d)}_{>0} \geq 0 \tag{4.19}$$

   holds. As a consequence we get a solution if $a+b \leq c+d$ holds. As due to case 1, one of the arms is the longest bar, this is only possible for the special case $a+b = c+d$. But on the other hand there exists a pose where the coupler and the base are parallel for $c+d \leq a+b$. Now this equation is always fulfilled which proves the second part of the lemma.

If $c+d = a+b$ and condition (iii) or (iv) are fulfilled we get a folded pose. It can easily be seen that the solution of the linear system of equations is a deltoid. □

[3] Note that $(a-b+c+d)(a-b-c-d) = 0$ would yield that the mechanism is rigid.

4.4.3 Main theorem

In this section we give the complete classification of flexible octahedra with two opposite vertices at infinity.

Theorem 4.6. *In the projective extension of E^3 any octahedron, where exactly two opposite vertices (V_3, V_6) are ideal points, is flexible in one of the following cases:*

(I) The remaining two pairs of opposite vertices (V_1, V_4) and (V_2, V_5) are symmetric with respect to a common line as well as the edges of the prisms through V_3 and V_6, respectively.

(II) (i) One pair of opposite vertices (V_2, V_5) is symmetric with respect to a plane which contains the remaining pair of opposite vertices (V_1, V_4). Moreover also the edges of the prisms through V_3 and V_6 are symmetric with respect to this plane.

(ii) The remaining 4 vertices V_1, V_2, V_4, V_5 are coplanar and form an antiparallelogram and its plane of symmetry is parallel to the edges of the prisms through V_3 and V_6, respectively.

(III) This type is characterized by the existence of two flat poses and consists of two prisms where the orthogonal cross sections are congruent antiparallelograms. For the construction of these octahedra see Fig. 4.12.

(IV) The remaining 4 vertices V_1, V_2, V_4, V_5 are coplanar and form

(i) a deltoid and the edges of the prisms through V_3 and V_6 are orthogonal to the deltoids line of symmetry,

(ii) a parallelogram.

Proof: For the notation used in this proof we refer to Fig. 4.9. Moreover the corresponding prisms through the points V_3 and V_6 are denoted by Π_3 and Π_6, respectively. The faces through the remaining vertices V_i in E^3 always form 4-sided pyramids Λ_i for $i = 1, 2, 4, 5$.

We can stop the discussion of cases if the points V_1, V_2, V_4, V_5 are permanently coplanar during the flex because then by Lemma 4.3 we can only get a solution of type (II,ii) and (IV) or special cases of them. The following proof is split into three parts:

1st Part:

In this part we apply the conditions of case (d) of Theorem 4.1 over the octahedron in such a way that the corresponding two cosine equalities hold if any of the 8 faces is considered as central triangle. Up to the relabeling of vertices this yields the following case:

$$c\varphi_3 = c\chi_2, \quad c\varphi_1 = c\chi_3, \quad c\kappa_3 = c\psi_3, \quad c\kappa_1 = c\psi_1. \tag{4.20}$$

If additionally $c\chi_2 = c\kappa_3$ holds we get a special case of item (A) of the 3rd part treated later.

Therefore we can assume w.l.o.g. that the orthogonal cross section of Π_3 and Π_6 are deltoids (and not parallelograms or antiparallelograms). Moreover it can easily be seen that a flat pose of Π_3 or Π_6 implies a flat pose of the whole octahedron. Therefore the spherical image of $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ are spherical deltoids or isograms.

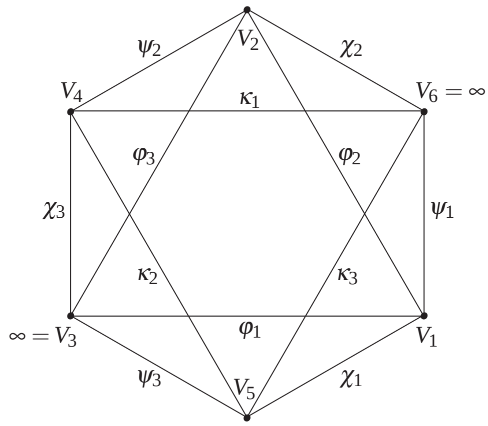

Figure 4.9: Schematic sketch of the octahedron $V_1, \ldots, V_6$ with dihedral angles $\varphi_i, \psi_i, \chi_i, \kappa_i$, $i = 1, 2, 3$.

1. If Λ_1 or Λ_4 are of isogram type then $c\chi_3 = c\kappa_1$ holds, which already implies that the orthogonal cross section of Π_3 and Π_6 are similar deltoids. Now we distinguish two cases:

   a. In the first case we assume that in both flat poses $V_3 \neq V_6$ holds. Due to the similarity the intersection points V_1, V_2, V_4, V_5 of corresponding prism edges are located on a line. As two such flat poses exist the line can only be orthogonal to the edges of the prism. Therefore V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 4.3.

   b. If in one of the flat poses $V_3 = V_6$ holds then the deltoids are congruent. As a consequence there exists an Euclidean motion such that Π_3 and Π_6 coincides. Moreover we can assume w.l.o.g. that this is a rotation about $[V_2, V_5]$. Due to the rotational symmetry and the symmetry of the deltoid the line spanned by the intersection points V_1 and V_4 of the other edges has to intersect the rotational axis $[V_2, V_5]$ (see Fig. 4.10(a)). Therefore V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 4.3.

2. If Λ_1 and Λ_4 are of deltoid type then $c\psi_2 = c\kappa_2$ and $c\chi_1 = c\varphi_2$ hold. We distinguish two cases:

   a. If $c\psi_2 = c\varphi_2$ holds, then Λ_2 and Λ_5 are of isogram type. In the flat poses $V_2 = V_5$ holds and we see that the corresponding faces of the pyramids Λ_2 and Λ_5 are congruent. This already implies with $c\psi_2 = c\varphi_2$ that the orthogonal cross sections of Π_3 and Π_6 are parallelograms/antiparallelograms which yields the contradiction.

   b. In the other case Λ_2 and Λ_5 are of deltoid type. This already implies that in the flat poses $V_3 = V_6$ holds. Therefore the orthogonal cross sections of Π_3 and Π_6 are congruent deltoids ($\Rightarrow c\chi_3 = c\kappa_1$). This yields a contradiction as Λ_1 and Λ_4 are of isogram type.

2nd Part:

As for the one case of item (c) of Theorem 4.1 the four points $\in E^3$ are already coplanar during the flex, we are done due to Lemma 4.3.

Therefore we apply the conditions of the other case of item (c) of Theorem 4.1 over the octahedron in such a way that the corresponding two cosine equalities hold if any of the 8 faces is

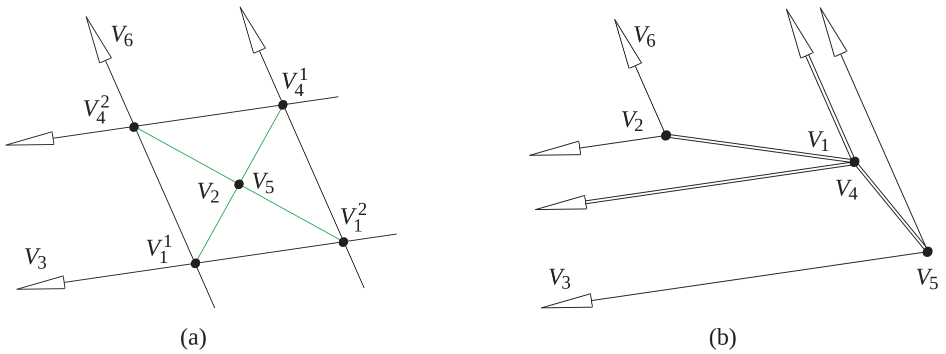

Figure 4.10: (a) Rotation of Π_3 about the projecting line $[V_2, V_5]$. The connecting lines $[V_1^1, V_4^1]$ or $[V_1^2, V_4^2]$ of possible intersection points intersect $[V_2, V_5]$. (b) Flat pose of the octahedron where Λ_1 and Λ_4 are congruent.

considered as central triangle. Up to the relabeling of vertices this yields the following case:

$$c\varphi_2 = c\psi_2, \quad c\varphi_1 = c\chi_3, \quad c\kappa_2 = c\chi_1, \quad c\kappa_1 = c\psi_1. \tag{4.21}$$

Moreover as the cosines of the dihedral angles through V_1 and V_4 are pairwise the same we can apply *Kokotsakis' theorem* (*Satz über zwei Vierkante*) given in §12 of [14] which implies that the pyramids Λ_1 and Λ_4 are congruent. Now we have to distinguish two cases because they can be congruent with respect to an orientation preserving or a non-orientation preserving isometry:

1. Orientation preserving isometry: As $[V_1, V_6] \parallel [V_4, V_6]$ and $[V_1, V_3] \parallel [V_4, V_3]$ has to hold the rigid body motion can only be a composition of a half-turn about a line l orthogonal to the plane $[X, V_3, V_6]$ plus a translation along the axis, where X stands for any point of E^3. Moreover l has to be located within the plane $[V_1, V_2, V_5]$ because otherwise there does not exist a translation such that the remaining pairs of corresponding edges intersect in V_2 and V_5, respectively. This already yields that V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 4.3.

2. Non-orientation preserving isometry: Here we are left with three possibilities:

   a. The Euclidean motion is a composition of a reflection on $\varepsilon := [X, V_3, V_6]$ and a translation parallel to this plane. If $[V_1, V_2, V_5]$ is orthogonal to ε then V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 4.3.

      In any other case the translation vector has to be the zero vector ($\Rightarrow V_2$ and V_5 are located on ε) such that the other corresponding edges intersect in V_2 and V_5, respectively. As the orthogonal cross sections of Π_3 is at least a deltoid, the flat poses of this prism imply flat poses of the whole structure as all vertices are located on ε. Therefore the spherical images of Λ_1 and Λ_4 have to be spherical deltoids:

      i. If $c\psi_2 = c\kappa_2$ holds the flat poses immediately imply that V_1, V_2, V_4, V_5 has to be coplanar during the flex and we are done due to Lemma 4.3.

      ii. For the other possibility $c\chi_3 = c\kappa_1$ the points V_3 and V_6 coincide in the flat poses and therefore the deltoidal cross sections of Π_3 and Π_6 are congruent. This case was already discussed in item (1b) of the 1st part.

b. The Euclidean motion is a composition of a reflection on $\varepsilon := [X, V_3, V_6]$ and a half-turn about a line l orthogonal to ε. Applying such a transformation all pairs of corresponding edges of the pyramids are parallel. Therefore V_2 and V_5 are also ideal points which contradicts our assumptions. [4]

c. Under the assumption that $\sphericalangle V_3XV_6$ is constant $\pi/2$ during the flex the Euclidean motion could also be composed of a reflection on one of the planes $\varepsilon_1 := [l, V_3]$ or $\varepsilon_2 := [l, V_6]$ plus a translation parallel to it. This case cannot yield a solution as any octahedron with $\sphericalangle V_3XV_6 = \text{const.} \neq 0$ has to be rigid. The proof is left to the reader.

Kokotsakis' theorem cannot be applied if the spherical image of Λ_1 and Λ_4 are isograms. In this case ($c\psi_2 = c\kappa_2$, $c\chi_3 = c\kappa_1$) such an octahedron already has two flat poses. Now the orthogonal cross sections of Π_3 and Π_6 are deltoids, parallelograms or antiparallelograms and the spherical image of Λ_2 and Λ_5 are spherical isograms or spherical deltoids. As not both spherical images of Λ_2 and Λ_5 can be isograms (otherwise we get item (B) of the 3rd part) we can assume w.l.o.g. that Λ_2 has a deltoidal spherical image.

1. If at least one further structure of Π_3 and Π_6 is of deltoid type then V_1 has to coincide with V_4 in the flat pose (see Fig. 4.10(b)). This already shows that also in this case Λ_1 and Λ_4 are congruent and therefore we can apply the same argumentation as given above.

2. If the orthogonal cross sections of Π_3 and Π_6 are parallelograms or antiparallelograms then we can only get a special case of item (A) of the following 3rd part.

3rd Part:

We are left with the possibilities given in item (a) and (b) of Theorem 4.1. W.l.o.g. we take V_1, V_2, V_3 as representative triangle. Then the motion transmission from Σ_3 to Σ_2 via V_3 and V_1 is reducible if:

- the orthogonal cross section of Π_3 is a parallelogram or an antiparallelogram,
- the spherical image of Λ_1 is an isogram,
- case (b) holds.

Analogous possibilities hold for the motion transmission from Σ_1 to Σ_2 via V_3 and V_2. Now combinatorial aspects show that one of the following cases has to hold:

A. the orthogonal cross section of Π_3 is a parallelogram or an antiparallelogram,

B. the spherical image of Λ_1 and Λ_2 are isograms,

C. both motion transmissions are reducible due to case (b),

D. the motion transmission from Σ_1 to Σ_2 (or Σ_3 to Σ_2) is reducible due to case (b) and the spherical image of Λ_1 (or Λ_2) is an isogram. [5]

---

[4] We get a special case of a flexible octahedron of Theorem 4.5.

[5] Note that we get the case in the parentheses from the other one just by a relabeling.

Therefore the remaining flexible octahedra with opposite vertices on the plane at infinity can only belong to one of these four cases. As a consequence the reducible composition implied by these flexible octahedra has to be of the same type independent of the choice of the central triangle. This yields the following conditions:

ad A. $c\varphi_1 = c\chi_3,\ c\varphi_3 = c\psi_3,\ c\psi_1 = c\kappa_1,\ c\chi_2 = c\kappa_3.$

ad B. $c\varphi_1 = c\psi_1,\ c\varphi_3 = c\chi_2,\ c\kappa_1 = c\chi_3,\ c\kappa_3 = c\psi_3,\ c\varphi_2 = c\psi_2 = c\kappa_2 = c\chi_1.$

ad C. $c\varphi_1 = c\kappa_1,\ c\varphi_3 = c\kappa_3,\ c\psi_3 = c\chi_2,\ c\chi_3 = c\psi_1.$

ad D. $c\varphi_1 = c\psi_1,\ c\varphi_3 = c\kappa_3,\ c\psi_3 = c\chi_2,\ c\chi_3 = c\kappa_1,\ c\kappa_2 = c\psi_2,\ c\chi_1 = c\varphi_2.$

In the following these four cases are discussed in detail:

ad (C) If the orthogonal cross section of Π_3 and Π_6 are parallelograms or antiparallelograms then we get a special case of item (A). Therefore we can assume that this is not the case.

This assumption together with the property that the cosine of the dihedral angles of Π_3 and Π_6 are the same, already imply that these prisms are related by an Euclidean similarity transform. Now we consider the orthogonal cross section (four-bar mechanism a,b,c,d) of one of these prisms:

1. $l+s < p+q$: If we choose s as base then *Grashof's theorem* is fulfilled and we get a double-crank mechanism. Such a mechanism has two poses where the coupler is parallel to the base.
   a. If in one of these two poses the parallel planes of both prisms do not coincide or if $V_3 = V_6$ holds then the condition that the corresponding edges of the prisms intersect each other in this pose, already yields that the coupler and the base must have the same length. But this already contradicts $l+s < p+q$.
   b. If in one of the two flat poses the parallel planes of both prisms coincide (but $V_3 \neq V_6$), then this already implies that V_1,V_2,V_4,V_5,V_3 and V_4,V_5,V_1,V_2,V_6 are congruent. Moreover it can be seen from this pose that the pyramids Λ_1 and Λ_4 are congruent with respect to an orientation preserving isometry. Due to $c\chi_3 = c\psi_1$ and $c\kappa_1 = c\varphi_1$ this property has to hold during the whole flex.[6] As the corresponding rigid body motion also has to interchange the ideal points V_3 and V_6 we are left with two possibilities:
      i. The rigid body motion is a composition of a half-turn about one of the two bisectors of $\sphericalangle V_3XV_6$ plus a translation along this axis. If the axis is located within the plane $[V_1,V_2,V_5]$ the points V_1,V_2,V_4,V_5 are coplanar during the flex and we are done due to Lemma 4.3. In any other case the translation vector has to be the zero vector such that the other corresponding edges intersect in V_2 and V_5, respectively. This yields solution (I).
      ii. The angle $\sphericalangle V_3XV_6$ is constant $\pi/2$ during the flex. Then a 90°-rotation about a line orthogonal to $[X,V_3,V_6]$ plus a translation along the axis yields a further possibility. This case cannot yield a solution for the same reason as case (2c) of the 2nd part.

[6]The same holds for the pyramids Λ_2 and Λ_5.

2. $l+s \geq p+q$: Now there exist the two special poses of Lemma 4.4. Analogous considerations as in the case $l+s < p+q$ yield one of the following cases:

   a. $a=b$, $c=d$: Now the four-bar mechanism a,b,c,d is a parallelogram or an antiparallelogram. We get a special case of item (A).

   b. We get the above discussed item (1b) and therefore solution (I).

   c. The orthogonal cross section of Π_3 and Π_6 are similar deltoids. This can only yield a case discussed in item (1) of the 1st part.

ad (D) If the spherical image of Λ_2 and Λ_5 are isograms we get item (B). Therefore we can assume w.l.o.g. that this is not the case and we can apply *Kokotsakis' theorem* which yields that Λ_2 and Λ_5 are congruent. Again we have to distinguish two cases:

1. Non-orientation preserving isometry: We can transform the two pyramids into each other by a reflection on one of the bisecting planes ε_i $(i=1,2)$ of $\sphericalangle V_3XV_6$ plus a translation parallel to ε_i.[7]

   If $[V_2,V_1,V_4]$ is orthogonal to ε_i then V_1,V_2,V_4,V_5 are coplanar during the flex (cf. Lemma 4.3).

   In any other case the translation vector has to be the zero vector ($\Rightarrow V_1$ and V_4 are located on ε_i) such that the other corresponding edges intersect in V_1 and V_4. We get solution (II,i).

2. Orientation preserving isometry: As the corresponding rigid body motion also has to interchange the ideal points V_3 and V_6 we are left with two possibilities:

   a. The rigid body motion is a composition of a half-turn about one of the two bisectors of $\sphericalangle V_3XV_6$ plus a translation along this rotary axis. In order to guarantee that the remaining vertices V_2 and V_5 exist, the corresponding edges have to intersect the axis of rotation. This already shows that all vertices of E^3 are coplanar during the flex (cf. Lemma 4.3).

   b. The angle $\sphericalangle V_3XV_6$ is constant $\pi/2$ during the flex. Then the 90°-rotation about a line l orthogonal to $[X,V_3,V_6]$ plus a translation along l yields a further possibility. This case cannot yield a solution for the same reason as case (2c) of the 2nd part.

ad (B) In this case the spherical image of the faces through each vertex $\in E^3$ is an isogram. Now the conditions $\mathrm{c}\varphi_2 = \mathrm{c}\psi_2 = \mathrm{c}\kappa_2 = \mathrm{c}\chi_1$ yield for φ_2 equal 0 or π that the octahedron has two flat poses. Therefore the orthogonal cross section of the prisms Π_3 and Π_6 can only be a deltoid, a parallelogram or an antiparallelogram.

It can easily be seen that the deltoid case does not fit with both folded positions of the spherical focal mechanism composed of two spherical isograms. Therefore Π_3 and Π_6 have to be of parallelogram type or antiparallelogram type.

Note that opposite edges of a pyramid with an isogram as spherical image are symmetric with respect to a common line in a flat pose. The same holds for the flat pose of a prisms with a parallelogram or antiparallelogram as orthogonal cross section. Beside the scaling factor these two properties already determine the octahedron in the flat pose up to 3 parameters, namely the

[7]The only possible rotation is a half-turn about a line orthogonal to $[X,V_3,V_6]$. But this rotation is only the transition between the two possible reflections.

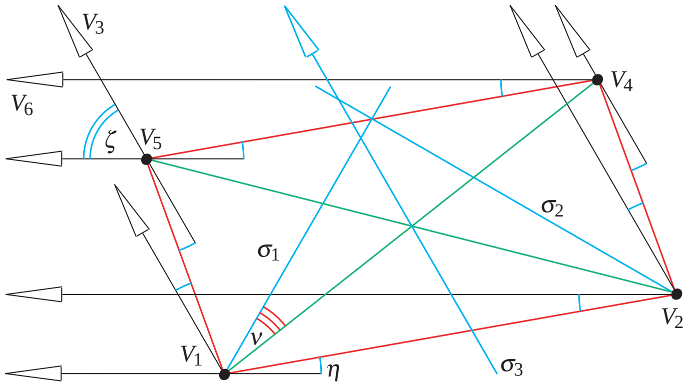

Figure 4.11: Illustration of the 3 free design parameters ζ, η, ν beside the scaling factor.

angles ζ, η, ν (see Fig. 4.11). Now this structure is flexible if we flex one of the prisms out of the flat pose in such a way that the orthogonal cross section is a parallelogram because then we get a special case of type (IV,ii).

In the other case (antiparallelogram) the octahedron is not even infinitesimal flexible. According to Kokotsakis (cf. §3 and §13 of [14]) this condition is fulfilled if the bisectors σ_i $i = 1,2,3$ have a point in common. [8] It can easily be seen (cf. Fig. 4.11) that this is the case if ν is zero. This already implies the construction of type (III) octahedra which equals the construction of Bricard's type 3 octahedra with two opposite vertices at infinity (see Fig. 4.12).

<u>*Remark*</u> 4.3. Note that in each flat pose of a type (III) octahedron flexion a bifurcation into a type (IV,ii) octahedron flexion is always possible. ◇

ad (A) In the first case we assume that the orthogonal cross section of Π_3 is a parallelogram. Then we consider one of the two possible configurations where Π_6 is in a flat pose. In this pose it can immediately be seen that V_1, V_2, V_4, V_5 is a parallelogram.[9] Then the flexion of Π_3 already implies type (IV,ii).

Therefore we can assume for the last case that the orthogonal cross section of both prisms are antiparallelograms. We have to distinguish two cases:

1. In both flat poses of Π_3, Π_6 is also flat and has the same carrier plane ε as the folded prism Π_3. Therefore this is an octahedron with two flat poses. As a consequence the spherical image of the pyramids $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ can only be a spherical isogram or a spherical deltoid.

   Assume the triangle V_1, V_2, V_3 as central triangle. If Λ_1 is of isogram type then we have a focal mechanism composed of Λ_1 and Π_3 as Eq. (4.5) holds.[10] Moreover, this is a reducible composition with a spherical coupler component. The corresponding spherical coupler can only be of isogram type because the deltoid case does not fit with both folded positions of the focal mechanism.

[8] The σ_i's are the limit of the intersection of two opposite faces of the respective pyramids and prism, respectively.
[9] This parallelogram can even degenerate into a folded one.
[10] The orthogonal cross section of Π_3 (an antiparallelogram) cannot have the additional property of a deltoid as then we get a flipped over rhombus which contradicts footnote 1.

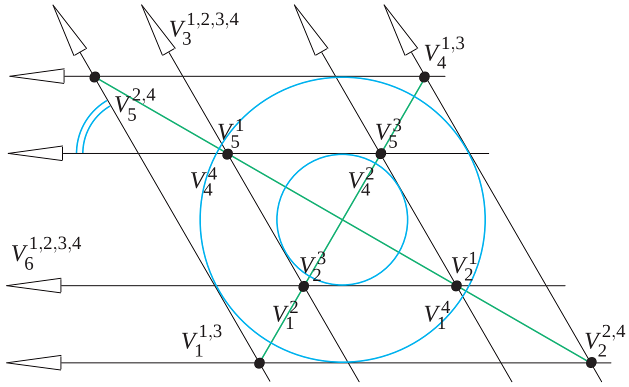

Figure 4.12: Construction of flexible octahedra of type (III): In the above given construction four flexible octahedra $V_1^i, \ldots, V_6^i$ $(i = 1,2,3,4)$ are hidden, where those with indices $i = 1,2$ are of type (III): The sides of the three quadrangles spanned by two pairs of opposite vertices touch three concentric circles (which cannot degenerate into the midpoint).
The octahedra with indices $i = 3,4$ cannot be of type (III) because in the second flat pose the points $V_1^i, V_2^i, V_4^i, V_5^i$ also have to form a rhombus. This is only possible if the orthogonal cross sections of Π_3 and Π_6 are flipped over rhombi which contradicts footnote 1. Therefore the octahedra $i = 3,4$ can only have a trivial flexion (the relative motion of Π_3 and Π_6 is a rotation with axis $[V_1^i, V_2^i, V_4^i, V_5^i]$; cf. footnote 1) beside the flexibility of type (IV,ii).

As a consequence of this consideration all pyramids $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ are either isograms (which yields case (B)) or they are all of deltoid type. For the latter case we have to distinguish two principal cases:

a. $c\kappa_2 = c\chi_1 = c\varphi_2 = c\psi_2$: In this case the points V_1, V_2, V_4, V_5 have to be collinear in both flat poses which already yield that these points are coplanar during the flex.

b. $c\chi_3 = c\kappa_1$, $c\psi_2 = c\varphi_2$, $c\chi_1 = c\kappa_2$: This case can only yield special cases of the 2nd part as Eq. (4.21) holds under consideration of $c\varphi_1 = c\chi_3$, $c\varphi_3 = c\psi_3$, $c\chi_2 = c\kappa_3$, $c\psi_1 = c\kappa_1$. [11]

2. Assuming there exists a flat pose of Π_3 and Π_6 is not in a flat pose sharing the same carrier plane ε of the folded Π_3. Then we can reflect Π_6 on ε and we get Π_6' with the ideal point V_6'. If $\Pi_6 = \Pi_6'$ holds then this already implies that the points V_1, V_2, V_4, V_5 are coplanar during the flex.

   Therefore we can assume w.l.o.g. $\Pi_6 \neq \Pi_6'$. If $V_1, \ldots, V_6$ is a flexible octahedron then also the octahedron $V_1, V_2, V_4, V_5, V_6, V_6'$ has to be flexible due to the symmetry. For the same reason the pyramids $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ of the octahedron $V_1, V_2, V_4, V_5, V_6, V_6'$ are of deltoid type, which already implies that the points V_1, V_2, V_4, V_5 are coplanar during the flex.

This finishes the proof of the necessity of the conditions given in Theorem 4.6.

[11] The remaining possibility $c\chi_2 = c\varphi_3$, $c\psi_2 = c\kappa_2$, $c\chi_1 = c\varphi_2$ can be done analogously because it can be obtained from this case by an appropriate relabeling.

The sufficiency for the flexibility of both types of item (IV) as well as of type (II,ii) follows directly from Lemma 4.3. As the types (I), (II) and (III) can be constructed from the corresponding types of Bricard flexible octahedra by a limiting process, the sufficiency for these types follows immediately from the flexibility of Bricard's octahedra. This finishes the proof of Theorem 4.6. □

4.5 Conclusion and future research

In this paper we completed the classification of flexible octahedra in the projective extension of the Euclidean 3-space. If all vertices are finite we get the well known Bricard flexible octahedra. There exist flexible octahedra of type 2 (cf. Theorem 4 of [18]) and type 3 (cf. Theorem 2 of [18]) with one vertex at infinity. Moreover there do not exist further flexible octahedra with one vertex on the plane at infinity (cf. Theorem 3 of [18]).

All flexible octahedra with at least three vertices at infinity are trivially flexible and listed in Theorem 4.5 (see also Theorem 4.3).

Finally we presented all types of flexible octahedra with two vertices at infinity in Theorem 4.6 (see also Theorem 4.4). The types (I), (II) and (III) of this theorem can be generated from the corresponding *Bricard octahedra* by a limiting process. The remaining octahedra of type (IV) do not have a flexible analogue in E^3; they are flexible without self-intersection.

For a practical application one can think of an open serial chain composed of prisms $\Pi_0, \ldots \Pi_n$ where each pair of neighboring prisms Π_i, Π_{i+1} $(i = 0, \ldots, n-1)$ forms a flexible octahedron of Theorem 4.6. Note that such a structure admits a constrained motion. Moreover, if we additionally assume that $\Pi_0 = \Pi_n$ holds, we get a closed serial chain which is in general rigid. It would also be interesting under which geometric conditions such structures are still flexible. Clearly, some aspects of this question are connected with the problem of nR overconstrained linkages (e.g. the spatial $4R$ overconstrained linkage is the Bennett mechanism). Finally, it should be noted that the *Renault style polyhedron* presented by I. PAK [19] can be seen as a trivial example for the case $n = 4$.

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