Stewart Gough platforms with linear singularity surface

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***Abstract*—In the general case the singularity loci of parallel manipulators of Stewart Gough type is a cubic surface in the space of translations. In [G. Nawratil, Stewart Gough platforms with non-cubic singularity surface, Technical Report No. 204, Geometry Preprint Series, Vienna University of Technology (2010)] all manipulators with a non-cubic singularity surface were characterized. Based on this result we determine the whole set of parallel manipulators of Stewart Gough type possessing a linear singularity surface. These manipulators have the advantage, that their singularity surface can easily be visualized because it is a plane for any orientation of the platform. We also give a geometric interpretation of these manipulators.**

I. INTRODUCTION

A parallel manipulator of Stewart Gough type consists of two systems, namely the platform Σ and the base Σ_0, which are connected via six Spherical-Prismatic-Spherical (or Spherical-Prismatic-Universal) joints. The geometry of such a manipulator is given by the six base anchor points $\mathsf{M}_i \in \Sigma_0$ with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ and by the six platform anchor points $\mathsf{m}_i \in \Sigma$ with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$. By using Euler parameters (e_0, e_1, e_2, e_3) for the parametrization of the spherical motion group $\mathsf{SO}(3)$ the coordinates $\mathbf{m}'_i$ of m_i with respect to the fixed space can be written as $\mathbf{m}'_i = H^{-1}\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t}$ with the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$, the homogenizing factor $H := e_0^2 + e_1^2 + e_2^2 + e_3^2$ and the rotation matrix $\mathbf{R} := (r_{ij})$ given by

$$\begin{bmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_0e_2) \\ 2(e_1e_2+e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{bmatrix}.$$

It is well known that a Stewart Gough platform (SG platform) is singular if and only if the carrier lines of the prismatic legs belong to a linear line complex. This is the case if $Q := det(\mathbf{Q}) = 0$ holds, whereas the i^{th} row of the 6×6 matrix $\mathbf{Q}$ equals the Plücker coordinates $\underline{\mathbf{l}}_i := (\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{m}'_i - \mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$ of the i^{th} carrier line.

Moreover it should be mentioned that we denote the coefficients of $t_1^i t_2^j t_3^k$ of Q by Q^{ijk}. The coefficients of $e_0^a e_1^b e_2^c e_3^d$ of Q^{ijk} are denoted by Q^{ijk}_{abcd}.

A. Motivation and related work

From the design point of view, it is desirable to have a graphical representation of the singularity set of SG platforms, because it simplifies the identification of the singular loci within the given workspace. As it is impossible to represent the singularity surface in 6-dimensional space graphically, only the visualization of 3-dimensional subspaces make sense. This can be done for a general SG platform according to [11].

Usually one fixes the orientational part and visualizes the singularity surface which is in the general case a cubic surface in the translation parameters t_1, t_2, t_3. The drawback of surfaces of degree 3 is that they can have very complicated shapes. Therefore it is desirable for designers to handle only with parallel manipulators of SG type which have a simple singularity surface for any orientation of the platform.

The best, one can think of in this context, are non-architecturally singular parallel manipulators those singularity set for any orientation of the platform is a cylindrical surface with rulings parallel to a given fixed direction p in the space of translations. In this case the singularity set can easily be visualized as curve by choosing p as projection direction. In addition the computation of singularity free zones reduces to a 5-dimensional task (cf. LIE ET AL. [12]). In NAWRATIL [13] and [14] it was shown, that planar SG platforms possessing such a cylindrical singularity surface must have 4 collinear anchor points. Finally it was proven by the author in [16] that there only exist two planar manipulator designs with such a property, namely:

(i) $m_1 = m_2$, $m_3 = m_4$, $m_5 = m_6$ and $[M_1, M_2] \parallel [M_3, M_4] \parallel [M_5, M_6] \parallel p$,

(ii) $[M_5, M_6] \parallel [M_1, \ldots, M_4] \parallel p$, $m_5 = m_6$, M_1, M_2, M_3, M_4 and m_1, m_2, m_3, m_4 are collinear.

The determination of the whole set of non-planar parallel manipulators with a cylindrical singularity surface is still an open problem, but there are good reasons to conjecture that this set consists of only one element, namely manipulator (i) without planar base.

Unfortunately, the set of SG platforms with a cylindrical singularity surface has only a very limited variety due to the above given conditions on the design parameters. Therefore one has to look for manipulators with other simple singularity surfaces. KARGER [7] suggested to use SG platforms with a quadratic singularity surface because all types of quadrics have well known and rather simple shapes. Moreover this property simplifies the computation of singularity free zones in the space of translations (for a fixed orientation) considerably, as the problem reduces to the minimization of a quadratic function under a quadratic constraint (cf. FLETCHER [3]).

In [19] the following main theorem of SG platforms with non-cubic (i.e. linear or quadratic) singularity surface was given by the author:

Theorem 1. *A non-architecturally singular SG platform possesses a non-cubic singularity surface if and only if* $rk(\mathbf{M}) < 5$ *holds with*

$$\mathbf{M} = \begin{pmatrix} 1 & a_1 & b_1 & c_1 & A_1 & B_1 & C_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_6 & b_6 & c_6 & A_6 & B_6 & C_6 \end{pmatrix}. \quad (1)$$

In addition it was shown in [19] that this main theorem possesses the following geometric interpretation:

Theorem 2. *A non-architecturally singular SG platform possesses a non-cubic singularity surface if and only if there exists a affine correspondence between the platform and the base or if the manipulator is planar with* $rk(\mathbf{M}) = 4$[1].

Remark 1. For the characterization of architecturally singular planar SG platforms we refer to KARGER [6], NAWRATIL [15], RÖSCHEL AND MICK [20] as well as WOHLHART [21]. For the non-planar case we refer to KARGER [8] and NAWRATIL [17]. ⋄

Based on these theorems we determine all manipulators with a linear singularity surface, whereas we distinguish between planar SG platforms (cf. Section 2) and non-planar ones (cf. Section 3).

II. PLANAR SG PLATFORMS

Theorem 3. *A non-architecturally singular SG platform with planar platform and base possesses a linear singularity surface if and only if there exists a affinity between corresponding anchor points.*

Proof: The fact that planar parallel manipulators of SG type with affine equivalent platform and base possess a linear singularity surface was already demonstrated by KARGER [7]. The proof that these are the only planar manipulators with this property is split up into the following 2 cases:

• No four anchor points are collinear:

For the proof of this part we will not invest that $rk(\mathbf{M}) = 4$ must hold due to Theorem 2.

Without loss of generality (w.l.o.g.) we can choose coordinate systems in the platform and the base such that $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$ and $c_i = C_i = 0$ for $i = 1, \ldots, 6$ hold. Moreover due to Lemma 1 of [6] and Lemma 2 of [18] we can assume

$$a_2 A_2 B_3 B_4 B_5 (a_3 - a_4)(b_3 - b_4) coll(\mathsf{m}_3, \mathsf{m}_4, \mathsf{m}_5) \neq 0, \quad (2)$$

whereas $coll(\mathsf{x}, \mathsf{y}, \mathsf{z}) \neq 0$ denotes the collinearity condition of the points x, y and z.

Therefore we can perform the elementary matrix manipulation of $\mathbf{Q}$ given by KARGER [6]. We end up with $\underline{\mathfrak{l}}_6 := (v_1, v_2, v_3, 0, -w_3, w_2)$ with

$$v_i := r_{i1} K_1 + r_{i2} K_2, \quad w_j := r_{j1} K_3 + r_{j2} K_4,$$

and

$$\begin{aligned} K_1 &= |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{a}|, \quad K_3 = |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Aa}|, \\ K_2 &= |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{b}|, \quad K_4 = |\mathbf{A}, \mathbf{B}, \mathbf{Ba}, \mathbf{Bb}, \mathbf{Ab}|, \end{aligned} \quad (3)$$

and

$$\mathbf{X} = \begin{bmatrix} X_2 \\ \vdots \\ X_6 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_2 \\ \vdots \\ y_6 \end{bmatrix}, \ \mathbf{Xy} = \begin{bmatrix} X_2 y_2 \\ \vdots \\ X_6 y_6 \end{bmatrix}. \quad (4)$$

Due to Eqs. (12) and (13) of [13] the expression K_1 and K_2 must vanish. Therefore we set K_1 and K_2 equal to zero and compute Q in dependency of K_3 and K_4. Then we get

$$\begin{aligned} Q^{200} &= a_2 A_2 r_{13}(r_{31} K_3 + r_{32} K_4) F[24], \\ Q^{020} &= A_2 r_{23}(r_{31} K_3 + r_{32} K_4) G[48], \end{aligned} \quad (5)$$

whereas the numbers in the brackets give the numbers of terms of the non-explicitly given factors F and G. Now we denote the coefficients of $e_0^a e_1^b e_2^c e_3^d$ of F resp. G by F_{abcd} resp. G_{abcd}. W.l.o.g. we can solve $F_{1010} = 0$ for b_5. Then $F_{1100} = 0$ can only vanish without contradiction (w.c.) for $b_3 = b_4 B_3 / B_4$. W.l.o.g. we can express a_5 from $G_{1100} = 0$. Then we can compute A_4 from the only non-contradicting factor of $G_{1010} = 0$. Now $K_2 = 0$ implies $b_6 = b_4 B_6 / B_4$ and $K_1 = 0$ can be solved for a_6 w.l.o.g.. As a consequence we get $rk(\mathbf{M}) = 3$ which indicates that there exists a affinity between the platform and the base (cf. [9]). This finishes the discussion of this case.

• Four anchor points are collinear:

First of all, we compute those planar SG platforms with four collinear anchor points which yield $rk(\mathbf{M}) = 4$ (cf. Theorem 2). W.l.o.g. we can set $A_1 = B_1 = B_2 = B_3 = B_4 = a_1 = b_1 = b_2 = 0$, i.e. $\mathsf{M}_1, \ldots, \mathsf{M}_4$ are collinear. An elementary case study shows that there exist the following two non-architecturally singular cases (up to renumbering):

a) $b_3 = b_4 = 0$ and $b_5 = b_6 B_5 / B_6$,
b) $A_4 = [b_4(a_2 A_3 - a_3 A_2) + a_4 b_3 A_2]/(a_2 b_3)$ and $A_5 = [a_2 b_3 A_6 B_5 + A_2 B_6 (a_5 b_3 - a_3 b_5) - A_2 B_5 (a_6 b_3 - a_3 b_6) + a_2 A_3 (b_5 B_6 - b_6 B_5)]/(a_2 b_3 B_6)$.

For these two cases we show that the coefficients Q^{ijk} with $i + j + k = 2$ cannot vanish w.c.:

ad a) For this case Q^{020} can only vanish w.c. for $a_5 = a_6$ and $B_5 = B_6$ ($\Rightarrow \mathsf{m}_5 = \mathsf{m}_6$). Then $Q^{002} = 0$ already yields the contradiction.

ad b) In this case Q^{200} factors into

$$r_{13} B_5 B_6 ((a_5 - a_6) r_{31} + (b_5 - b_6) r_{32}) L[20]. \quad (6)$$

1. $B_5 = 0$: Then Q^{101} cannot vanish w.c..
2. $a_5 = a_6$, $b_5 = b_6$ ($\Rightarrow \mathsf{m}_5 = \mathsf{m}_6$): Q^{101} cannot vanish w.c..
3. $L[20] = 0$: Computation of L_{1100} yields

$$b_4 [A_2 (a_4 b_3 - a_3 b_4) - a_2 A_3 (b_3 - b_4)]. \quad (7)$$

As for $b_4 = 0$ we get the contradiction from $L_{1010} = 0$ we set the remaining factor equal to zero.

[1] According to KARGER [9] no special geometric properties for these planar parallel manipulators of SG type were known so far.

i. $A_2 \neq 0$: Under this assumption we can express a_4 from this condition. Again $L_{1010} = 0$ yields the contradiction.

ii. $A_2 = 0$: Now the condition can only vanish w.c. for $b_3 = b_4$. Again $L_{1010} = 0$ yields the contradiction. This finishes the proof of Theorem 3. □

Remark 2. In the case with 4 collinear anchor points we only end up with contradictions as planar SG manipulators with affine equivalent platform and base and 4 collinear anchor points are already architecturally singular. The reason for this is that the carrier lines of the legs through these 4 collinear anchor points always belong to a regulus (cf. item 8 of Theorem 3 in [8]). ⋄

III. NON-PLANAR SG PLATFORMS

Due to Theorem 2 we know that non-planar parallel manipulators of SG type possess a non-cubic singularity surface if and only if there is a affine correspondence between the platform and the base anchor points. Moreover the following two lemmata hold:

Lemma 1. *A non-planar parallel manipulator of SG type possessing a linear singularity surface has to have 3 collinear anchor points.*

Proof: KARGER proved in Theorem 1 of [8] that a non-planar architecturally singular SG platform has to possess 3 collinear anchor points. He only used coefficients of Q^{ijk} with $i+j+k > 1$ for the proof of this theorem and therefore Lemma 1 is valid. □

Lemma 2. *A non-planar parallel manipulator of SG type possessing a linear singularity surface has to have 4 collinear anchor points.*

Proof: KARGER proved in Theorem 2 of [8] that a non-planar architecturally singular SG platform has to possess 4 collinear anchor points. In his proof (done by contradiction) he only used in one case the coefficients of $Q^{100} = 0$. In all other cases he used coefficients of Q^{ijk} with $i + j + k > 1$. Therefore we only have to prove that in the one mentioned case the contradiction can also be concluded from coefficients of Q^{ijk} with $i + j + k > 1$. This can be done as follows:

This special case equals case (IIa) in the proof of Theorem 2 of [8]. The case is given by:

$$\begin{aligned} &A_1 = B_1 = B_2 = B_3 = a_1 = b_1 = b_2 = b_3 = 0, \\ &C_i = c_1 = c_2 = c_3 = c_4 = 0 \quad \text{for} \quad i = 1, \ldots, 6, \\ &A_3 = a_3A_2/a_2, B_5 = B_6, A_5 = A_4(c_6 - c_5)/c_6, \\ &\text{and} \quad x_6 = (x_5 - x_4)c_6/c_5 + x_4 \quad \text{for} \quad x = a, b. \end{aligned} \tag{8}$$

Now Q factors into:

$$A_2a_3B_6(a_2 - a_3)S[10705]/(a_2c_5c_6). \tag{9}$$

Then S^{003} can only vanish w.c. for $A_6 = 0$. Then S^{101} yields the contradiction. This finishes the proof of Lemma 2. □

Theorem 4. *There do not exist non-planar parallel manipulators of SG type with a linear singularity surface.*

Proof: W.l.o.g. can choose coordinate systems in the platform and base such that

$$\begin{aligned} A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0, \\ C_1 = C_2 = C_3 = c_1 = c_2 = c_3 = 0, \end{aligned} \tag{10}$$

hold. Due to Theorem 2 there exists a affinity κ between corresponding anchor points. W.l.o.g. we can assume that $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4$ are not coplanar, i.e. $A_2B_3C_4 \neq 0$. Now the matrix $\mathbf{K}$ of κ mapping $\mathsf{M}_i \mapsto \mathsf{m}_i$ for $i = 1, \ldots, 4$ equals:

$$\begin{bmatrix} \frac{a_2}{A_2} & \frac{a_3A_2 - a_2A_3}{A_2B_3} & \frac{B_3(a_4A_2 - a_2A_4) + B_4(a_2A_3 - a_3A_2)}{A_2B_3C_4} \\ 0 & \frac{b_3}{B_3} & \frac{b_4B_3 - b_3B_4}{B_3C_4} \\ 0 & 0 & \frac{c_4}{C_4} \end{bmatrix}. \tag{11}$$

Due to Lemma 2 there also exist four collinear anchor points. If these four points are base anchor points then the manipulator is architecturally singular for the same reason as given in Remark 2.

Therefore κ has to be a singular affinity with 4 collinear platform anchor points. W.l.o.g. we can assume that the planar platform is located in the xy-plane; i.e. $c_4 = 0$. Moreover we can assume that 3 points from $\mathsf{m}_1, \ldots, \mathsf{m}_4$ are not collinear. W.l.o.g. we can assume that these points are $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_4$, i.e. $a_2b_4 \neq 0$. Now we are left with the following possibilities:

a. $\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_5, \mathsf{m}_6$ collinear,
b. $\mathsf{m}_i, \mathsf{m}_j, \mathsf{m}_3, \mathsf{m}_k$ collinear,
c. $\mathsf{m}_i, \mathsf{m}_3, \mathsf{m}_5, \mathsf{m}_6$ collinear,

with $i, j \in \{1, 2, 4\}$, $i \neq j$ and $k \in \{5, 6\}$.

• Discussion of case a:

W.l.o.g. we can set $i = 1$ and $j = 2$. We solve equation $b_k = 0$ for B_k which yields: $B_k = (b_4B_3 - b_3B_4)C_k/(b_3C_4)$ for $k = 5, 6$.[2] Then the computation of Q^{200}_{5001} yields $a_2b_4A_2B_3^2C_4^2C_5C_6T[10]$ with

$$\begin{aligned} T := a_2b_3[C_4(A_5 - A_6) - A_4(C_5 - C_6)] \\ +[A_2(a_4b_3 - a_3b_4) + a_2b_4A_3](C_5 - C_6). \end{aligned} \tag{12}$$

Therefore we have to distinguish the following 2 cases:

1. $C_5C_6 = 0$: W.l.o.g. we can set $C_5 = 0$. Now Q^{020}_{5010} can only vanish w.c. for $a_3 = a_2A_3/A_2$. Then $Q^{200}_{4020} = 0$ implies $b_4 = b_3B_4/B_3$. Finally $Q^{200}_{4011} = 0$ yields the contradiction.
2. $T = 0$, $C_5C_6 \neq 0$: We can solve $T = 0$ for A_5 w.l.o.g.. Then $Q^{200}_{4011} = 0$ cannot vanish without contradiction.

• Discussion of case b:

W.l.o.g. we can set $i = 1$, $j = 2$ and $k = 5$. We get $b_3 = C_5 = 0$. Then Q^{200}_{5010} yields $a_2b_4A_2B_3^2C_4^2B_5C_6(C_4 - C_6)R[4]$ with

$$R := a_2(A_3B_5 - A_5B_3) + a_3A_2(B_3 - B_5). \tag{13}$$

Therefore we have to distinguish the following 4 cases:

1. $C_6 = 0$: This yields the collinearity of $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3, \mathsf{m}_5, \mathsf{m}_6$. Moreover Q^{200}_{0303} cannot vanish w.c..

[2] As b_3 is in the denominator of B_k we loose the case $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3, \mathsf{m}_5, \mathsf{m}_6$ collinear, but this case is discussed in case b, item 1.

2. $B_5 = 0$, $C_6 \neq 0$: Now Q^{200}_{4020} can only vanish w.c. for $C_4 = C_6$. Then Q^{020}_{4020} yields:

$$a_2 b_4 A_2 B_3^2 C_6^3 A_5 (A_2 - A_5)(A_4 - A_6)(a_2 A_3 - a_3 A_2). \quad (14)$$

We must distinguish two cases:

i. $A_4 = A_6$: Then $Q^{020}_{4110} = 0$ implies $a_3 = a_2 A_3 / A_2$. Now Q^{200}_{3030} can only vanish w.c. for $a_4 = a_2 A_3 / A_2$. Finally $Q^{200}_{3120} = 0$ yields the contradiction.

ii. $a_3 = a_2 A_3 / A_2$, $A_4 \neq A_6$: Then $Q^{200}_{3030} = 0$ can only vanish w.c. for

$$a_2[B_6(A_3 + A_4) - B_4(A_3 + A_6)] + a_4 A_2 (B_4 - B_6) = 0.$$

Assuming $B_4 \neq B_6$ we can express a_4 from this equation. Then $Q^{200}_{3120} = 0$ yields the contradiction.

For $B_4 = B_6$ this condition can only vanish w.c. for $B_6 = 0$. In this case $Q^{200}_{3021} = 0$ yields the contradiction.

3. $C_4 = C_6$, $B_5 C_6 \neq 0$: Now Q^{200}_{4110} yields

$$a_2 b_4 A_2 B_3^2 B_5 C_6^3 (B_4 - B_6) R[4]. \quad (15)$$

We must distinguish two cases:

i. $B_4 = B_6$: Now Q^{200}_{4020} and Q^{020}_{4110} can only vanish for $R = 0$ or $A_i = -a_i$ for $i = 2, 3$. As for $R = 0$ we get the contradiction from $Q^{200}_{3021} = 0$ we set $A_2 = -a_2$ and $A_3 = -a_3$. Then $Q^{200}_{3120} = 0$ implies $A_5 = A_3$ and finally $Q^{200}_{3021} = 0$ yields the contradiction.

ii. $R = 0$, $B_4 \neq B_6$: W.l.o.g. we can solve $R = 0$ for A_5. Then $Q^{200}_{3120} = 0$ cannot vanish w.c..

4. $R = 0$, $B_5 C_6 (C_4 - C_6) \neq 0$: In this case we get the contradiction from $Q^{200}_{4020} = 0$.

• Discussion of case c:

If $\mathsf{m}_1, \ldots, \mathsf{m}_4$ form a quad then we can renumerate the points such that $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_5, \mathsf{m}_6$ are collinear (see case a). The same can be done if m_3 is located on a side of the triangle $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_4$ whereas m_3 does not coincide with one of the vertices. Therefore we only have to discuss the case: $\mathsf{m}_i = \mathsf{m}_3, \mathsf{m}_5, \mathsf{m}_6$ collinear with $i \in \{1, 2, 4\}$. W.l.o.g. we can set $i = 1$ which implies $a_3 = b_3 = 0$. We compute $a_5 b_6 - a_6 b_5$ which can only vanish w.c. for

$$C_6(A_3 B_5 - A_5 B_3) + C_5(A_6 B_3 - A_3 B_6) = 0. \quad (16)$$

For $C_5 = C_6 = 0$ we get a special case of case b, item 1. For the case $C_i = 0$ and $A_3 B_i - A_i B_3 = 0$ with $i \in \{5, 6\}$ we already get an architecturally singular manipulator as $\mathsf{m}_1 = \mathsf{m}_3 = \mathsf{m}_i$ and $\mathsf{M}_1, \mathsf{M}_3, \mathsf{M}_i$ are collinear (cf. item 6 of Theorem 3 in [8]). Therefore only the discussion of the cases $C_5 C_6 \neq 0$ remains:

Under this assumption we can solve Eq. (16) for A_4 w.l.o.g.. Then Q^{200}_{5001} can only vanish w.c. for $(C_5 - C_6)P[4] = 0$ with

$$P := C_6(A_3 B_4 - A_4 B_3) + C_4(A_6 B_3 - A_3 B_6) = 0. \quad (17)$$

1. $C_5 = C_6$: Now Q^{020}_{5010} can only vanish w.c. for:

i. $A_3 = 0$: As for $C_4 = C_6$ the coefficient Q^{200}_{4101} yields the contradiction, Q^{200}_{5010} can only vanish for:

$$a_2(A_6 C_4 - A_4 C_6) + a_4 A_2 C_6 = 0. \quad (18)$$

W.l.o.g. we can express A_6 from this condition. Then $Q^{200}_{4101} = 0$ implies $a_4 = 0$ and from $Q^{200}_{4110} = 0$ we get $A_4 = 0$. Finally $Q^{200}_{4011} = 0$ yields the contradiction.

ii. $P = 0$, $A_3 \neq 0$: W.l.o.g. we can solve $P = 0$ for A_6. As for $C_4 = C_6$ the coefficient Q^{020}_{5010} yields the contradiction, Q^{200}_{5010} can only vanish for:

$$a_2(B_6 C_4 - B_4 C_6) + b_4 A_2 C_6 = 0. \quad (19)$$

From this equation we can express B_6. Then $Q^{020}_{5010} = 0$ implies $a_4 = A_3 b_4 / B_3$ and $Q^{200}_{4020} = 0$ yields $A_4 = B_4 A_3 / B_3$. Finally Q^{200}_{4011} cannot vanish w.c..

2. $P = 0$, $C_5 \neq C_6$: W.l.o.g. we can express A_6 from $P = 0$. Then Q^{200}_{5010} can only vanish w.c. for $a_4 = A_3 b_4 / B_3$. Then $Q^{200}_{4011} = 0$ yields the contradiction. □

IV. Main Theorem

The results of section 2 and 3 can be summed up into the so-called main theorem of SG platforms with a linear singularity surface:

Theorem 5. *A parallel manipulator of SG type possesses a linear singularity surface if and only if the platform and base are planar and affine equivalent and neither the base nor the platform anchor points are located on a (degenerated) conic section.*

Proof: Due to Theorem 3 and 4 there exists a affinity κ between corresponding anchor points of the planar platform and planar base. κ has to be regular because otherwise one set of anchor points is at least located on a line, which corresponds to an architecturally singular design.

Moreover we can again choose coordinate systems in the platform and the base such that $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$ and $c_i = C_i = 0$ for $i = 1, \ldots, 6$ hold. We can also assume that $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are not collinear, i.e. $A_2 B_3 \neq 0$. Then the matrix $\mathbf{K}$ of κ mapping $\mathsf{M}_i \mapsto \mathsf{m}_i$ for $i = 1, 2, 3$ equals:

$$\begin{bmatrix} \frac{a_2}{A_2} & \frac{a_3 A_2 - a_2 A_3}{A_2 B_3} & 0 \\ 0 & \frac{b_3}{B_3} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (20)$$

Now the determinant Q of the Jacobian factors into $a_2 b_3 H W_1[48] W_2[12] W_3[32] = 0$, whereby $a_2 b_3 = 0$ implies a singular affinity κ.

W_1 only depends on the design parameters and its geometric meaning is that the six base anchor points are located on a conic section. Therefore $W_1 = 0$ corresponds to an architecturally singular design. This is also true if one replaces the affinity κ by a projectivity. This even more general statement is very well known from the beginning of the 20th century (cf. Borel [1] and Bricard [2]).

The factor W_2 is independent of the translation parameters and is given by:

$$r_{13}r_{31}U + r_{23}r_{31}a_2B_3 - r_{13}r_{32}A_2b_3 \quad (21)$$

with $U := a_2A_3 - a_3A_2$. The factor W_3 is linear in the translation parameters and it can be written as $\alpha t_1 + \beta t_2 + \gamma t_3$ with:

$$\alpha = a_2(r_{13}b_3 + r_{31}B_3), \ \ \beta = b_3(a_2r_{23} + A_2r_{32}) - r_{31}U,$$
$$\gamma = r_{21}U + A_2(B_3H - b_3r_{22}) - a_2(B_3r_{11} - b_3r_{33}).$$

It can immediately be seen that W_2 or W_3 cannot vanish for all poses of the platform if κ is regular (see also KONG AND GOSSELIN [10]). □

Remark 3. It was shown by KARGER in [5] that planar parallel manipulators with affinely equivalent platform and base (for the special cases of equiform and congruent platforms see [4] and [9], respectively) have self-motions only if they are architecturally singular, i.e. if the anchor points are located on a conic section. Moreover it should be noted, that such manipulators can only have translatory self-motions. For more details on these self-motions we refer to the above cited papers of KARGER and to [10]. ◇

It should also be noted that the computation of singularity free zones in the space of translations is a very simple task for planar manipulators with affinely equivalent platform and base. For a fixed orientation the signed distance to the next singularity is given by the Hessian normal form of the plane $\alpha t_1 + \beta t_2 + \gamma t_3$. Therefore the radius $r > 0$ of the largest singularity free sphere is given by

$$r := \left| \frac{\alpha t_1 + \beta t_2 + \gamma t_3}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}} \right|. \quad (22)$$

As a consequence of Theorem 2 and 5 we can also characterize manipulators with a quadratic singularity surface:

Corollary 1. *A non-architecturally singular SG platform possesses a quadratic singularity surface if and only if the manipulator is planar and* $rk(\mathbf{M}) = 4$ *holds or if the manipulator is non-planar and there is a (regular or singular) affinity between corresponding anchor points.*

V. CONCLUSION

In the general case the singularity loci of parallel manipulators of Stewart Gough type is a cubic surface in the space of translations. Based on the set of SG platforms with a non-cubic singularity surface given by the author in [19], we determined all parallel manipulators of Stewart Gough type possessing a linear singularity surface. These manipulators are characterized by the fact that there exists a regular affinity between corresponding anchor points of the planar platform and planar base (cf. Theorem 5).

Moreover, these manipulators have the advantage, that their singularity surface can easily be visualized because it is a plane for any orientation of the platform. The radius of the largest singularity free sphere in the space of translations (for a fixed orientation) can be given explicitly (cf. Eq.(22)).

As a side product of Theorem 5 and the results of [19] we also characterized parallel manipulators of Stewart Gough type with a quadratic singularity surface (cf. Corollary 1).

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REFERENCES

[1] BOREL, E.: *Mémoire sur les déplacements à trajectoires sphérigues.* Mém. présenteés par divers savants, Paris(2), 33, 1–128 (1908).

[2] BRICARD, R.: *Mémoire sur les déplacements à trajectoires sphérigues.* Journ. École Polyt.(2), 11, 1–96 (1906).

[3] FLETCHER, R.: *Practical Methods of Optimization.* Wiley (1987).

[4] KARGER, A.: *Singularities and self-motions of equiform platforms.* Mechanism and Machine Theory **36** (7) 801–815 (2001).

[5] KARGER, A.: *Singularities and self-motions of a special type of platforms.* Advances in Robot Kinematics - Theory and Applications (J. Lenarcic, F. Thomas eds.), 155–164, Springer (2002).

[6] KARGER, A.: *Architecture singular planar parallel manipulators.* Mechanism and Machine Theory **38** (11) 1149–1164 (2003).

[7] KARGER, A.: *Stewart-Gough platforms with simple singularity surface.* Advances in Robot Kinematics - Mechanisms and Motion (J. Lenarcic, B. Roth eds.), 247–254, Springer (2006).

[8] KARGER, A.: *Architecturally singular non-planar parallel manipulators.* Mechanism and Machine Theory **43** (3) 335–346 (2008).

[9] KARGER, A.: *Parallel Manipulators with Simple Geometrical Structure.* Proceedings of the 2nd European Conference on Mechanism Science EuCoMeS'08 (M. Ceccarelli ed.), 463–470, Springer (2008).

[10] KONG, X., AND GOSSELIN, C.M.: *Generation of Architecturally Singular 6-SPS Parallel Manipulators with Linearly Related Platforms.* Electronic Journal of Computational Kinematics **1** (1) (2002).

[11] LI, H., GOSSELIN, C.M., RICHARD, M.J., AND MAYER ST-ONGE, B.: *Analytic Form of the Six-Dimensional Singularity Locus of the General Gough-Stewart Platform.* Journal of Mechanical Design **128** 279–287 (2006).

[12] LI, H., GOSSELIN, C.M., AND RICHARD, M.J.: *Determination of the maximal singularity-free zones in the six-dimensional workspace of the general Gough-Stewart platform.* Mechanism and Machine Theory **42** (4) 497–511 (2007).

[13] NAWRATIL, G.: *Results on Planar Parallel Manipulators with Cylindrical Singularity Surface.* Advances in Robot Kinematics - Analysis and Design (J. Lenarcic, P. Wenger eds.), 321–330, Springer (2008).

[14] NAWRATIL, G.: *Main Theorem on Planar Parallel Manipulators with Cylindrical Singularity Surface.* Proceedings of 33. Süddeutsches Differentialgeometrie Kolloquium (H. Havlicek, F. Manhart, B. Odehnal eds.), Vienna, Austria, 35–57, TU Vienna (2008).

[15] NAWRATIL, G.: *On the degenerated cases of architecturally singular planar parallel manipulators.* Journal of Geometry and Graphics **12** (2) 141-149 (2008).

[16] NAWRATIL, G.: *All Planar Parallel Manipulators with Cylindrical Singularity Surface.* Mechanism and Machine Theory **44** (12) 2179–2186 (2009).

[17] NAWRATIL, G.: *A new approach to the classification of architecturally singular parallel manipulators.* Computational Kinematics (A. Kecskemethy, A. Müller eds.), 349–358, Springer (2009).

[18] NAWRATIL, G.: *A remarkable set of Schönflies-singular planar Stewart Gough platforms.* Technical Report No. 198, Geometry Preprint Series, Vienna University of Technology (2009).

[19] NAWRATIL, G.: *Stewart Gough platforms with non-cubic singularity surface.* Technical Report No. 204, Geometry Preprint Series, Vienna University of Technology (2010).

[20] RÖSCHEL, O., AND MICK, S.: *Characterisation of architecturally shaky platforms.* Advances in Robot Kinematics - Analysis and Control (J. Lenarcic, M.L. Husty eds.), 465–474, Kluwer (1998).

[21] WOHLHART, K.: *From higher degrees of shakiness to mobility.* Mechanism and Machine Theory **45** (3) 467–476 (2010).